PREFACE

New Syllabus Mathematics is a series of four books. These books follow the Mathematics Syllabus for Secondary Schools, implemented from 2001 by the Ministry of Education, Singapore. The whole series covers the complete syllabus for the Singapore-Cambridge GCE 'O' Level Mathematics.

The fifth edition of New Syllabus Mathematics 4 retains the goals and objectives of the previous edition, but has been revised to meet the requests of users of the fourth edition and to keep materials up-to-date as well as to give students a better understanding of the contents.

All topics are comprehensively dealt with to give students a firm grounding in the subject. Explanations of concepts and principles are concise and written in clear language with supportive illustrations and examples. Examples and exercises have been carefully graded to aid students in progressing within, as well as up, each level. Those exercises marked with a ✶ are either tricky or involve more calculations. "Problem Solving" and "Exploration", placed at the end of the chapter, contain more difficult and challenging questions requiring students to apply their knowledge and experience in solving them.

Numerous revision exercises are provided at appropriate intervals to enable students to recapitulate what they have learnt. In addition, there are mid-year and final-year examination specimen papers.

Important features which have been retained in this edition to facilitate learning are:

- an interesting introduction at the beginning of each chapter complete with photographs or graphics
- brief specific instructional objectives for each chapter
- in-class activities (investigation / discussion / problem solving)
- activities and interesting information in the marginal text (clip-notes, "Down Memory Lane", "Back In Time", "Investigate", "Check This Out!", "It's A Fact", "Just For Fun", "Are You Game Enough?", "For Your Information", "Library Corner" and "Problems")

Problem-solving heuristics are subsequently introduced at appropriate sections of the book to reinforce problem-solving skills. In addition, questions which call for problem-solving skills are also set in the margin for students to do at their own pace and time.

Ample opportunities are also provided for mathematical investigative and communicative activities.

It is hoped that these features will help students learn mathematics with more zest and excel in the subject.

CONTENTS

CHAPTER

1

Cumulative Frequency Distribution

In this chapter, you will learn how to

- construct a cumulative frequency table;
- draw a cumulative frequency curve;
- estimate the median and quartiles from the cumulative frequency curve.

Preliminary Problem

Do you know how many babies have been born in Singapore since 1983? Such figures are recorded by the Singapore Population Census Office. Each year, the number of babies that are born is added to the accumulated number of the previous years. In this way, the government is able to analyse the population growth.

Cumulative Frequency Table

So far, we have learnt different ways of presenting data. Another way of presenting a set of data is by using the **table of cumulative frequencies**. The cumulative frequency table provides information such as how many students obtained less than a certain mark, how many days in a month the fluoride level in drinking water was below a certain value, how many people weighed less than a certain weight and so on.

A cumulative frequency table can be represented by a **cumulative frequency curve** which is also known as an **ogive**.

The following examples explain how cumulative frequency tables and cumulative frequency curves are constructed and applied to solve problems.

Example 1

The lengths of 40 insects of a certain species were measured correct to the nearest millimetre. The frequency distribution is given below:

Length (mm)	*25–29*	*30–34*	*35–39*	*40–44*	*45–49*	*50–54*	*55–59*
Frequency	*1*	*3*	*6*	*12*	*10*	*6*	*2*

(a) Construct a cumulative frequency table for the given data.
(b) Draw a cumulative frequency curve for the data.
(c) Estimate from the curve
  (i) the number of insects that were less than 43.5 mm long,
  (ii) the percentage of insects that were of length 37.5 mm or more,
  (iii) the value of l, if 75% of the insects were less than l mm long.

Solution

(a) The cumulative frequency table is constructed below. The table shows the cumulative frequency distribution of the lengths of the 40 insects:

Length (mm)	*Upper class boundaries*	*Frequency*	*Length less than*	*Cumulative frequency*
25–29	29.5	1	29.5	1
30–34	34.5	3	34.5	1 + 3 = 4
35–39	39.5	6	39.5	4 + 6 = 10
40–44	44.5	12	44.5	10 + 12 = 22
45–49	49.5	10	49.5	22 + 10 = 32
50–54	54.5	6	54.5	32 + 6 = 38
55–59	59.5	2	59.5	38 + 2 = 40

(b) The cumulative frequency curve is drawn by plotting the cumulative frequencies against the upper class boundaries, i.e. plotting the points corresponding to the ordered pairs (29.5, 1), (34.5, 4), (39.5, 10), (44.5, 22), (49.5, 32), (54.5, 38) and (59.5, 40). Fig. 1.1 shows the cumulative frequency curve representing the cumulative frequency distribution of the lengths of the 40 insects. The points plotted are joined by a smooth curve.

Note: The ordered pair (24.5, 0) indicates that there were no insects with lengths less than 24.5 cm.

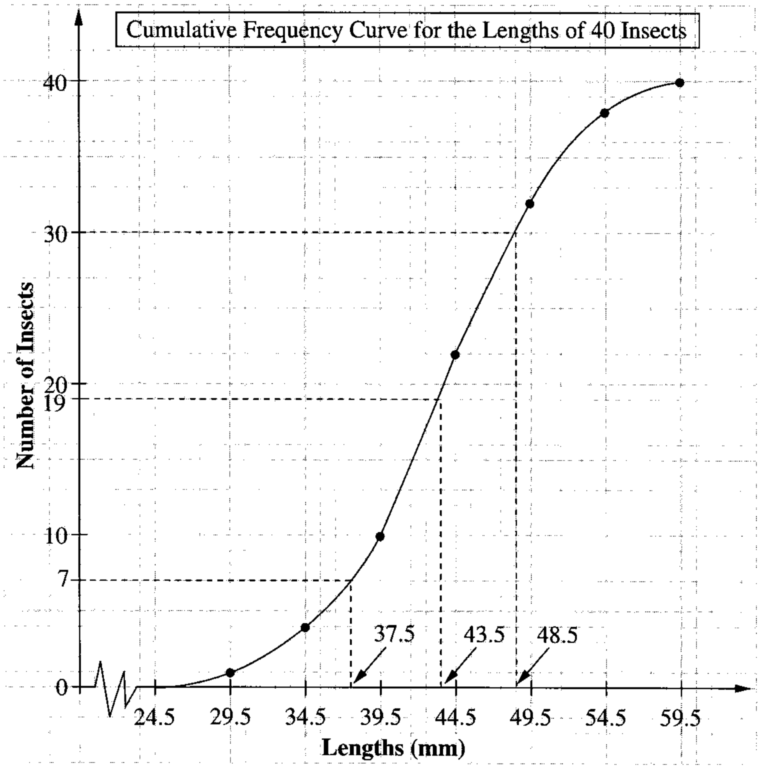


Fig. 1.1

(c) (i) To estimate the number of insects that were less than 43.5 mm long, locate the length 43.5 mm on the horizontal axis. Draw a vertical line to meet the curve followed by a horizontal line to meet the vertical axis or cumulative frequency axis as shown in the diagram.

From the graph, the number of insects that were *less than* 43.5 mm long is 19.

(ii) Find 37.5 on the horizontal axis and draw a vertical line to meet the curve and then draw a horizontal line to meet the vertical axis (see Fig. 1.1).

From the graph, 7 insects were less than 37.5 mm long.

$\therefore$ the number of insects that were of length 37.5 mm or more = 40 – 7 = 33.

The percentage of insects that were of length 37.5 mm or more is

$$\frac{33}{40} \times 100\% = 82.5\%$$

(iii) 75% of $40 = \frac{3}{4} \times 40 = 30$

$\therefore$ 30 insects were less than l mm long.

From 30 on the vertical axis, draw a horizontal line to meet the curve followed by a vertical line to meet the horizontal axis. *From the graph,* 30 insects were less than 48 mm long.

$\therefore$ $l = 48.5$

Besides cumulative frequency curve, the cumulative frequency polygon can be used to find the ***median****. However, by this method, the median obtained is only an* ***estimate*** *of the true median.*

If the plotted points in Fig. 1.1 are joined instead by a straight line as shown in Fig. 1.2, a **cumulative frequency polygon** is obtained.

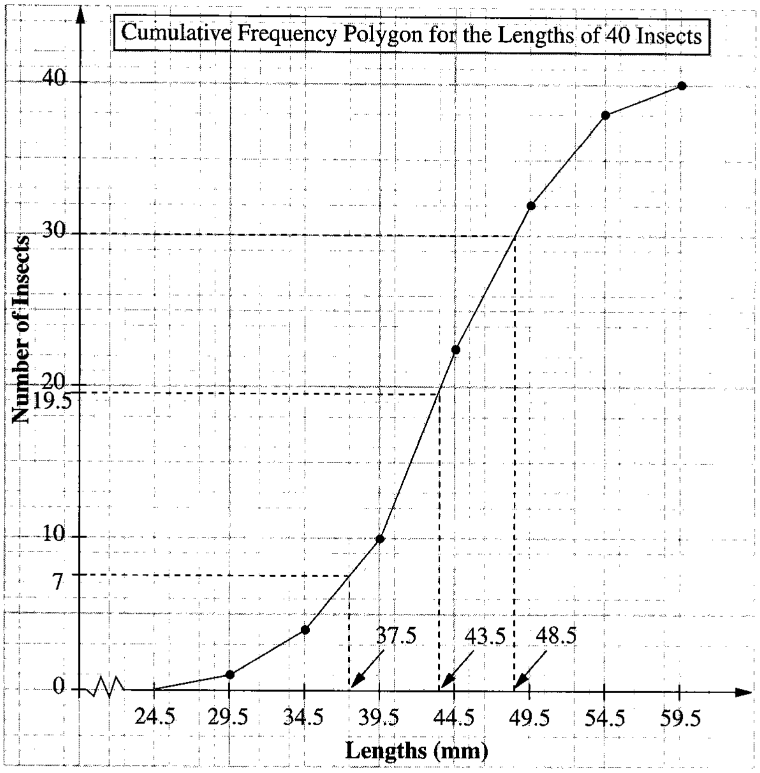


Fig. 1.2

The values estimated from the cumulative frequency curve and the cumulative frequency polygon may be slightly different (compare the values shown in Fig. 1.1 and Fig. 1.2). For most purposes, they are acceptable although the values are merely *estimates.*

Example 2

The masses of 300 apples were measured. The table gives the cumulative frequency distribution of the masses:

Mass (x g)	*Number of apples*
$x \leqslant 60$	*0*
$x \leqslant 70$	*8*
$x \leqslant 80$	*19*
$x \leqslant 90$	*57*
$x \leqslant 95$	*89*
$x \leqslant 100$	*141*
$x \leqslant 105$	*216*
$x \leqslant 110$	*266*
$x \leqslant 120$	*290*
$x \leqslant 130$	*300*

(a) Draw a cumulative frequency curve.

(b) Estimate from the curve

- *(i) the number of apples having masses 98 g or less,*
- *(ii) the value of m given that 20% of the apples had masses more than m g.*

(c) Taking class intervals $60 < x \leqslant 70$, $70 < x \leqslant 80$, $80 < x \leqslant 90$, $90 < x \leqslant 95$, *..., construct a frequency distribution and draw a histogram.*

Solution

(a) The graph shows the cumulative frequency curve.

(b) (i) *From the curve*, we estimate that 120 apples have masses 98 g or less.

(ii) 20% of 300

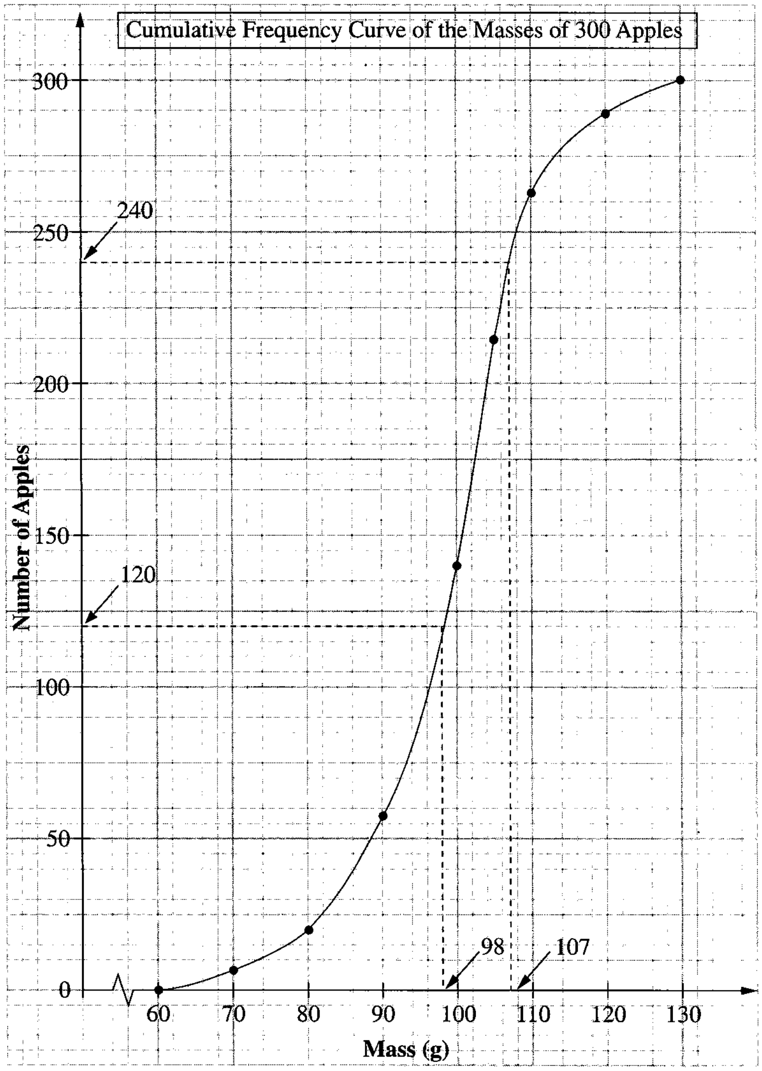
$= \frac{20}{100} \times 300$

$= 60$

$\therefore$ 60 apples have masses more than m g, i.e. $300 - 60 = 240$ apples have masses m g or less.

From the curve, 240 apples have masses 107 g or less.

$\therefore \quad m = 107$.



(c) The frequency distribution is constructed as shown in the following table:

Mass (x g)	*Cumulative frequency*	*Mass (x g)*	*Frequency*	*Frequency density*
$x \leqslant 70$	8	$60 < x \leqslant 70$	8	$8 \div 10 = 0.8$
$x \leqslant 80$	19	$70 < x \leqslant 80$	$19 - 8 = 11$	$11 \div 10 = 1.1$
$x \leqslant 90$	45	$80 < x \leqslant 90$	$45 - 19 = 26$	$26 \div 10 = 2.6$
$x \leqslant 95$	89	$90 < x \leqslant 95$	$89 - 45 = 44$	$44 \div 5 = 8.8$
$x \leqslant 100$	141	$95 < x \leqslant 100$	$141 - 89 = 52$	$52 \div 5 = 10.4$
$x \leqslant 105$	216	$100 < x \leqslant 105$	$216 - 141 = 75$	$75 \div 5 = 15$
$x \leqslant 110$	266	$105 < x \leqslant 110$	$266 - 216 = 50$	$50 \div 5 = 10$
$x \leqslant 120$	290	$110 < x \leqslant 120$	$290 - 266 = 24$	$24 \div 10 = 2.4$
$x \leqslant 130$	300	$120 < x \leqslant 130$	$300 - 290 = 10$	$10 \div 10 = 1$
			Total = 300	

Figure 1.3 shows the histogram.

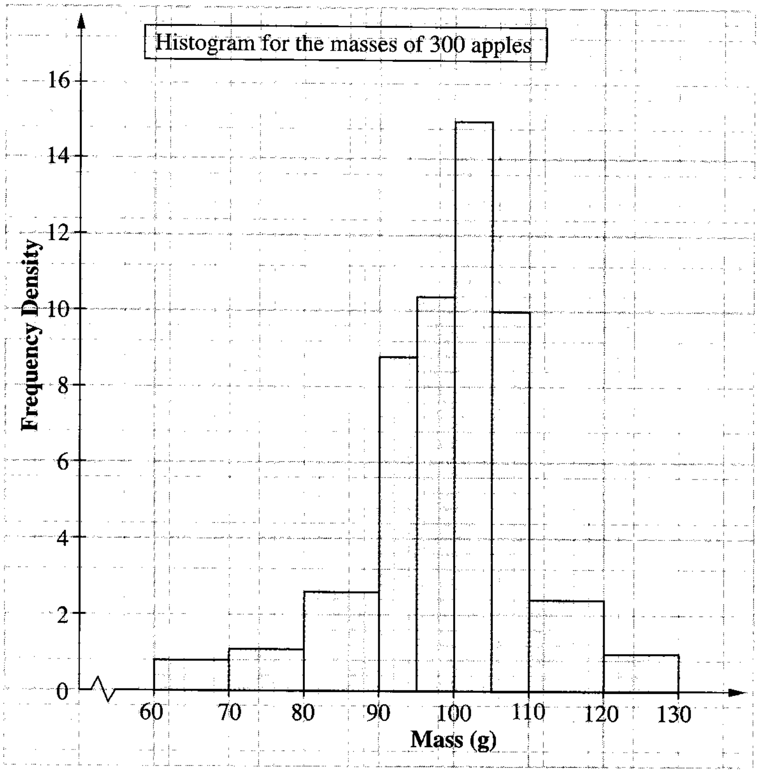


Fig. 1.3

Exercise 1a

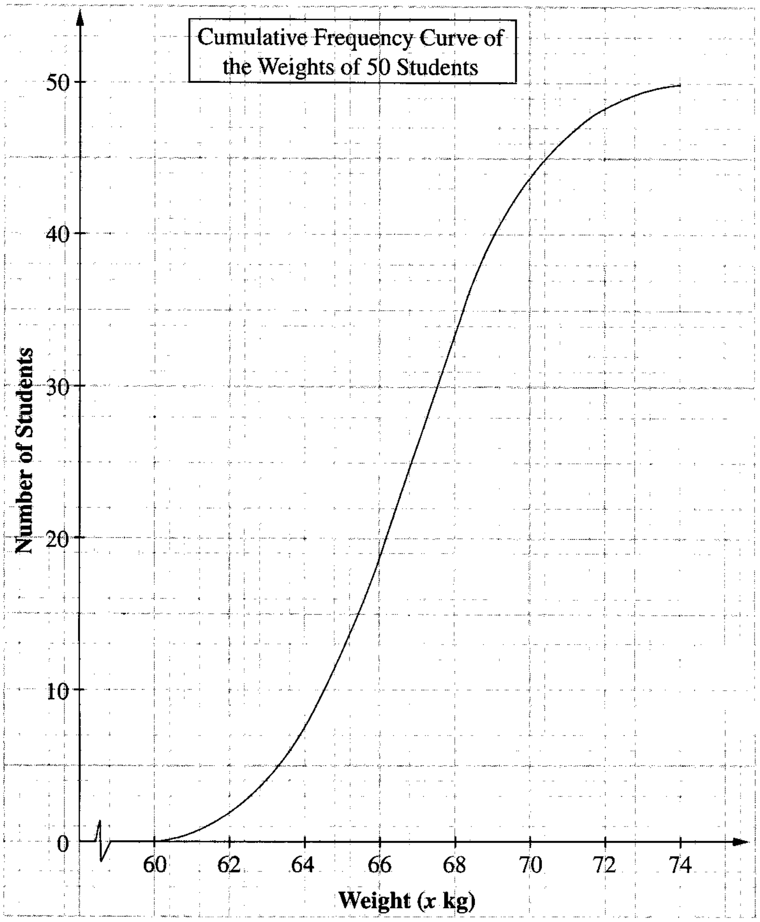
1. The weights, in kg, of 50 students are measured. The cumulative frequency curve shows the weight, x kg, and the number of students whose weights are less than or equal to x kg. (As an example, 20 students have weights of 66.2 kg or less.)

   Use the curve to estimate

   (a) the number of students whose weights are less than or equal to 65 kg,

   (b) the number of students whose weights are more than 68.6 kg,

   (c) the percentage of the total number of students whose weights are more than 64.4 kg.



2. The speeds of 100 motor vehicles passing a certain point in a busy street are recorded.

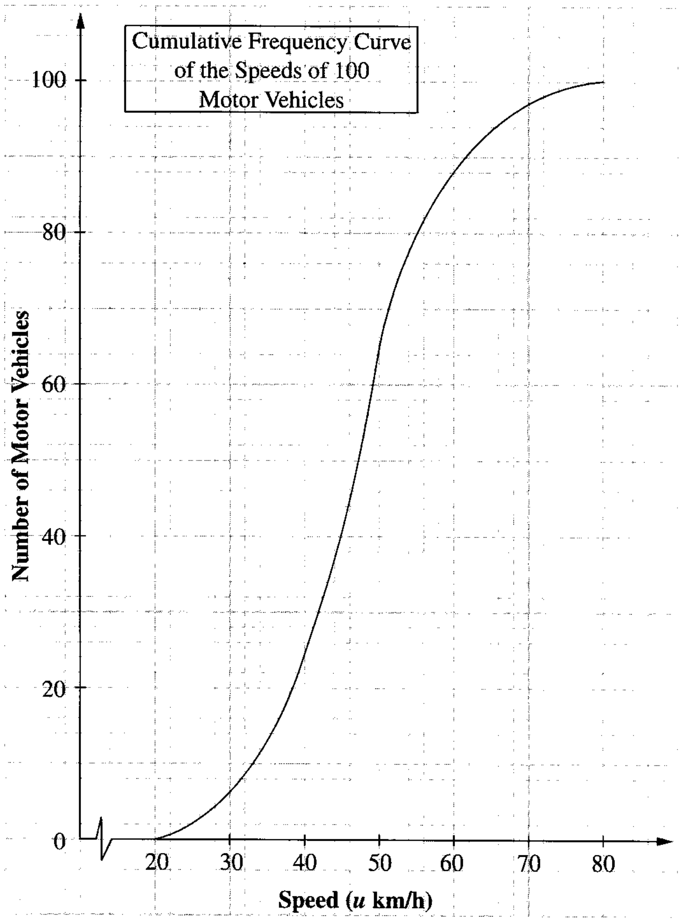
   The cumulative frequency curve shows the speed, u km/h and the number of vehicles, whose speeds are less than u km/h. (As an example, 74 vehicles have speeds less than 53 km/h.)

   Use the curve to estimate

   (a) the number of vehicles whose speeds are less than 34 km/h,

   (b) the fraction of the total number of vehicles whose speeds are greater than or equal to 59 km/h,

   (c) the value of v, if 40% of the vehicles have a speed less than v km/h.



3. The results of a music examination taken by 160 pupils are shown in the cumulative frequency table below:

Mark	< 10	< 20	< 30	< 40	< 50	< 60	< 70	< 80
Number of pupils	0	8	21	55	103	135	150	160

   (a) Using a *horizontal* scale of 2 cm to represent 10 marks and a *vertical* scale of 1 cm to represent 10 pupils, draw a cumulative frequency curve for the results.

   (b) Use your graph to estimate

   (i) the number of pupils who scored less than 45 marks,

   (ii) the fraction of the total number of pupils who failed the music examination given that 34 is the lowest mark to pass the examination,

   (iii) the value of x if 22.5% of the pupils obtained at least x marks in the music examination.

4. Fifty loaves of bread from a bakery are weighed. Their weights are distributed as shown in the cumulative frequency table below:

Weight (g)	445	446	447	448	449	450	451	452	453	454	455
Number of loaves of this weight or less	0	2	6	11	20	30	37	42	45	48	50

(a) Draw a cumulative frequency curve for these results using the following scales.

On the *horizontal* axis, take values of weight from 445 g to 455 g and use a scale of 2 cm to represent 1 g.

On the *vertical* axis, take values of the cumulative frequency from 0 to 50 and use a scale of 2 cm to represent 10 loaves.

(b) Use your graph to estimate

(i) the number of loaves of bread whose weights are less than or equal to 450.4 g,

(ii) the number of loaves rejected either because they are underweight or overweight, given that a loaf is underweight if it weighs 446.3 g or less and overweight if it weighs more than 453.7 g.

(iii) the value of x if $\frac{3}{10}$ of the loaves weigh more than x g.

5. 500 earthworms were collected from a sample of soil. Their lengths were recorded and the results are given in the following table:

Length (mm)	*Number of worms*
$10 < x \leqslant 20$	10
$20 < x \leqslant 30$	20
$30 < x \leqslant 40$	50
$40 < x \leqslant 50$	90
$50 < x \leqslant 60$	150
$60 < x \leqslant 70$	100
$70 < x \leqslant 80$	50
$80 < x \leqslant 90$	20
$90 < x \leqslant 100$	10

(a) Copy and complete the following cumulative frequency table:

Length (mm)	$\leqslant 10$	$\leqslant 20$	$\leqslant 30$	$\leqslant 40$	$\leqslant 50$	$\leqslant 60$	$\leqslant 70$	$\leqslant 80$	$\leqslant 90$	$\leqslant 100$
Number of worms	0	10								500

(b) Draw a cumulative frequency curve to represent the results by using 2 cm to represent 100 worms on the *vertical* axis and taking values of the cumulative frequency from 0 to 500. On the *horizontal* axis, take values of the length from 10 mm to 100 mm and use a scale of 1 cm to represent 10 mm.

(c) Use your graph to estimate

(i) the number of earthworms whose lengths are less than or equal to 58 mm,

(ii) the percentage of earthworms whose lengths are greater than 76 mm,

(iii) the value of x if 18% of the earthworms are of length x mm or less.

6. The amount of time spent by 750 pupils of a certain school to travel from home to school on a particular morning is given in the following table:

Time taken in minutes	*Number of pupils*
$0 \leqslant x < 5$	30
$5 \leqslant x < 10$	50
$10 \leqslant x < 15$	70
$15 \leqslant x < 20$	120
$20 \leqslant x < 25$	260
$25 \leqslant x < 30$	130
$30 \leqslant x < 35$	40
$35 \leqslant x < 40$	30
$40 \leqslant x < 45$	20

(a) Copy and complete the following cumulative frequency table:

Time taken in minutes	0	5	10	15	20	25	30	35	40	45
Number of pupils who spend less than this time	0	30								750

(b) Draw a cumulative frequency curve to illustrate the results using the following scales:

On the *horizontal* axis, take values of time from 0 minute to 45 minutes and use a scale of 2 cm to represent 5 minutes.

On the *vertical* axis, take values of the cumulative frequency from 0 to 750 and use a scale of 2 cm to represent 100 pupils.

(c) Use your graph to estimate

(i) the number of pupils who take less than 17.5 minutes to travel to school,
(ii) the fraction of the 750 pupils who take at least 27 minutes to travel to school,
(iii) the value of x given that 40% of the 750 pupils take at least x minutes to travel to school.

7. The lengths of 600 leaves from a tree are measured. The following table gives the cumulative frequency distribution of these lengths:

Length (x mm)	$x \leqslant 20$	$x \leqslant 25$	$x \leqslant 30$	$x \leqslant 35$	$x \leqslant 40$	$x \leqslant 45$	$x \leqslant 50$
Number of leaves	0	20	80	260	500	580	600

(a) Draw a cumulative frequency curve to represent these results using the following scales:
On the *horizontal* axis, take values of the length from 20 mm to 50 mm and use a scale of 2 cm to represent 5 mm.

On the *vertical* axis, take values of the cumulative frequency from 0 to 600 and use a scale of 2 cm to represent 100 leaves.

(b) Use your graph to estimate

(i) the number of leaves whose lengths are less than or equal to 41.5 mm,
(ii) the percentage of leaves whose lengths are greater than 33 mm.

(c) Copy and complete the following frequency distribution table:

Length (x mm)	*Number of leaves*
$20 < x \leqslant 25$	20
$25 < x \leqslant 30$	60
$30 < x \leqslant 35$	
$35 < x \leqslant 40$	
$40 < x \leqslant 45$	
$45 < x \leqslant 50$	

(d) Draw a histogram to represent the frequency distribution in (c).

8. The table below shows the amount of milk (in kg) obtained from each of the 70 cows of a dairy farm on a particular day:

Amount of milk (x kg)	*Number of cows*
$0 \leqslant x < 4$	7
$4 \leqslant x < 6$	11
$6 \leqslant x < 8$	17
$8 \leqslant x < 10$	20
$10 \leqslant x < 12$	10
$12 \leqslant x < 14$	5

(a) Construct a cumulative frequency table and draw a cumulative frequency curve.

(b) Use your curve to estimate

- **(i)** the number of cows that give less than 9.4 kg of milk,
- **(ii)** the fraction of the 70 cows that give at least 7.4 kg of milk,
- **(iii)** the value of x if 70% of the cows give at least x kg of milk.

Median, Quartiles and Percentiles

We recall that the median is the **middle value** when a set of data is arranged in order of increasing magnitude, i.e. the median divides the set of data into two equal halves. For example, the following numbers are arranged in ascending order:

$$2,\ 5,\ \underset{\substack{\uparrow\\ \textbf{lower quartile}}}{10},\ 12,\ 16,\ \underset{\substack{\uparrow\\ \textbf{median}}}{24},\ 30,\ 40,\ \underset{\substack{\uparrow\\ \textbf{upper quartile}}}{54},\ 65,\ 70$$

The median 24 divides the set of numbers into 2 equal halves. We extend the idea of the median to find the middle value of each half. The middle value of the lower half is 10 and is called the **lower quartile**. The middle value of the upper half is 54 and is called the **upper quartile**. The lower quartile, the median and the upper quartile which divide the set of numbers into four equal parts are called **quartiles**.

In the set of numbers

⑧ ⑮ ㉒

4 7 8 8 10 14 16 17 20 24 25 28,

↑ ↑ ↑

lower quartile (Q_1) median (Q_2) upper quartile (Q_3)

Q_1, Q_2 and Q_3 are 8, 15 and 22 respectively.

Similarly, the values dividing a set of data into 100 equal parts are called **percentiles** and are denoted by $P_1, P_2, \ldots, P_{99}$.

Recall that for ungrouped data, the median can be easily obtained from a set of observations which is arranged in *order of increasing magnitude* and is dependent on the number of observations in the set. For grouped data, the median of a frequency distribution can be estimated graphically from the cumulative frequency curve of the distribution.

Fig. 1.4 shows the cumulative frequency curve representing the distribution of the time taken by 60 pupils to solve a mathematics problem.

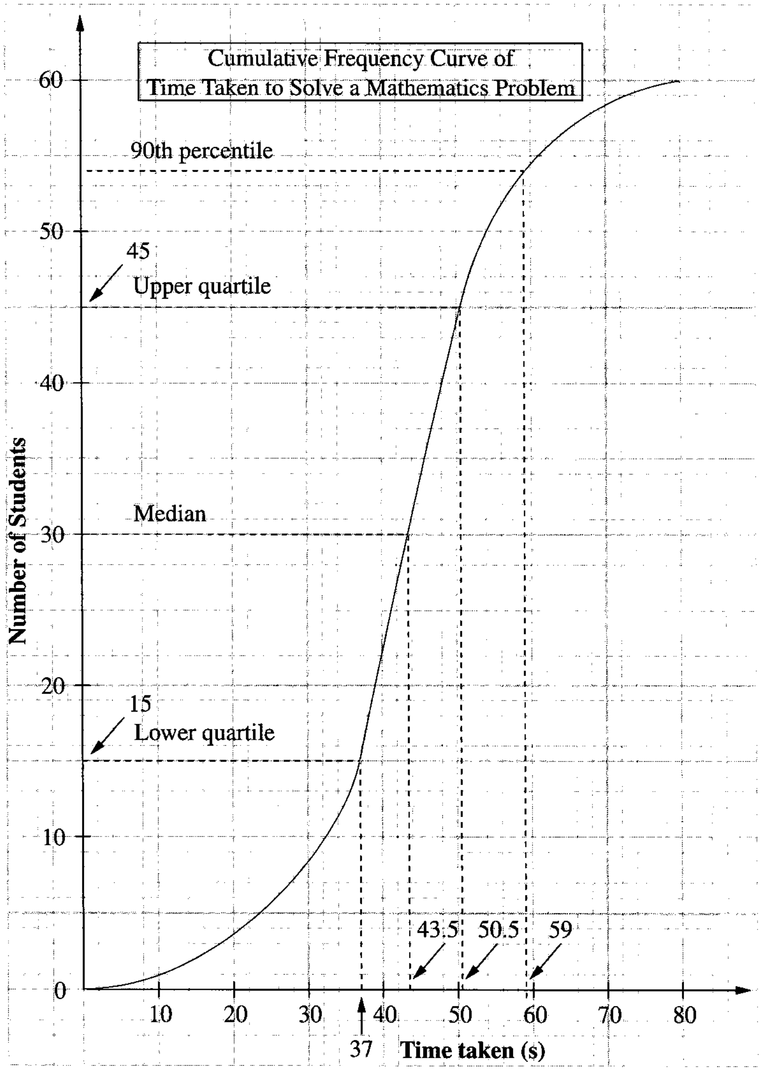


Fig. 1.4

To estimate the median time taken from the cumulative frequency, we note that half or 50% of the 60 pupils take less than or equal to the median time to solve the problem.

Median corresponds to the **50th percentile**, i.e. $Q_2 = P_{50}$.

One half of the total frequency = $\frac{1}{2} \times 60 = 30$.

From 30 on the vertical axis, draw a horizontal line to meet the curve and then draw a vertical line to meet the horizontal axis as shown in Fig. 1.4. *From the curve*, the median time = 43.5 seconds.

Similarly, to estimate the lower quartile, we obtain one-quarter of the total frequency = $\frac{1}{4} \times 60 = 15$. *From the curve*, the lower quartile = 37 seconds.

$\frac{1}{4}$ or 25% of the pupils take less than or equal to 37 seconds to solve the problem. The **lower quartile** corresponds to the **25th percentile**, i.e. $Q_1 = P_{25}$.

75% of the total frequency = $\frac{75}{100} \times 60 = 45$. *From the curve*, the upper quartile = 50.5 seconds.

$\frac{3}{4}$ or 75% of the pupils take less than or equal to 50.5 seconds to solve the problem. The **upper quartile** corresponds to the **75th percentile**, i.e. $Q_3 = P_{75}$.

To estimate the 90th percentile, we note that 90% of the total frequency = $\frac{90}{100} \times 60 = 54$. *From the curve*, the 90th percentile = 59 seconds.

i.e. 90% of the pupils take less than or equal to 59 seconds to solve the problem.

Example 3

The examination marks of 100 pupils are given in the table:

Mark	*Number of pupils*
$x \leqslant 10$	2
$10 < x \leqslant 20$	12
$20 < x \leqslant 30$	25
$30 < x \leqslant 40$	29
$40 < x \leqslant 50$	15
$50 < x \leqslant 60$	10
$60 < x \leqslant 70$	4
$70 < x \leqslant 80$	3

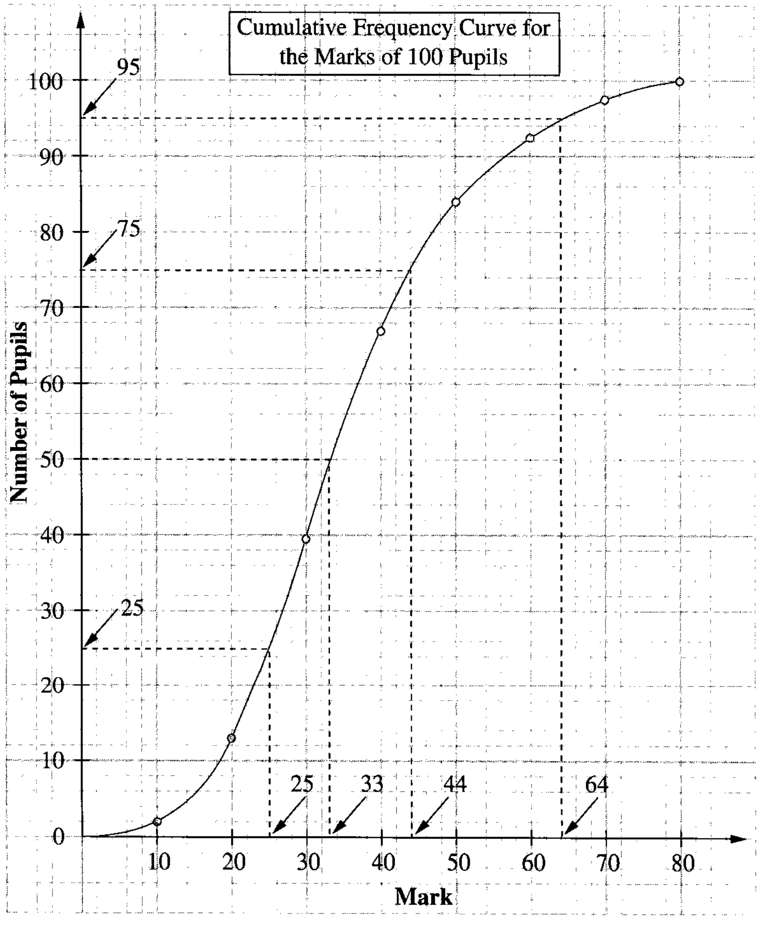
(a) Construct a cumulative frequency table, using the classes "≤ 10", "≤ 20", and so on.
(b) Draw the cumulative frequency curve for the results obtained.
(c) Use your curve to estimate
  (i) the median mark,
  (ii) the upper quartile,
  (iii) the lower quartile,
  (iv) the minimum mark required to gain a distinction if the top 5% of the pupils are awarded a distinction.

Solution

(a) The table below shows the cumulative frequency table.

Mark	*Number of pupils*
⩽ 10	2
⩽ 20	14
⩽ 30	39
⩽ 40	68
⩽ 50	83
⩽ 60	93
⩽ 70	97
⩽ 80	100

(b) The graph below shows the cumulative frequency curve for the results:



(c) (i) 50% of the total frequency = $\frac{50}{100} \times 100 = 50$

From the curve, the median mark = 33.

(ii) $\frac{3}{4}$ of the total frequency = $\frac{3}{4} \times 100 = 75$

From the curve, the upper quartile = 44.

(iii) 25% of the total frequency = $\frac{25}{100} \times 100 = 25$

From the curve, the lower quartile = 25.

(iv) 95% of the total frequency = $\frac{95}{100} \times 100 = 95$

From the curve, 95% of the pupils obtained 64 marks or less.

∴ the minimum mark required for distinction = 65.

Interquartile Range

The interquartile range of a set of numbers is the *difference* between the upper quartile and the lower quartile,

i.e. **Interquartile range = Upper quartile – Lower quartile = $Q_3 - Q_1$.**

The interquartile range of the distribution of the time taken by 60 pupils to solve a mathematics problem (see Fig. 1.4) is given by $Q_3 - Q_1 = 50.5 - 37 = 13.5$ seconds.

The interquartile range of the distribution of the marks of 100 pupils in an examination discussed in Example 3 is given by $Q_3 - Q_1 = 44 - 25 = 19$ marks.

The **interquartile range** gives an indication of how the numbers in a set of data are spread about the median of the set. A *small interquartile range* indicates that the numbers cluster closely around the median while a *large interquartile range* indicates that the data is spread across a wide range of values.

Example 4

Instantaneous speeds of 100 motor vehicles at a point on a busy street were recorded as shown below:

Speed (v km/h)	*Frequency*
$20 \leqslant v < 30$	6
$30 \leqslant v < 40$	18
$40 \leqslant v < 50$	39
$50 \leqslant v < 60$	25
$60 \leqslant v < 70$	9
$70 \leqslant v < 80$	3

(a) Construct a cumulative frequency table.

(b) Draw a cumulative frequency curve.

(c) Use the curve to estimate

(i) the quartiles, *(ii) the interquartile range,*

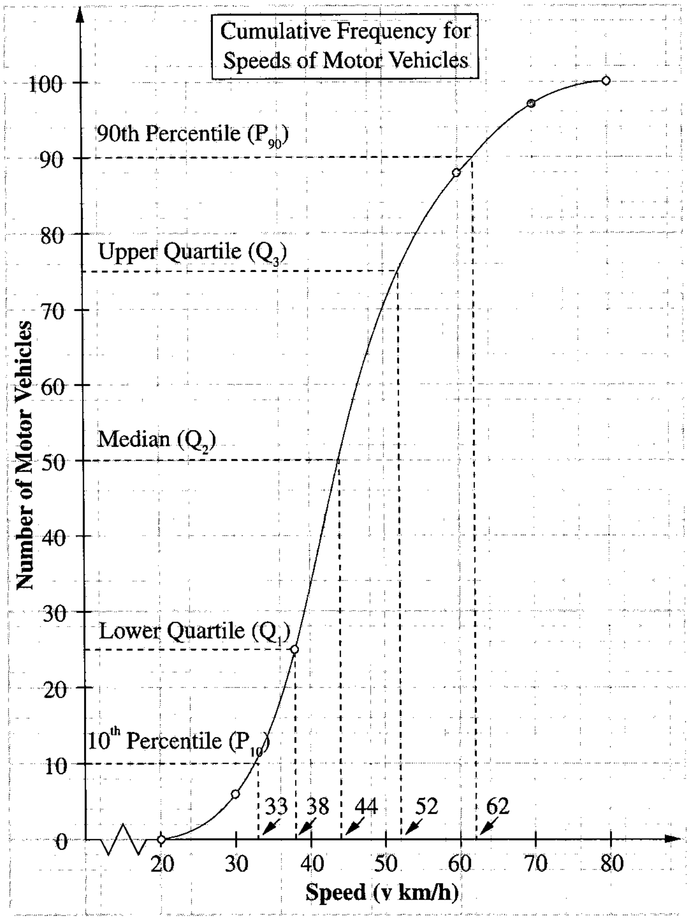
(iii) the 10th and 90th percentiles.

Solution

(a) The cumulative frequency table is shown below:

Speed (less than)	20	30	40	50	60	70	80
Cumulative frequency	0	6	24	63	88	97	100

(b) The graph below shows the cumulative frequency curve representing the results:



(c) (i) *From the graph,*

the median speed, Q_2 = 44 km/h.

the lower quartile, Q_1 = 38 km/h.

the upper quartile, Q_3 = 52 km/h.

(ii) The interquartile range = $Q_3 - Q_1 = 52 - 38 = 14$ km/h.

(iii) 10% of the total frequency = $\frac{10}{100} \times 100 = 10$

From the graph, the 10th percentile = 33 km/h.

90% of the total frequency = $\frac{90}{100} \times 100 = 90$

From the graph, the 90th percentile = 62 km/h.

Example 5

The following information gives the weekly expenditure on food for 80 households in a certain area:

- *5 households had weekly expenditure ≤ 30 dollars.*
- *None spent more than 100 dollars on food per week.*
- *The median weekly expenditure on food was 56 dollars.*
- *The lower quartile of the distribution was 45 dollars.*
- *The interquartile range of the distribution was 21 dollars.*

(a) Use this information to draw a smooth cumulative frequency curve for the weekly expenditure on food for the 80 households.

(b) Use your graph to estimate

(i) the 40th percentile,

(ii) the number of households with a weekly expenditure of between \$33 and \$73 on food.

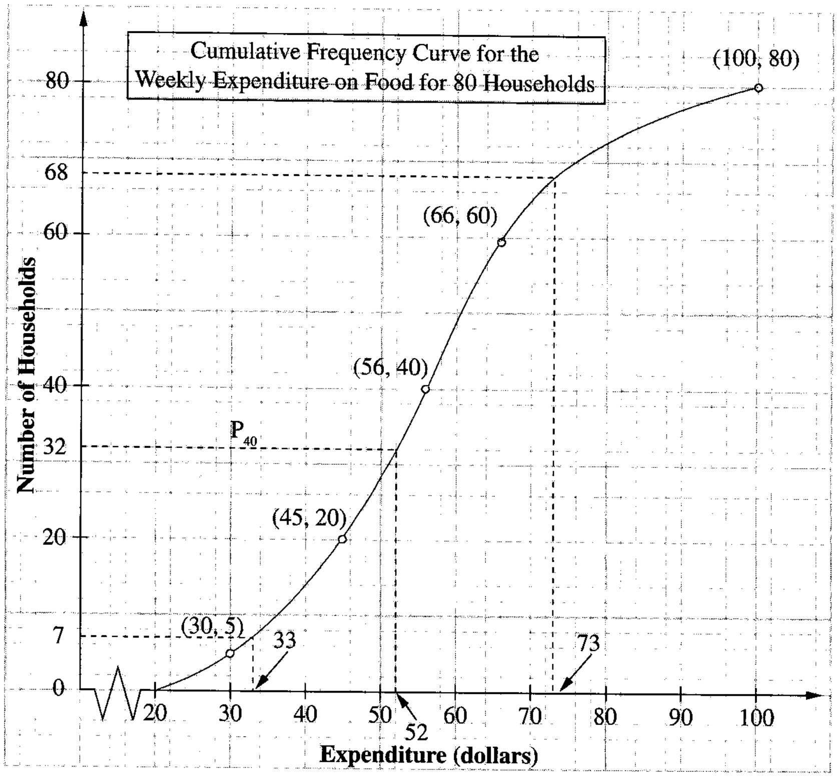
Solution

(a) Interquartile range = $Q_3 - Q_1 = 21$

$\therefore \quad Q_3 = 21 + Q_1 = 21 + 45 = 66.$

The given information indicates the following points on the cumulative frequency curve: (30, 5), (100, 80), (56, 40), (45, 20) and (66, 60).

$\uparrow$ Q_2 $\uparrow$ Q_1 $\uparrow$ Q_3



(b) (i) The total frequency is 80.

$$40\% \text{ of the total frequency} = \frac{40}{100} \times 80 = 32$$

From the graph, the 40th percentile is approximately $52.

(ii) *From the graph*, the estimated number of households with a weekly expenditure on food between $33 and $73 is 68 – 7 = 61.

Example 6

In an agricultural experiment, the lengths of 124 ears of barley were measured. The data obtained is expressed in the following table:

Length (mm)	*Number of ears of barley*
$x \leqslant 20$	*1*
$20 < x \leqslant 30$	*8*
$30 < x \leqslant 40$	*35*
$40 < x \leqslant 50$	*50*
$50 < x \leqslant 60$	*25*
$60 < x \leqslant 70$	*5*

(a) Construct a cumulative frequency table, using the classes "20 or less", "30 or less", and so on.

(b) Draw the cumulative frequency curve for the results.

(c) Use your graph to estimate the

(i) median,
(ii) interquartile range,

(d) From the graph, find the number of ears of barley with lengths

(i) greater than 55 mm,
(ii) either not greater than 25 mm or greater than 64 mm.

(e) It was discovered later that all the lengths were wrongly recorded such that all lengths should be 5 mm more.

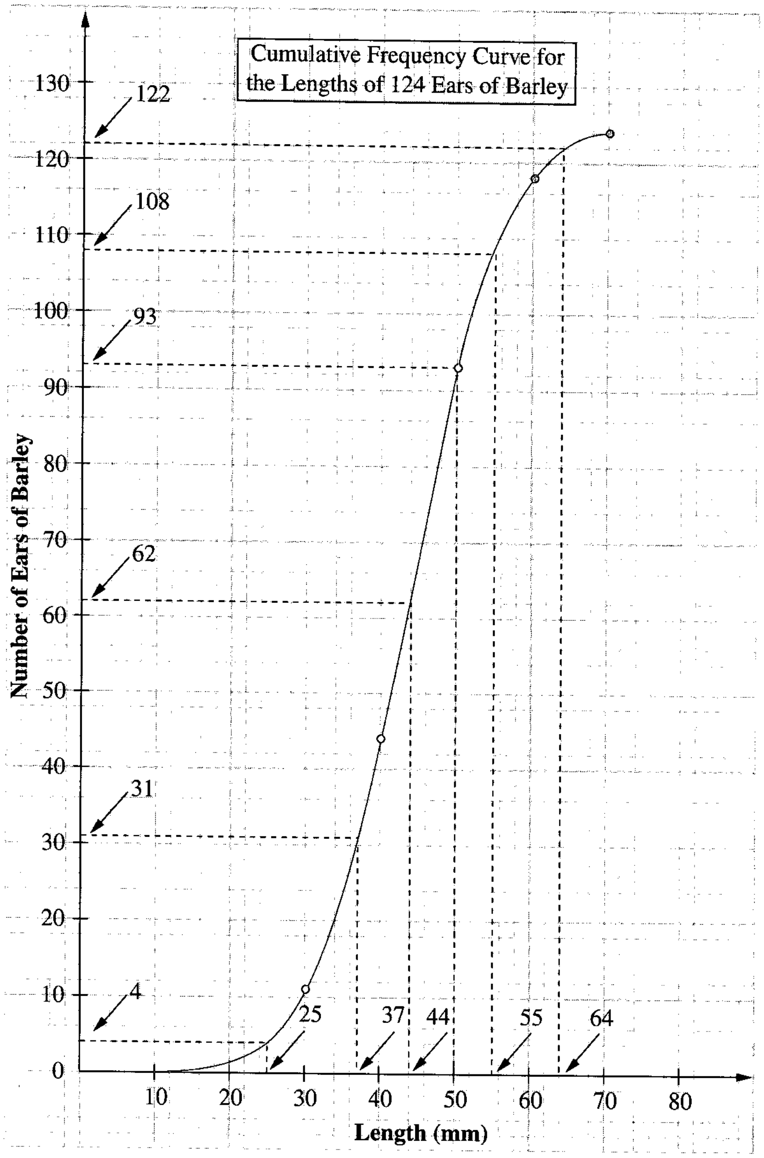
Find the correct value of (i) the median, (ii) the interquartile range.

Solution

(a) The table below shows the cumulative frequency table.

Length (mm)	0	20	30	40	50	60	70
Number of ears of this length or less	0	1	9	44	94	119	124

(b) The following graph shows the cumulative frequency curve representing the results:



(c) *From the graph,*

(i) the median = 44 mm

(ii) the lower quartile = 37 mm
the upper quartile = 50 mm

∴ the interquartile range = (50 – 37) mm = 13 mm.

(d) (i) Approximately 108 ears of barley have lengths 55 mm or less.

∴ the number of ears of barley with lengths greater than 55 mm is 124 – 108 = 16.

(ii) Number of ears of barley with lengths not greater than 25 mm = 4

Numbers of ears of barley with lengths greater than 64 mm = 124 – 122 = 2

∴ the number of ears of barley with lengths not greater than 25 mm or greater than 64 mm = 4 + 2 = 6.

(e) Since all lengths should be 5 mm more, the correct value of

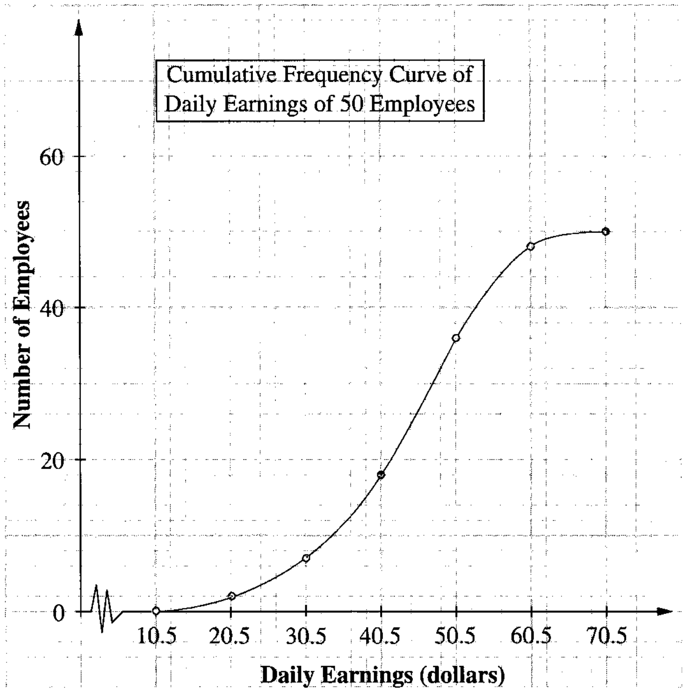
(i) the median = (44 + 5) mm = 49 mm,

(ii) the lower quartile = (37 + 5) mm, the upper quartile = (50 + 5) mm, and thus the interquartile range = (50 + 5) – (37 + 5) = 13 mm.

Note: The same adjustment was made to the lower and upper quartiles and so the interquartile range remained unchanged, i.e. no adjustment was needed to give the correct value of the interquartile range.

Exercise 1b

✯**1.** The graph shows the cumulative frequency curve of the daily earnings of 50 employees in a company.



Use the graph to estimate

(a) the median,

(b) the interquartile range.

Two murder suspects, A and B, were on trial. Four witnesses were questioned.

***First witness** answered: I only know that A is innocent.*

***Second witness** said: I only know that B is innocent.*

***Third witness** replied: At least one of the first two witnesses is telling the truth.*

***Fourth witness** said: I am positively sure that the third witness gave false evidence.*

After the investigation, it was confirmed that what the fourth witness said was true. Who was the murderer?

✯**2.** The graph shows a cumulative frequency curve of the daily travelling expenses of 1 000 pupils in school.

(a) Use the graph to estimate

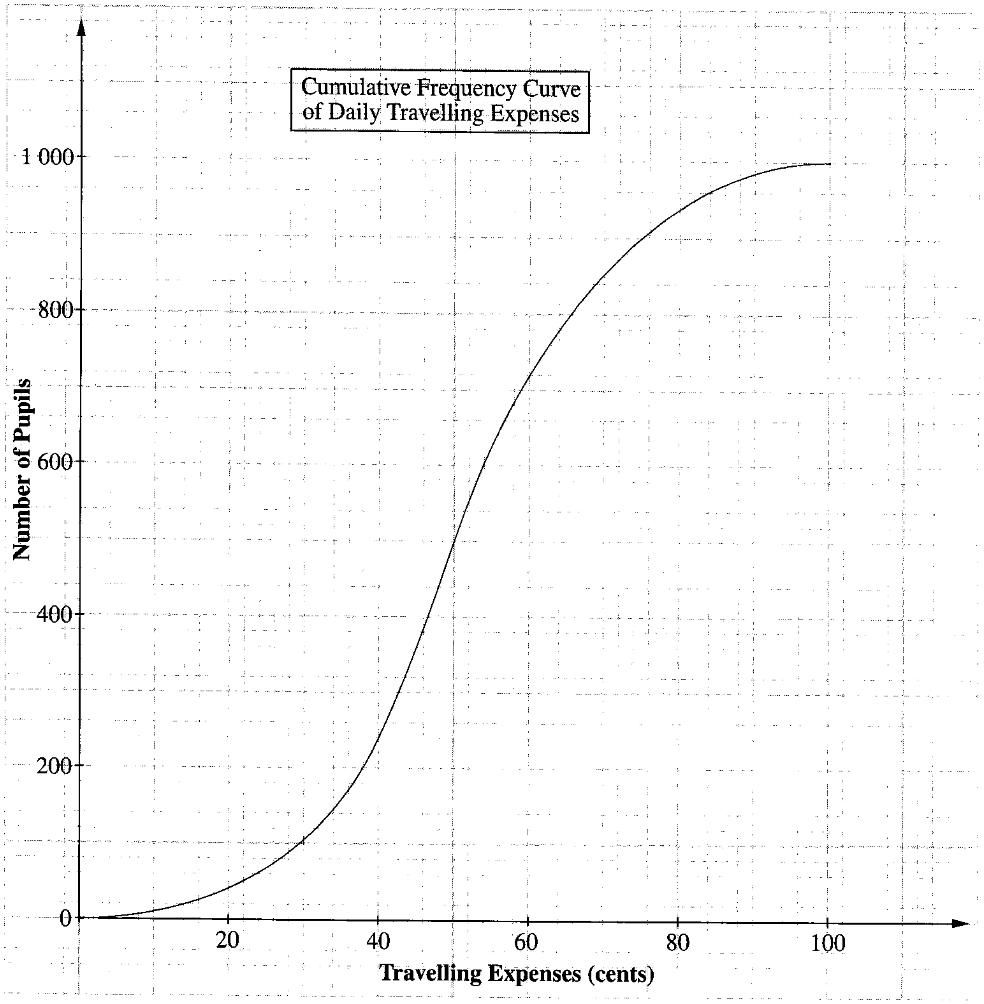
(i) the median,

(ii) the interquartile range.

(b) Estimate the number of pupils who spend

(i) 60 cents or less,

(ii) more than 50 cents but not more than 80 cents.



∗**3.** The table below shows the distribution of the marks scored by 600 pupils in an examination:

Marks	10	20	30	40	50	60	70	80	90	100
Number of pupils scoring less than this mark	9	27	88	180	308	415	497	568	590	600

Using a *vertical* scale of 1 cm for 50 pupils and a *horizontal* scale of 1 cm for 10 marks, plot these values and draw a smooth curve through your plotted points.

(a) Use your graph to estimate
  (i) the median mark,
  (ii) the pass mark such that 60% of the pupils will pass the examination.
(b) Indicate clearly the upper and lower quartiles on your graph and write down the interquartile range.

∗**4.** 80 pupils participated in a general knowledge quiz. The table below shows the cumulative frequency of the marks scored:

Number of marks scored	5	10	15	20	25	30	35	40
Number of participants obtaining this score or less	3	8	20	50	67	75	79	80

(a) Calculate how many participants scored marks between 26 and 30 inclusive.

(b) Using a *vertical* scale of 1 cm to represent 5 participants and a *horizontal* scale of 1 cm to represent 5 marks, plot these values on a sheet of graph paper and join the plotted points with a smooth curve.

(c) Estimate from your graph, the median mark and the interquartile range.

(d) Estimate from your graph, the number of participants
- **(i)** scoring less than or equal to 17 marks,
- **(ii)** scoring more than 28 marks.

$\ast$**5.** 64 adults were asked to indicate the weekly number of hours they spend watching television. The table below shows the information obtained:

Time in hours	*Number of adults*
$x \leqslant 5$	2
$5 < x \leqslant 10$	8
$10 < x \leqslant 15$	22
$15 < x \leqslant 20$	16
$20 < x \leqslant 25$	10
$25 < x \leqslant 30$	4
$30 < x \leqslant 35$	2

(a) Copy and complete the cumulative frequency table below:

Time (h)	$\leqslant 5$	$\leqslant 10$	$\leqslant 15$	$\leqslant 20$	$\leqslant 25$	$\leqslant 30$	$\leqslant 35$
Number of adults	2						64

(b) Using a *vertical* scale of 2 cm to represent 10 adults and a *horizontal* scale of 2 cm to represent 5 hours, draw a cumulative frequency curve to display the information.

(c) Use your graph to estimate
- **(i)** the median,
- **(ii)** the interquartile range of the distribution,
- **(iii)** the number of adults who spend more than 25 hours per week watching television.

$\ast$**6.** The results of 56 students in an examination are tabulated below:

Mark (x)	*Frequency*
$0 \leqslant x < 10$	1
$10 \leqslant x < 20$	3
$20 \leqslant x < 30$	4
$30 \leqslant x < 40$	5
$40 \leqslant x < 50$	7
$50 \leqslant x < 60$	8
$60 \leqslant x < 70$	11
$70 \leqslant x < 80$	9
$80 \leqslant x < 90$	6
$90 \leqslant x < 100$	2

(a) Copy and complete the following cumulative frequency table:

Marks	< 10	< 20	< 30	< 40	< 50	< 60	< 70	< 80	< 90	< 100
Number of students	1	4								56

(b) Draw a cumulative frequency curve representing this distribution.
From the graph, estimate the median, the quartiles and write down the interquartile range.

(c) Estimate the percentage of students who scored a mark

(i) greater than or equal to 65, **(ii)** less than 34.

✶7. Table (a) is the frequency distribution of the lifespan (in weeks) of a certain insect. (The interval 0–2 indicates all lifespans greater than 0 and up to and including 2 units).

Lifespan (Weeks)	*Frequency*
0–2	1
2–4	2
4–6	4
6–8	8
8–10	14
10–12	24
12–14	35
14–16	50
16–18	47
18–20	15
Total	200

(a)

Lifespan (less than or equal to)	*Cumulative Frequency*
2	1

(b)

(a) Copy and complete the corresponding cumulative frequency in Table (b).

(b) Draw a cumulative frequency curve.

(c) Estimate from your graph the

(i) median lifespan, **(ii)** upper and lower quartiles,

(iii) number of insects which will live for more than 10 weeks.

✶8. Table (A) shows the age distribution of 60 members of a club.

Age (x years)	*Frequency*
$19 < x \leqslant 24$	4
$24 < x \leqslant 29$	5
$29 < x \leqslant 34$	7
$34 < x \leqslant 39$	11
$39 < x \leqslant 44$	11
$44 < x \leqslant 49$	8
$49 < x \leqslant 52$	9
$54 < x \leqslant 59$	5
Total	60

(A)

Age (less than or equal to)	*Cumulative Frequency*
24	4
59	60

(B)

(a) Copy and complete Table (B).

(b) Using a *vertical* scale of 2 cm to represent 10 members and a *horizontal* scale of 1 cm to represent 5 years, draw the cumulative frequency curve.

(c) Use your curve to estimate
  (i) the median age,
  (ii) the interquartile range,
  (iii) the number of members with age above 36 years,
  (iv) the number of members with age between 36 and 52 years.

☆**9.** A businessman travels between two towns quite often. He timed the trips on 100 random occasions. The table below shows the information he collected:

Time of trip (x min)	*Number of occasions*
$140 < x \leqslant 144$	2
$144 < x \leqslant 148$	6
$148 < x \leqslant 152$	17
$152 < x \leqslant 156$	20
$156 < x \leqslant 160$	24
$160 < x \leqslant 164$	16
$164 < x \leqslant 168$	11
$168 < x \leqslant 172$	4

(a) Construct a cumulative frequency table.

(b) Draw the cumulative frequency curve.

(c) Estimate from your curve
  (i) the median value of the distribution,
  (ii) the interquartile range,
  (iii) the number of occasions when the time taken fell between 145 and 158 minutes.

☆**10.** In a survey of 40 families, the number of children in each family is recorded as follows:

Number of children	0–2	2–4	4–6	6–8	8–10	10–12
Number of families	10	14	6	5	4	1

[4–6, for example, is taken to mean more than 4 children but not more than 6 children.]

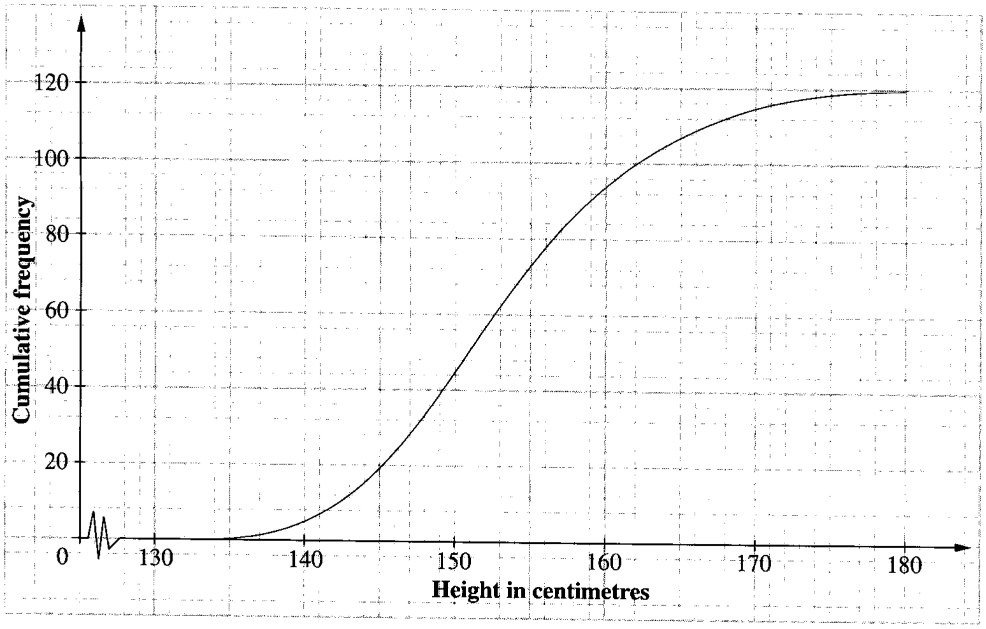
(a) Construct a cumulative frequency table.

(b) Draw a cumulative frequency curve representing the distribution.

(c) Estimate from the graph **(i)** the median, **(ii)** the interquartile range.

11. A long ruler was fastened to a wall and used to measure the heights of 120 children. The following diagram shows the cumulative frequency graph of these heights:

(a) Use the graph to estimate
  (i) the median,
  (ii) the interquartile range,
  (iii) the number of children whose height is greater than 170 cm.



(b) Several days later it was noticed that the ruler had been wrongly positioned, and that all heights should be 3 cm less.

State what adjustment, if any, should be made to your results for parts **(a)(i)** and **(a)(ii)** in order to give the correct value of

(i) the median,

(ii) the interquartile range. *(C)*

12. *(Answer the whole of this question on a sheet of graph paper.)*

160 electric light bulbs of brand *A* were tested to find the life span of each bulb (i.e., the time it lasted before it failed). The results are given in the table below:

Life span (t hours)	*Number of bulbs*
$t \leqslant 500$	2
$500 < t \leqslant 1000$	4
$1000 < t \leqslant 1500$	13
$1500 < t \leqslant 2000$	68
$2000 < t \leqslant 2500$	51
$2500 < t \leqslant 3000$	18
$3000 < t \leqslant 3500$	3
$3500 < t \leqslant 4000$	1

(a) Copy and complete the following cumulative frequency table:

Life in hours	$\leqslant 500$	$\leqslant 1000$	$\leqslant 1500$	$\leqslant 2000$	$\leqslant 2500$	$\leqslant 3000$	$\leqslant 3500$	$\leqslant 4000$
Number of bulbs	2	6						160

(b) Using a *horizontal* scale of 2 cm to represent 500 hours and a *vertical* scale of 2 cm to represent 20 bulbs, draw a smooth cumulative frequency curve for these results.

(c) Showing your method clearly, use your graph to estimate
  (i) the median,
  (ii) the 10th percentile of the distribution.

160 brand B bulbs were also tested and a report on the test gave the following information:
  4 bulbs had a life span $\leqslant$ 500 hours.
  None lasted beyond 3200 hours.
  The median life span was 2300 hours.
  The upper quartile of the distribution was 2600 hours.
  The interquartile range of the distribution was 600 hours.

(d) Use this information to draw, on the same axes, a smooth cumulative frequency curve for the brand B bulbs.

(e) Use your graphs to estimate the number of bulbs with a life span 2750 hours or less
  (i) for brand A,
  (ii) for brand B.

(f) Both brands cost the same price.
Which do you think is a better buy? Give a reason for your choice. (*C*)

Summary

1. The **cumulative frequency table** provides another way of representing a set of data. It can be obtained from the frequency table. The cumulative distribution can be displayed graphically by a **cumulative frequency curve**.

2. A cumulative frequency curve can be used to estimate the median, quartiles and percentiles of a distribution. It can also be used to estimate information like how many pupils obtained less than a certain mark, how many people weigh less than a certain weight, and so on.

Review Questions 1

*1. The table below shows the distribution of marks scored by 500 cadets in a physical test:

  (a) Calculate the mean mark.
  (b) Construct a cumulative frequency table.
  (c) Draw a cumulative frequency curve representing the distribution.
  (d) Estimate from the graph,
    (i) the median,
    (ii) the 70th percentile,
    (iii) the interquartile range,
    (iv) the number of cadets who scored less than 43 marks,
    (v) the pass mark given that 60% of the cadets passed the physical test.

Mark (x)	*Number of cadets*
$0 \leqslant x < 10$	9
$10 \leqslant x < 20$	17
$20 \leqslant x < 30$	63
$30 \leqslant x < 40$	65
$40 \leqslant x < 50$	86
$50 \leqslant x < 60$	112
$60 \leqslant x < 70$	68
$70 \leqslant x < 80$	55
$80 \leqslant x < 90$	17
$90 \leqslant x < 100$	8

∗2. The masses of 200 eggs were measured and the results are given in the following table:

Mass (x g)	*Number of eggs*
$40 < x \leqslant 45$	8
$45 < x \leqslant 50$	24
$50 < x \leqslant 55$	80
$55 < x \leqslant 60$	64
$60 < x \leqslant 65$	16
$65 < x \leqslant 70$	8

(a) Draw a histogram to represent these data.

(b) Calculate the mean mass.

The eggs are graded according to their masses as follows:

Grade 1 : $62 \text{ g} < x \leqslant 70 \text{ g}$

Grade 2 : $51 \text{ g} < x \leqslant 62 \text{ g}$

Grade 3 : $40 \text{ g} < x \leqslant 51 \text{ g}$

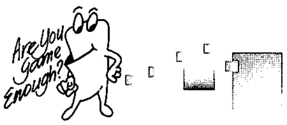
(c) Construct the cumulative frequency table for the distribution of the masses of 200 eggs.

(d) Draw the cumulative frequency curve representing the distribution.

(e) Use your curve to estimate

(i) the median mass, **(ii)** the interquartile range,

(iii) the percentage of eggs in each grade.



Jane, Karen, Susan and Shirley were the first four competitors to complete a cross-country race. The following remarks were heard from 4 spectators:

(a) "Jane was first; Susan was second."

(b) "Shirley was third; Karen was fourth."

(c) "Susan was first; Jane was fourth."

(d) "Karen was first; Susan was third."

Give the order in which they finished the race if for each pair of clues in the remarks, only one is true.

∗3. The table shows the heights, in cm, of 56 plants grown under experimental conditions:

Height (x cm)	*Number of plants*
$0 < x \leqslant 20$	3
$20 < x \leqslant 30$	4
$30 < x \leqslant 40$	6
$40 < x \leqslant 50$	15
$50 < x \leqslant 60$	20
$60 < x \leqslant 70$	8

(a) Draw a histogram to illustrate the information.

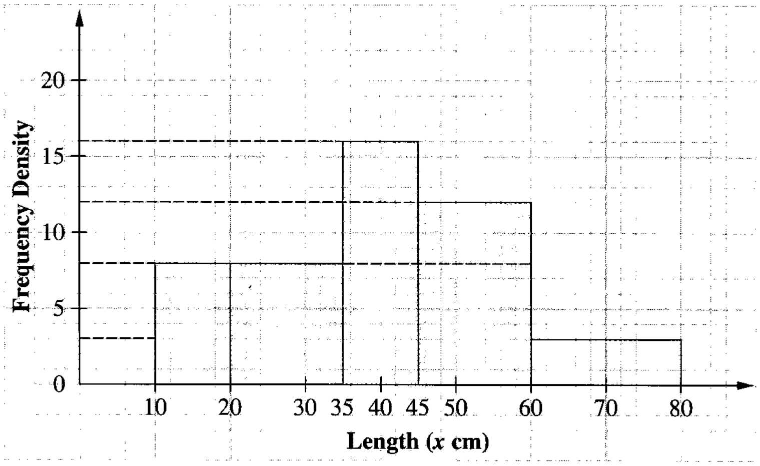
(b) Calculate the mean height.

(c) Construct the cumulative frequency table for the distribution and draw the cumulative frequency curve.

(d) Use your curve to estimate

- **(i)** the median,
- **(ii)** the upper quartile,
- **(iii)** the lower quartile,
- **(iv)** the number of plants having heights greater than 57 cm,
- **(v)** the value of x if 37.5% of the 56 plants have a height of less than or equal to x cm.

★4. Sixty children were asked to draw, without measuring, a line of length 0.4 m and their accuracy was checked by measurement. The histogram illustrates the results.



(a) Copy and complete the table.
(b) Calculate the mean length.
(c) Construct a cumulative frequency table for the data.
(d) Draw a smooth cumulative frequency curve and use it to estimate
- **(i)** the median length,
- **(ii)** the interquartile range,
- **(iii)** the fraction of the children who drew a line longer than 48 cm.

Length (x cm)	*Number of children*
$10 < x \leqslant 20$	8
$20 < x \leqslant 35$	
$35 < x \leqslant 45$	
$45 < x \leqslant 60$	18
$60 < x \leqslant 80$	

★5. The table gives the frequency distribution of marks obtained by 80 candidates in the Mathematics and English examinations:

Mark	*Mathematics*	*English*
$0 < x \leqslant 20$	8	2
$20 < x \leqslant 40$	12	10
$40 < x \leqslant 60$	18	33
$60 < x \leqslant 80$	25	31
$80 < x \leqslant 100$	17	4

(a) Copy and complete the table on the right showing the cumulative frequency distribution in each subject.

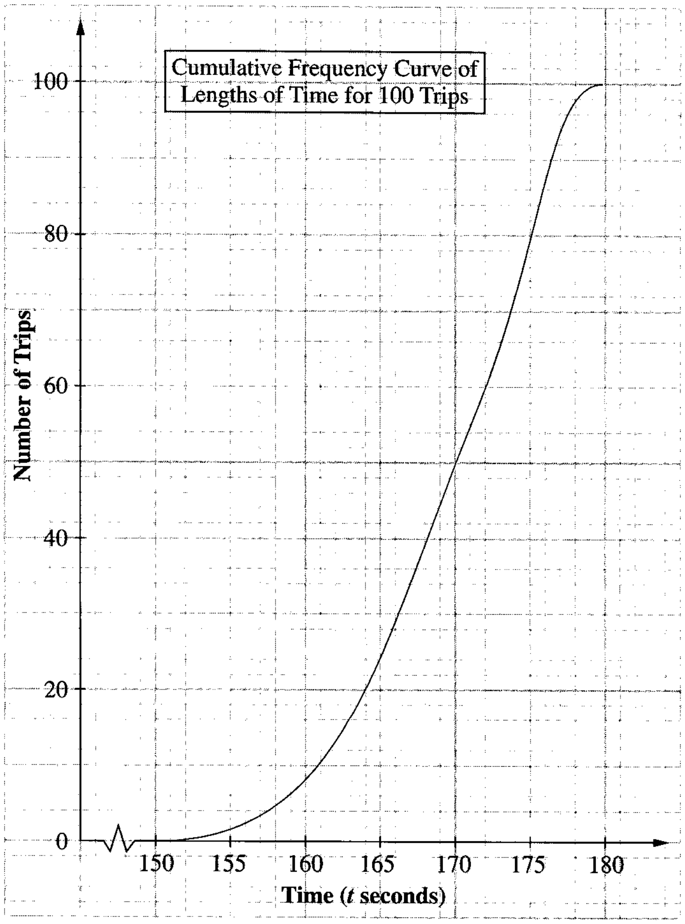
(b) Using a scale of 2 cm to represent 20 marks on the *horizontal* axis and 2 cm to represent 20 candidates on the *vertical* axis, draw separate cumulative frequency diagrams for each of the subjects, Mathematics and English.
Showing your method clearly, use your graphs to estimate
- **(i)** the median mark in Mathematics,
- **(ii)** the interquartile range in English,

	Number of candidates with this mark or less	
Mark	*Mathematics*	*English*
20	8	
40	20	
60		
80		
100	80	

(iii) the number of candidates who will obtain a distinction in English, if the minimum mark for a distinction is 76,
(iv) how many more candidates will fail to achieve a credit in Mathematics than in English if the minimum mark for a credit is 60 in each subject. (*C*)

***6.** A man travels up and down a certain expressway during off-peak hours. He timed the trip on 100 occasions. The following cumulative frequency curve gives the results:

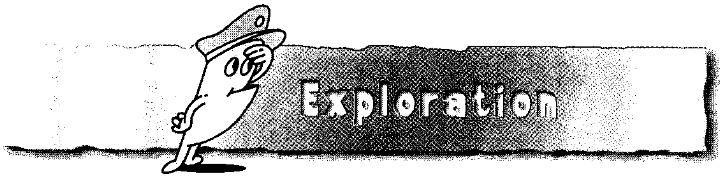
(a) Using the curve, estimate
  (i) the median time taken,
  (ii) the interquartile range,
  (iii) the number of trips lasting 164 seconds or less,
  (iv) the value of x if 20% of the 100 trips lasted longer than x seconds.



(b) Copy and complete the following frequency distribution table:

Time (t seconds)	*Number of trips*
$150 < t \leqslant 155$	2
$155 < t \leqslant 160$	6
$160 < t \leqslant 165$	
$165 < t \leqslant 170$	
$170 < t \leqslant 175$	
$175 < t \leqslant 180$	22

(c) Draw a histogram to represent this set of data.
(d) Calculate the mean of the distribution.



Exploration

1. The table below gives the distribution of the weights, in kg, of 80 members of a country club:

Weight (x kg)	*Number of members*
$40 < x \leqslant 50$	5
$50 < x \leqslant 60$	8
$60 < x \leqslant 70$	15
$70 < x \leqslant 80$	32
$80 < x \leqslant 90$	14
$90 < x \leqslant 100$	6

(a) Copy and complete the following cumulative frequency tables:

(i)

Weight (x kg)	$\leqslant 40$	$\leqslant 50$	$\leqslant 60$	$\leqslant 70$	$\leqslant 80$	$\leqslant 90$	$\leqslant 100$
Number of members							

(ii)

Weight (x kg)	> 40	> 50	> 60	> 70	> 80	> 90	> 100
Number of members							

(b) Draw on the same sheet of graph paper, the cumulative frequency curves corresponding to the two cumulative frequency tables.

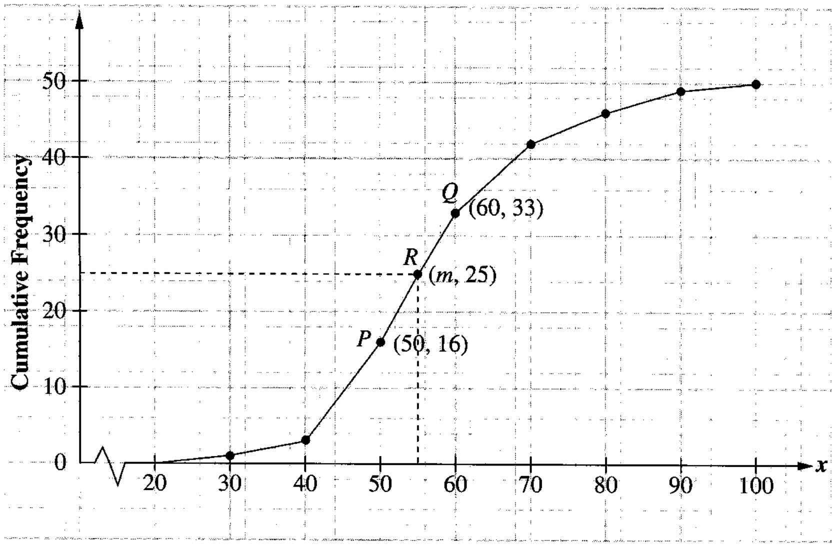
(c) Which quartile do you think will contain the point of intersection of the two curves?

(d) Identify two points one on each curve such that the horizontal distance between these two points is equal to the interquartile range. Are there any pairs of such points? Write down the interquartile range.

2. The table shows the frequency distribution and cumulative frequency distribution of 50 measurements of a variable x.

x	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100
f	1	2	13	17	9	4	3	1
x	$\leqslant 30$	$\leqslant 40$	$\leqslant 50$	$\leqslant 60$	$\leqslant 70$	$\leqslant 80$	$\leqslant 90$	$\leqslant 100$
Cumulative Frequency	1	3	16	33	42	46	49	50

The diagram shows the cumulative frequency polygon of the distribution.



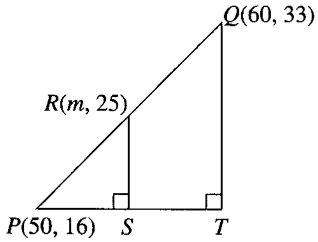
(a) The median can be calculated from the polygon by considering the section PQ of the polygon corresponding to the class interval 50 – 60 in which the median lies. (Explain why the median lies in this interval).

(b) With reference to the diagram on the right,

(i) write down the lengths of RS, QT and PS,

(ii) use similar triangles PRS and PQT to find the length of PS,

(iii) write down the value of m, the median. Compare your answer with the value which can be read directly from the graph.



(c) Use this method to calculate the lower and upper quartiles.

Problem Solving

Example 7

A lady meets her three nephews for the first time. She asks them for their ages. They reply as follows:

Alvin: *I'm 12 years old (A1);*
I'm 2 years younger than Kenneth (A2);
I'm 1 year older than Justin (A3).
Kenneth: *I'm not the youngest (K1);*
Justin's age and my age differ by 3 years (K2);
Justin is 15 years old (K3).
Justin: *I'm younger than Alvin (J1);*
Alvin is 13 years old (J2);
Kenneth is 3 years older than Alvin (J3).

Being playful, each boy gives the lady three clues, one of which is false. Can you help the poor lady unscramble the confusion?

Solution

Alvin's first clue A1 and Justin's second clue J2 cannot be true at the same time.

Suppose that A1 is true, then J2 is false and his two other clues are true. Thus Kenneth is 12 + 3 = 15 years old (J3) and Justin is below 12 years old (J1). However, this leads to the falsity of Kenneth's second clue K2 and third clue K3.

Thus, the supposition that A1 is true contradicts the fact that each boy gives only one false clue.

A1 must therefore be false and his two other clues A2 and A3 must be true. It follows that Justin's third clue J3, "Kenneth is 3 years older than Alvin", is false. Therefore, J2, "Alvin is 13 years old", is true.

Thus, we can conclude that Alvin is 13 years old, Kenneth is 15 years old (A2) and Justin is 12 years old (K2).

In solving this problem we used the strategies of *making suppositions, eliminating possibilities* and *making logical deductions.*

Example 8

Ahmad, Bala and Charles are good friends. One of them is in a Polytechnic (Poly), the other is in a Junior College (JC) while the third is doing his National Service (NS). Can you identify them from the following facts?

(1) Charles is older than the JC boy.
(2) Ahmad and the NS boy are not of the same age.
(3) The NS boy is younger than Bala.

Solution

First of all, we set up a table and eliminate each impossible condition by putting a '✗' and each possible condition by a '✓'.

	Poly	*JC*	*NS*
Ahmad			
Bala			
Charles			

(2) leads to

	Poly	*JC*	*NS*
Ahmad			✗
Bala			
Charles			

1. Amilah, Betty, Carol and Dorothy are the four top athletes of a school. Before the school's sports final, the Principal, the Vice-Principal and the Sports Secretary of the school make the following predictions:

Principal *– 'Carol will come in first and Betty second.'*
Vice-Principal *– 'Carol will come in second and Dorothy third.'*
Sports Secretary *– 'Amilah will come in first and Betty fourth.'*

On Sports Day, the four athletes occupy the top four positions. The predictions of the Principal, the Vice-Principal and the Sports Secretary are partly correct. Who occupies which position in the top four spots?

(3) leads to *then*

	Poly	*JC*	*NS*
Ahmad			✗
Bala			✗
Charles			

	Poly	*JC*	*NS*
Ahmad			✗
Bala			✗
Charles	✗	✗	✓

(1) and (3) lead to *then*

	Poly	*JC*	*NS*
Ahmad			✗
Bala		✗	✗
Charles	✗	✗	✓

	Poly	*JC*	*NS*
Ahmad			✗
Bala	✓	✗	✗
Charles	✗	✗	✓

This leads to the final result

	Poly	*JC*	*NS*
Ahmad	✗	✓	✗
Bala	✓	✗	✗
Charles	✗	✗	✓

i.e. Ahmad is in JC, Bala is in Poly and Charles is in NS.

This problem is solved *using tabulations and elimination.*

2. *Four men represent the University of Malaya (UM), National University of Singapore (NUS), University of Tasmania (UT) and Auckland University (AU) in a friendly sports games comprising the following events: swimming, athletics, football and tennis. Each of them only takes part in one event. Some information about the four men is given below:*

(a) Mr Teo comes from the northern hemisphere and plays only ball games.

(b) Mr Goh does not play ball games and he comes from the southern hemisphere.

(c) Mr Lee shares a room in the games village with 2 other people – the representative from UM and the tennis player.

(d) Mr Tay is not from UM and he is younger than both the representative from NUS and the swimmer.

(e) The representative from UT is not a swimmer.

Can you deduce which university Mr Teo, Mr Goh, Mr Lee and Mr Tay represent and what event each of them participates in respectively?

* * *

CHAPTER 2

Locus and Constructions

In this chapter, you will learn how to

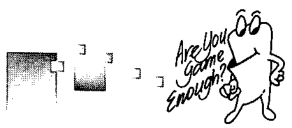
- △ construct simple loci of points in two dimensions;
- △ solve problems involving intersection of loci.

Preliminary Problem

The path taken by the woman and two little girls on the carousel follow a certain pattern and rhythm as the carousel rotates. The locus of their path traces a circle.

Likewise, the tip of a minute hand on a clock traces a circle when it goes through an hour cycle.

What other examples can you think of?



Using an eraser, a protractor, a pair of compasses and a pencil, construct on a sheet of white paper two line segments such that the sum and the square root of the product of their lengths are equal to the length of the pencil and the length of the eraser, respectively.

In-Class Activity

Work on the following questions with a partner.

1. When a lift in a tall building moves up or down, what path does it follow? Describe and make a sketch of the path.
2. Discuss the path of a swinging pendulum bob. Describe and make a sketch of the path.
3. Describe and make a sketch of the path followed by the tip of the second hand of a wall clock.
4. Discuss the movement of the Earth around the Sun. Describe and make a sketch of the path taken by the Earth.
5. Discuss the movement of the tip of a man's nose when he is walking along a straight line on level ground. What assumptions do you have to make before you can simplify the description of the path? Make a sketch of the path.
6. Do you find that the movements of the objects discussed above follow regular paths? Are these paths regulated by certain conditions? Discuss the condition that regulates each path.
7. Do movements of objects always follow regular paths? If not, can you give some examples of movements of objects which do not follow regular paths?
8. In your sketches of paths of the objects, are the paths represented by sets of points which show the different positions of the objects? In Mathematics, these sets of points are known as **loci** (singular – locus).

In general, the locus of a variable point can be defined as the path traced out by an object moving in a specific manner. More generally, we define the locus of a variable point as a set of points satisfying certain conditions.

Two loci which we will encounter often in Mathematics are straight lines and circles.

Loci in Two Dimensions

Let us study some commonly used loci in two dimensions through the following activities. You may like to discuss them with your classmates.

1. **(a)** In your sketch of the path followed by the tip of the second hand of a wall clock, what do you notice about the points representing different positions of the tip? Can you describe the path?

(b) Mark a point O on the centre of a piece of paper. Mark a number of points, each of which is exactly 2 cm from O. Imagine that you have marked a hundred or a thousand or even a million of such points. What can you conclude about these points, each of which is exactly 2 cm from O?

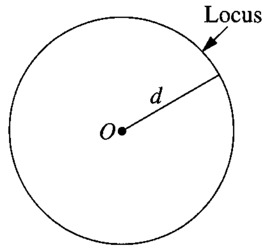


Fig. 2.1

In general, the locus of a point which is at a given distance *d* from a given point *O* is a circle with centre *O* and radius *d*. (See Fig. 2.1.)

2. (a) How will you walk so that you will always be 2 m from a straight wall? Try by walking a short distance, keeping the same distance away from one wall of the classroom.
   (b) On a sheet of paper, draw a line XY down the middle of the page. Mark a number of points 3 cm from XY with some of the points on the right and some on the left of XY. Imagine that you have marked a million of such points. What can you say about these points, each of which is exactly 3 cm from XY?

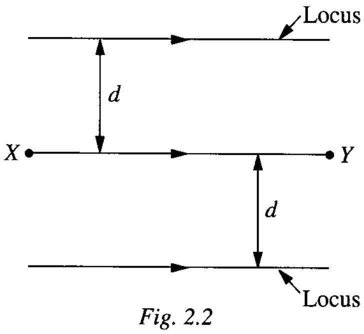


Fig. 2.2

In general, the loci of a point which is at a given distance *d* from a given straight line *XY* are two straight lines parallel to *XY* and at a distance *d* from *XY*. (See Fig. 2.2.)

3. (a) How will you walk so that you will always be the same distance from two desks in the classroom? Try walking a short distance such that you are equidistant from the two desks.
   (b) On a sheet of paper, draw a line XY 4 cm long across the middle of the page. Mark a number of points, each of which is equidistant from X and Y. Have some of the points above and some below the line XY. Imagine that you have marked a large number of such points. What can you say about these points, each of which is equidistant from X and Y?
   (c) Fig. 2.3 shows two given points X and Y. The perpendicular bisector of the line joining X and Y is constructed. The perpendicular bisector passes through M, the mid-point of XY. Are triangles PXM and PYM congruent? Do you agree that any point on the perpendicular bisector of XY is equidistant from X and Y?

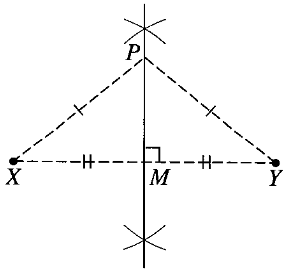


Fig. 2.3

In general, the locus of a point which is equidistant from two given points *X* and *Y* is the perpendicular bisector of the line *XY*. (See Fig. 2.3.)

4. (a) How will you walk so that you will always be at the same distance from two adjacent walls in your classroom? Try walking a short distance such that you are equidistant from the two walls.
   (b) On a sheet of paper draw two lines AB and XY to intersect at a point O. Mark a number of points which are equidistant from AB and XY. Imagine that you have marked a large number of such points. What can you say about these points, each of which is equidistant from AB and XY?

(c) Fig. 2.4 shows two straight lines AB and XY intersecting at the point O. Q is a point which is equidistant from AB and XY.

QR *and* QS are the perpendiculars from Q to AB and XY, respectively.

$\triangle QRO \equiv \triangle QSO$ (RHS)

$\therefore\ Q\hat{O}R = Q\hat{O}S$

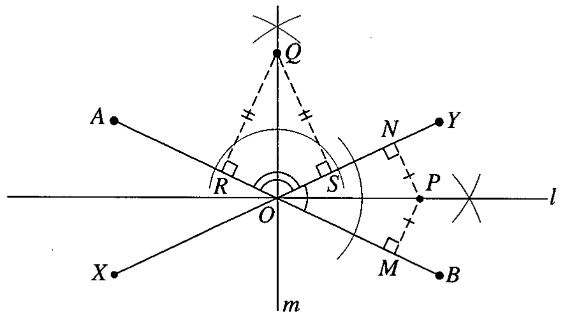


Fig. 2.4

This means Q lies on the bisector (line m) of $A\hat{O}Y$ between AB and XY.
Can you prove that any point P on the bisector (line l) of the other angle between AB and XY (i.e. $Y\hat{O}B$) is also equidistant from the two lines?

In general, the locus of a point which is equidistant from two given intersecting straight lines is a pair of straight lines which bisect the angles between the two given lines. (See Fig. 2.4.)

Exercise 2a

1. Describe the locus of a point P which moves in a plane so that it is always 4 cm from a fixed point O in the plane.

2. X is a fixed point in a given plane. Draw the locus of a point P which is always 3.5 cm from X.

3. Draw a line AB, 7 cm long. A point P moves such that it is always 3 cm from AB. Draw the locus of P.

4. Describe the locus of a point Q which moves in a plane so that it is always 5 cm from a given straight line l.

5. Two points A and B are 7.5 cm apart. Draw the locus of a point P equidistant from A and B.

6. Two straight lines AB and CD intersect at right angles at the point O. Draw, using separate diagrams,

- **(a)** the locus of a point 2.5 cm from O,
- **(b)** the locus of a point 3 cm from CD,
- **(c)** the locus of a point equidistant from C and O,
- **(d)** the locus of a point equidistant from OB and OD.

7. Draw two intersecting lines l and m. Draw the locus of a point P which moves such that it is equidistant from l and m.

8. Construct an angle XYZ equal to 60°. Draw the locus of a point P which moves such that it is equidistant from XY and YZ.

9. Construct $\triangle ABC$ in which $AB = 6$ cm, $BC = 7$ cm and $CA = 8$ cm. Draw the locus of P such that P is equidistant from A and C.

10. Construct $\triangle PQR$ in which $QR = 8$ cm, $P\hat{Q}R = 70°$ and $PR = 9$ cm. Construct the locus which represents points equidistant from PQ and QR.

11. A coin of radius 2 cm is placed flat on the floor and made to move inside a circular ring of radius 6 cm, also lying flat on the floor. The coin is always in contact with the ring as shown in Fig. 2.5. Describe and show diagramatically the locus of the centre of the coin.

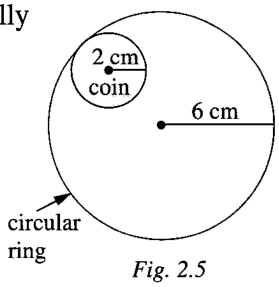


Fig. 2.5

12. A coin of radius 2 cm is placed flat on a table. Another coin of the same radius, also lying flat on the table, is made to roll round it. What is the locus of the centre of the moving coin? Show the path of the moving coin diagramatically. (See Fig. 2.6.)

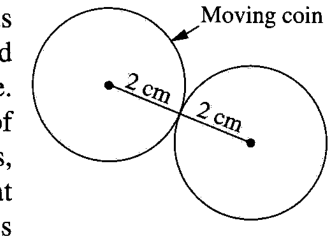


Fig. 2.6

13. A long stick leans vertically against a wall. The stick then slides in such a way that its upper end describes a vertical straight line down the wall, while the lower end crosses the floor in a straight line at right-angles to the wall. Construct a number of positions of the mid-point of the stick. Hence, draw its locus. (See Fig. 2.7.)

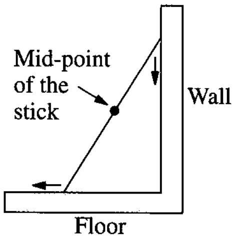


Fig. 2.7

14. P and Q are fixed points, and R is a variable point such that $\angle PRQ = 30°$. Using a 30° set-square, plot a number of positions of R. Hence, draw its locus. (See Fig. 2.8.)

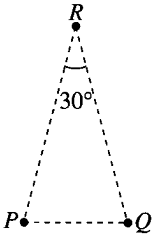


Fig. 2.8

15. On the circumference of a circle of radius 5 cm, mark a fixed point A. If R is a variable point on the circumference and Q is the mid-point of AR, plot a number of positions of Q and hence find its locus.

Intersections of Loci

If two or more loci intersect at a point P, then P satisfies the conditions of the loci simultaneously. Below are examples involving loci and intersections of loci.

Example 1

(a) Using ruler and compasses, construct $\triangle ABC$ in which $AB = 8.8$ cm, $BC = 7$ cm and $CA = 5.6$ cm.

(b) On the same diagram, draw

(i) the locus of a point which is 4.6 cm from A,

(ii) the locus of a point which is equidistant from BA and BC,

(c) Find the distance between two points which are 4.6 cm from A and equidistant from BA and BC. Give your answer in centimetres and correct to 1 decimal place.

Solution

Refer to Fig. 2.9 on next page.

(a) Construction steps for $\triangle ABC$ are as follows:
   (i) Draw AB 8.8 cm long.
   (ii) With B as centre and radius 7 cm, draw an arc.

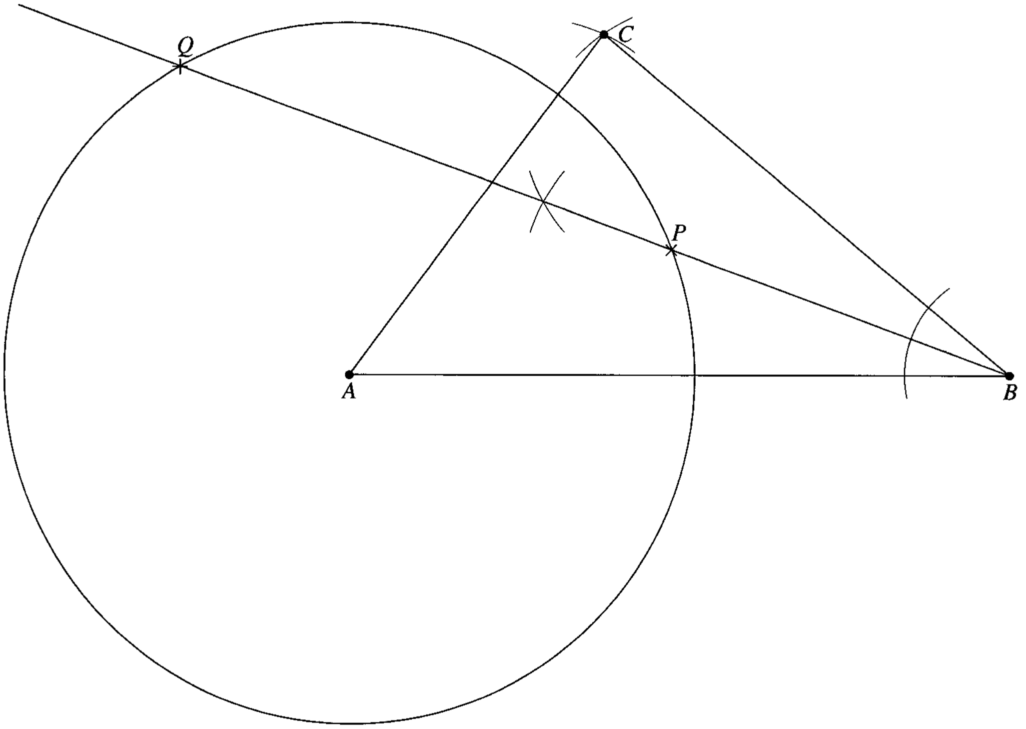


Fig. 2.9

(iii) With A as centre and radius 5.6 cm, draw an arc to cut the first arc at C.
(iv) Join AC and BC.

(b) (i) With A as centre and radius 4.6 cm, draw a circle. The locus of a point which is 4.6 cm from A is the circle drawn.
(ii) The locus of a point which is equidistant from BA and BC is the angle bisector of $A\hat{B}C$. The angle bisector is constructed as shown in Fig. 2.9.

(c) Two points which are 4.6 cm from A and equidistant from BA and BC are P and Q, the intersection points of the two loci in (b)(i) and (b)(ii). By measurement, $PQ = 7.0$ cm (correct to 1 decimal place).

Example 2

After an accident at sea, a search for survivors is carried out in the triangular region XYZ in which Y is 40 km due east of X, the bearing of Z from X is 030° and the bearing of Z from Y is 315°.

(a) Using a scale of 1 cm to represent 5 km, construct an accurate drawing of the triangular region XYZ.

(b) By making an appropriate measurement, find the distance, in kilometres, of Y from Z.

(c) A helicopter finds a life raft at a point P inside the triangular region XYZ. P is equidistant from the vertices X and Z. It is also equidistant from the sides ZX and ZY.

(i) On your scale drawing, construct the locus of points which are equidistant from
(a) X and Z, *(b) ZX and ZY.*
(ii) Mark clearly the position of P.

(d) A second life raft is spotted at a point Q between the vertex Z and side XY outside the triangular region XYZ. Q is 15 km from XY and equidistant from the vertices X and Z.

(i) On the same scale drawing, construct the locus of points which are 15 km from XY and on the same side of XY as Z.

(ii) Mark clearly the position of Q.

(e) By making appropriate measurements, find

(i) the distance, in kilometres, between P and Q,

(ii) the bearing of P from Q.

Solution

Refer to Fig. 2.10 below.

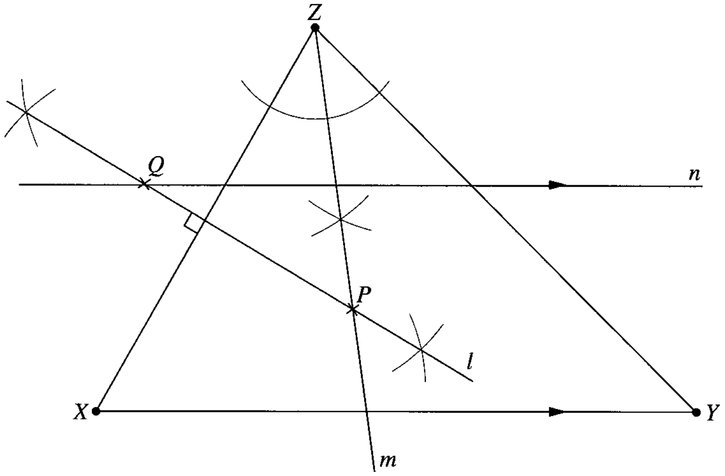


Fig. 2.10

(a) $XY = 8$ cm.

(b) In the scale drawing, by measurement, $YZ \approx 7.2$ cm,

$\therefore$ the distance of Y from $Z \approx 7.2 \times 5 = 36$ km.

(c) (i) (a) Draw the perpendicular bisector (the line l in Fig. 2.10) of XZ to obtain the locus of points equidistant from X and Z.

(b) Draw the angle bisector (the line m in Fig. 2.10) of XYZ to obtain the locus of points equidistant from ZX and ZY.

(ii) P, the point of intersection of l and m is marked clearly as shown.

(d) (i) The second raft is 15 km from XY. Using a scale of 1 cm to represent 5 km, draw a line n parallel to XY and 3 cm from XY as shown in the scale drawing in Fig. 2.10 to obtain the locus of points which represent the possible positions of the raft.

(ii) Q, the point of intersection of l and n is marked clearly as shown.

(e) In the scale drawing, by measurements,

(i) $PQ \approx 3.2$ cm and thus the distance of P from $Q \approx 3.2 \times 5 = 16$ km.

(ii) the angle between the line l and the line $n \approx 30°$ and thus the bearing of P from $Q \approx 120°$.

Example 3

Three towns, A, B and C, on an island are such that C is 90 km due east of B. Given that B and C are 70 km and 60 km from A respectively.

(a) Using a scale of 1 cm to represent 10 km, make an accurate drawing of the locations of the three towns.

(b) Use your drawing to find the bearing of A from B.

(c) An offshore fish farm P is 30 km from A and is equidistant from B and C. On your scale drawing, construct the locus of points which are

(i) 30 km from town A, *(ii) equidistant from town B and town C,*
Mark clearly the point P.

(d) The owner of the fish farm P plans to have a second fish farm at a site Q which is also 30 km from A but such that $Q\hat{A}C = A\hat{C}B$.

On the same scale drawing, construct the locus of X such that $X\hat{A}C = A\hat{C}B$.
Mark clearly the point Q.

(e) By making appropriate measurements, find the distance of Q from P.

Solution

Refer to Fig. 2.11.

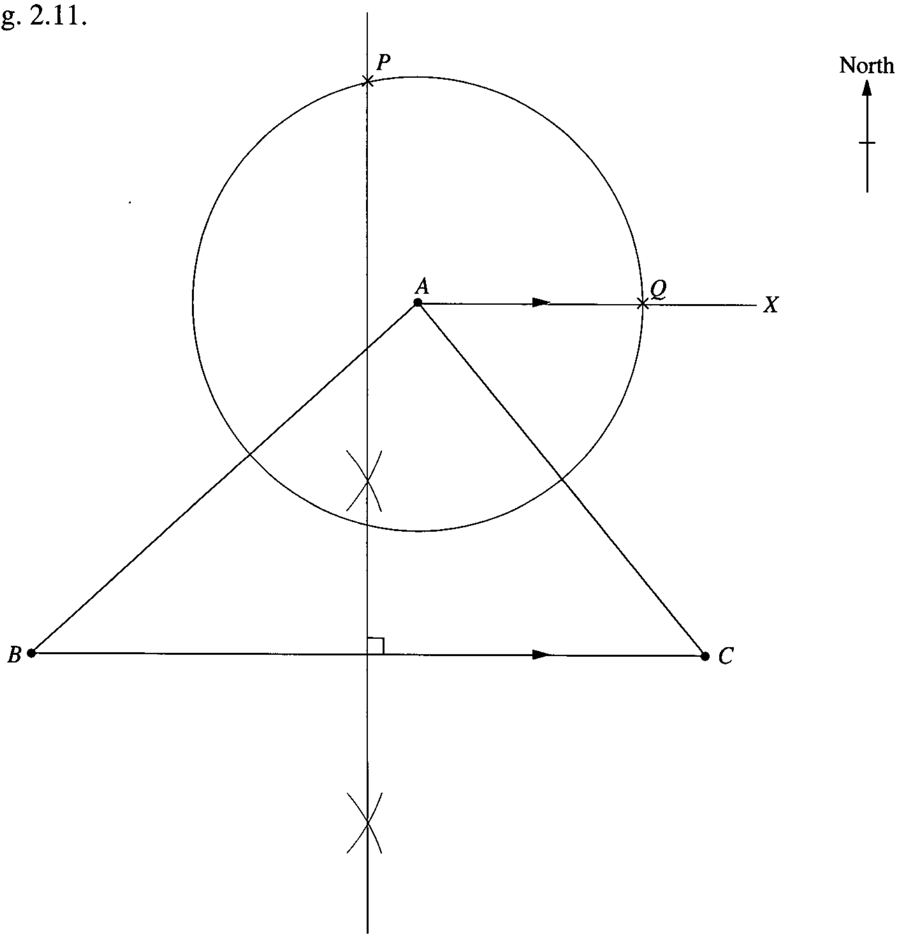


Fig. 2.11

(a) Refer to Example 1 for the construction of $\triangle ABC$.

(b) In the scale drawing, by measurement, $A\hat{B}C \approx 42°$.

$\therefore$ the bearing of A from $B \approx 048°$.

(c) In the diagram,

(i) taking A as the centre, draw a circle of radius 3 cm to obtain the locus of points 30 km from town A;

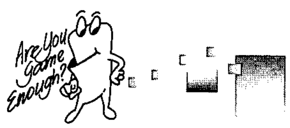
(ii) construct the perpendicular bisector of BC to obtain the locus of points equidistant from town B and town C;

(iii) P, the point of intersection of the circle and the perpendicular bisector, and which lies outside $\triangle ABC$ is clearly marked in the diagram.

(d) In the diagram, construct from A a line AX parallel to BC to obtain the locus of X.
Q, the point of intersection of AX and the circle, is clearly marked as shown in the diagram.

(e) In the drawing, by measurement, $PQ \approx 4.7$ cm.

$\therefore$ the distance of Q from $P \approx 47$ km.



Given two fixed points, A and B, construct two parallel lines, one passing through A and the other passing through B such that these two lines are l units apart, where l is less than the distance between A and B.

Exercise 2b

1. **(a)** Construct and label $\triangle XYZ$ in which XY = 10 cm, YZ = 7.5 cm and $X\hat{Y}Z$ = 60°. Measure and write down the length of XZ.

(b) On your diagram, construct the locus of a point

(i) 6 cm from Y,

(ii) equidistant from X and Z.

(c) The point P, inside $\triangle XYZ$, is 6 cm from Y and equidistant from the points X and Z.

(i) Label clearly, on your diagram, the point P.

(ii) Measure and write down the length of PX.

2. **(a)** Using ruler and compasses only, construct $\triangle PQR$ in which PQ = 11 cm, PR = 8 cm and QR = 6 cm.

(b) On your diagram, construct the locus of a point

(i) 3.8 cm from R,

(ii) equidistant from PQ and PR.

(c) The point X, inside $\triangle PQR$ is 3.8 cm from R and equidistant from PQ and PR.

(i) Label clearly, on your diagram, the point X.

(ii) Measure and write down the length of XQ.

3. A playground is in the shape of a triangle ABC in which $B\hat{A}C$ = 50°, AB = 70 m and BC = 80 m. A seesaw S in the playground is 55 m from A and equidistant from B and C.

(a) Using a scale of 1 cm to represent 10 m, make an accurate scale drawing of the playground.

(b) On the same diagram, draw the locus which represents all the points inside the triangle which are

(i) 55 m from A,

(ii) equidistant from B and C.

(c) Mark clearly on your diagram, the position of the seesaw S.

(d) Measure the length BS and find the distance of the seesaw S from B.

4. A school funfair is to be held on a regular hexagonal site $UVWXYZ$ with sides 50 m as shown in Fig. 2.12.

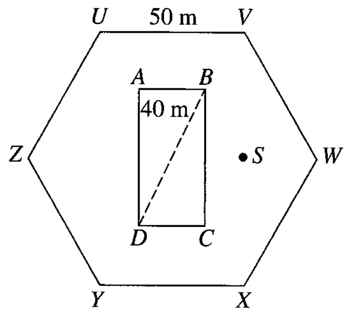


Fig. 2.12

(a) Using a scale of 1 cm to 10 m, construct a scale drawing of the site.

(b) The organiser wants to build a stage S, 25 m from W and equidistant from VW and WX. On your diagram,

- **(i)** construct the locus of a point 25 m from W,
- **(ii)** draw the locus of a point equidistant from VW and WX.

Mark the position of the stage S on your diagram.

(c) Rectangle $ABCD$ on Fig. 2.12 is the seating space for the audience and is positioned at the centre of the site with diagonals of 40 m. The corners A and C align with UX and the corners B and D with VY.

- **(i)** On your diagram, construct the region $ABCD$.
- **(ii)** What is the area of the seating space?

5. A factory occupies a quadrilateral site $ABCD$ in which AB = 110 m, $B\hat{A}D$ = 65°, AD = 90 m, $A\hat{D}C$ = 110° and DC = 60 m.

(a) Using a scale of 1 cm to represent 10 m, construct a plan of the quadrilateral $ABCD$. Measure $A\hat{B}C$.

Two fuel storage tanks, T_1 and T_2, are located 30 m from C and 15 m from BD respectively.

(b) On the same diagram, draw the locus which represents all the points inside the quadrilateral which are

(i) 30 m from C, **(ii)** 15 m from BD.

(c) Mark clearly on your diagram, the positions of the tanks T_1 and T_2.

(d) By measurement, find the distance between T_1 and T_2.

6. A metal sheet is in the form of a $\triangle XYZ$, where XY = 8.8 m, $X\hat{Y}Z$ = 64° and $Y\hat{X}Z$ = 41°.

(a) Using a scale of 1 cm to represent 1 m, construct an accurate scale drawing of the metal sheet.

A hole is to be drilled on the metal sheet at the point O which is equidistant from X, Y and Z.

(b) On the same diagram, draw the locus which represents points inside the triangle which are equidistant from

(i) X and Y, **(ii)** Y and Z.

(c) Mark clearly on your diagram, the position of the point O.

(d) What is the distance of the hole from the corners X, Y and Z of the metal sheet?

7. A collar badge is in the shape of a $\triangle PQR$ in which PQ = 14 mm, $Q\hat{P}R$ = 60° and PR = 12 mm.

(a) Using a scale of 1 cm to represent 1 mm, construct an accurate scale drawing of the collar badge.

A pin is soldered to the back of the badge at a point X which is equidistant from PQ, QR and RP.

(b) On the diagram, draw the locus which represents all the points inside the triangle which are equidistant from

(i) PQ and QR, **(ii)** QR and RP.

(c) Mark clearly on your diagram, the position of the point X.

∗8. A garden is in the shape of a $\triangle ABC$ in which AB = 110 m, AC = 86 m and $B\hat{A}C$ = 80°.

(a) Using a scale of 1 cm to represent 10 m, construct an accurate scale drawing of the garden.

(b) A pond in the garden is 65 m from C and is equidistant from A and B. On your diagram, draw the locus which represents points which are

(i) 65 m from C,

(ii) equidistant from A and B.

Label, with the letter P, the point representing the position of the pond.

(c) A durian tree in the garden is 18 m from AC and is equidistant from BA and BC. On your diagram, construct and label with the letter Q, the point representing the position of the durian tree.

9. In a single diagram, construct

- **(a)** $\triangle ABC$ with base $BC = 8$ cm, $AB = 7$ cm and $AC = 5$ cm,
- **(b)** the locus of points 2.4 cm from A,
- **(c)** the locus of points (on the same side of BC as A) 2.4 cm from BC,
- **(d)** the circle of radius 2.4 cm, which passes through A and touches BC, and whose centre is inside $\triangle ABC$.

∗10. The quadrilateral $ABCD$ represents a farmer's field in which $AB = 100$ m, $AD = 80$ m, $A\hat{D}C = 105°$, $B\hat{A}D = 90°$ and $A\hat{B}C = 75°$.

- **(a)** Using a scale of 1 cm to represent 10 m, construct an accurate scale drawing of the field.
- **(b)** There is a mango tree at B and another one at C. The farmer wishes to plant a third mango tree equidistant from B and C and 40 m from his house at A. On your diagram, draw the locus which represents points which are
  - **(i)** equidistant from B and C,
  - **(ii)** 40 m from A.

Label, with the letter M, the point where the farmer may plant the third mango tree. Find how far this mango tree is from the other two mango trees.

- **(c)** A water storage tank in the field is 50 m from AD and is also equidistant from CB and CD. By using the intersection of two loci, find and label the position of the water storage tank using the letter T.

Further Loci

In this section, we will study some more loci.

1. The locus of a point P which moves such that the area of $\triangle ABP$ remains constant is a set of points of the two lines l and m **parallel to** and **equidistant from** AB as shown in Fig. 2.13.

If $AB = 4$ cm, and the area of $\triangle ABP = 12$ cm^2, then the locus of P consists of two lines parallel to AB and 6 cm from AB as shown in Fig. 2.14.

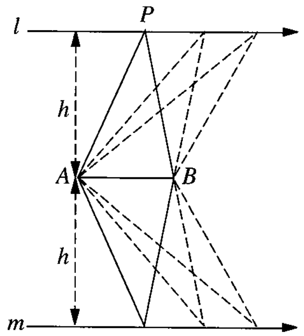


Fig. 2.13

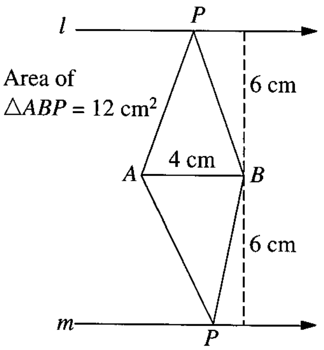
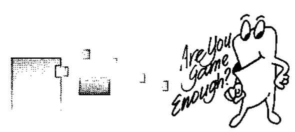


Fig. 2.14



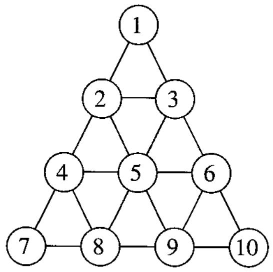


Fig. 1

Make a copy of Fig. 1 on a sheet of drawing paper. You require some counters. Put one counter on each numbered circle. In this game, you move a counter over an adjacent counter to go to an unoccupied circle and you remove the counter that has been "jumped" over. You are not allowed to *move a counter on an empty circle without "jumping" over an adjacent counter. You start the game by first removing one counter so as to create an unoccupied circle. Your aim is to remove all the remaining counters but one, making as few moves as possible. Investigate to find out which will be the best counter to be removed first and what are the subsequent moves to achieve the minimum number of moves. State this minimum number of moves. Repeat the same investigation for Fig. 2.*

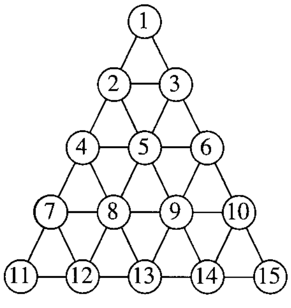


Fig. 2

2. If $XY = 6$ cm and $X\hat{P}Y = 90°$, the locus of P is the set of points excluding X and Y, on the circumference of the circle with XY as diameter. (See Fig. 2.15.)

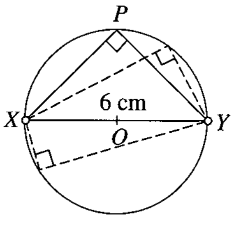


Fig. 2.15

3. If $XY = 5$ cm, $X\hat{P}Y = 70°$, the locus of P is the set of points on the arc XPY and its reflection in XY, excluding X and Y, as shown in Fig. 2.16. Note that

$$X\hat{O}Y = 2 \times 70° = 140°,$$
$$X\hat{Y}O = Y\hat{X}O = 20°.$$

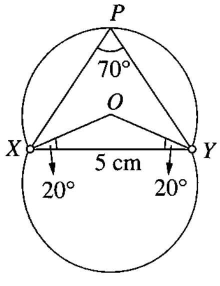


Fig. 2.16

4. **(a)** The locus of a point P, whose distance from a fixed point O is $OP \leqslant 2$ cm, is represented by the points inside and on the circumference of the circle with centre O and radius 2 cm. (See Fig. 2.17.)

(b) If $OP < 2$ cm, the locus of P will not include the points on the circumference and the circumference will be represented by a broken line. (See Fig. 2.18.)

(c) If $OP > 2$ cm, then the locus of P is the set of all the points outside the circle. (See Fig. 2.19.)

(d) If $OP \geqslant 2$ cm, then the locus of P is the set of all the points outside the circle, as well as the points on the circumference. (See Fig. 2.20.)

(e) If $1 \text{ cm} \leqslant OP \leqslant 2$ cm, then the locus of P is the set of all the points between the two concentric circles, centre O, radii 1 cm and 2 cm respectively and including the points on the circumferences. (See Fig. 2.21.)

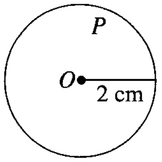


Fig. 2.17

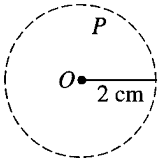


Fig. 2.18

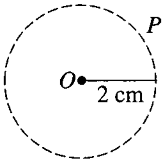


Fig. 2.19

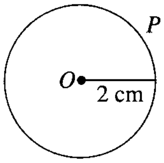


Fig. 2.20

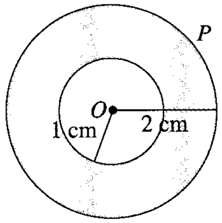


Fig. 2.21

5. **(a)** From Fig. 2.3, we learned that if X and Y are two fixed points, and if a point P moves in a plane such that $PX = PY$, then the locus of P is the perpendicular bisector of the line XY. (See Fig. 2.22.)

(b) If P moves such that $PX \leqslant PY$, the locus of P is the set of points shown in the shaded region including all the points on the perpendicular bisector which is represented by a solid line. (See Fig. 2.23.)

(c) If P moves such that $PX < PY$, the locus of P is the set of points shown in the shaded region, excluding all the points on the perpendicular bisector which is represented by a broken line. (See Fig. 2.24.)

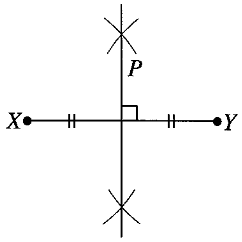


Fig. 2.22

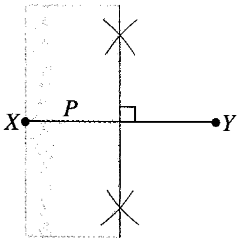


Fig. 2.23

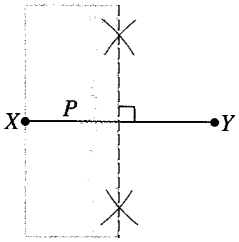


Fig. 2.24

Now let us look at some examples which involve the loci that we have learnt.

Example 4

Fig. 2.25 shows a circle, centre O. The diameter AB is 4 cm long. Indicate by shading, the locus of a point P which moves such that OP $\geqslant$ 2 cm and PA < PB.

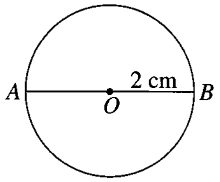


Fig. 2.25

Solution

The shaded region in Fig. 2.26 represents the locus of P where XY is the perpendicular bisector of AB.

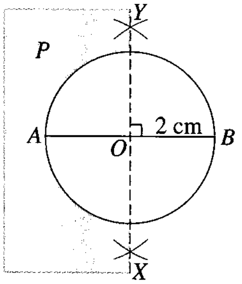


Fig. 2.26

Example 5

Fig. 2.27 shows a rectangle PQRS of length 6 cm and width 4 cm.

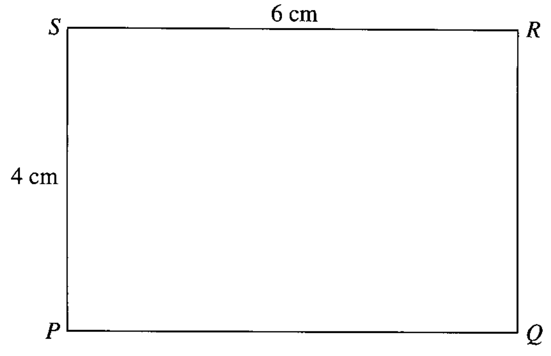


Fig. 2.27

(a) On the diagram, draw the locus of points within the rectangle which are

(i) 4 cm from P,
(ii) equidistant from P and Q
(iii) 1 cm from PQ.

(b) Mark and label on the diagram, a possible position of a point A within rectangle PQRS such that
$AP \geqslant 4$ cm, $AP \geqslant AQ$ and area of $\triangle PQA = 3$ cm².

(c) A point B within rectangle PQRS is such that $BP \leqslant 4$ cm, $BP \leqslant BQ$ and area of $\triangle PQB \geqslant 3$cm².
On the diagram, shade the region in which B must lie.

Solution

Refer to Fig. 2.28 below.

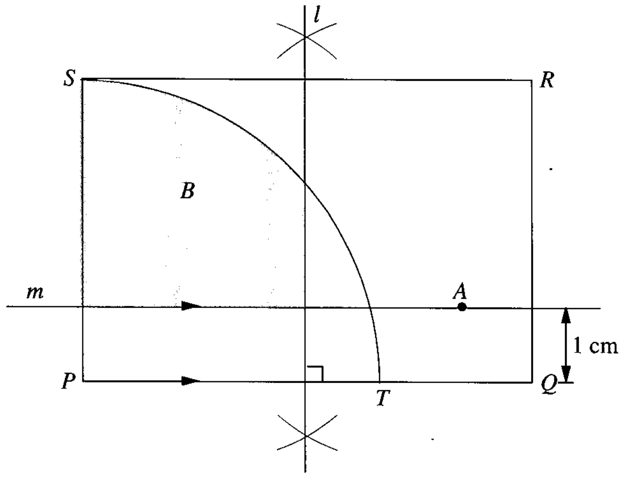


Fig. 2.28

If the area of $\triangle PQA$ = 3 cm², then

$$\frac{1}{2} \times 6 \times h = 3,$$

where h is the distance of A from PQ.

$$\therefore \quad h = 1.$$

Hence A lies on the line m.

(a) (i) With centre P and radius $PS = 4$ cm, draw an arc to cut PQ at T.
(ii) Construct the perpendicular bisector l, of the line PQ.
(iii) Draw a line m parallel to and 1 cm away from PQ.

(b) $AP \geqslant 4$ cm implies that A lies outside the quadrant SPT including the borders.
$AP \geqslant AQ$ implies that A lies inside the rectangle to the right of l including l.
Area of $\triangle PQA = 3$ cm² implies that A lies on the line m inside the rectangle.
A possible position of A is marked as shown in the diagram.

(c) $BP \leqslant 4$ cm implies that B lies inside the quadrant SPT including the borders.
$BP \leqslant BQ$ implies that B lies inside the rectangle to the left of l including l.
Area of $\triangle PQB \geqslant 3$ cm² implies that B lies inside the rectangle above and including m.
The shaded region in Fig. 2.28 is the region in which B must lie.

Example 6

Construct $\triangle PQR$ in which $PQ = 9.5$ cm, $Q\hat{P}R = 100°$ and $PR = 7.2$ cm.

(a) On the same diagram,

(i) draw the locus of a point equidistant from P and R,

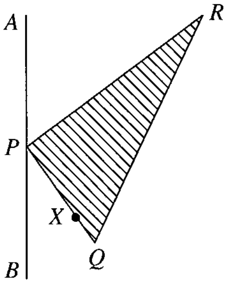
(ii) draw the locus of a point equidistant from Q and R,

(iii) draw the circle through P, Q and R.

(b) Measure and write down the radius of the circle.

(c) A is the point on the same side of QR as P such that $\triangle AQR$ is isosceles, with $QA = RA$ and $Q\hat{A}R = 100°$. Mark the point A clearly on your diagram.

CHECK THIS OUT



Draw a fixed vertical line AB and mark a fixed point X. Place a set square QPR (right-angled at P) with the vertex P on AB and the side PQ passing through X as shown in the diagram. Draw the line PR. Repeat this for a large number of different positions of P along the fixed line AB. What do you notice? Can you name the curve?

Solution

Refer to Fig. 2.29.

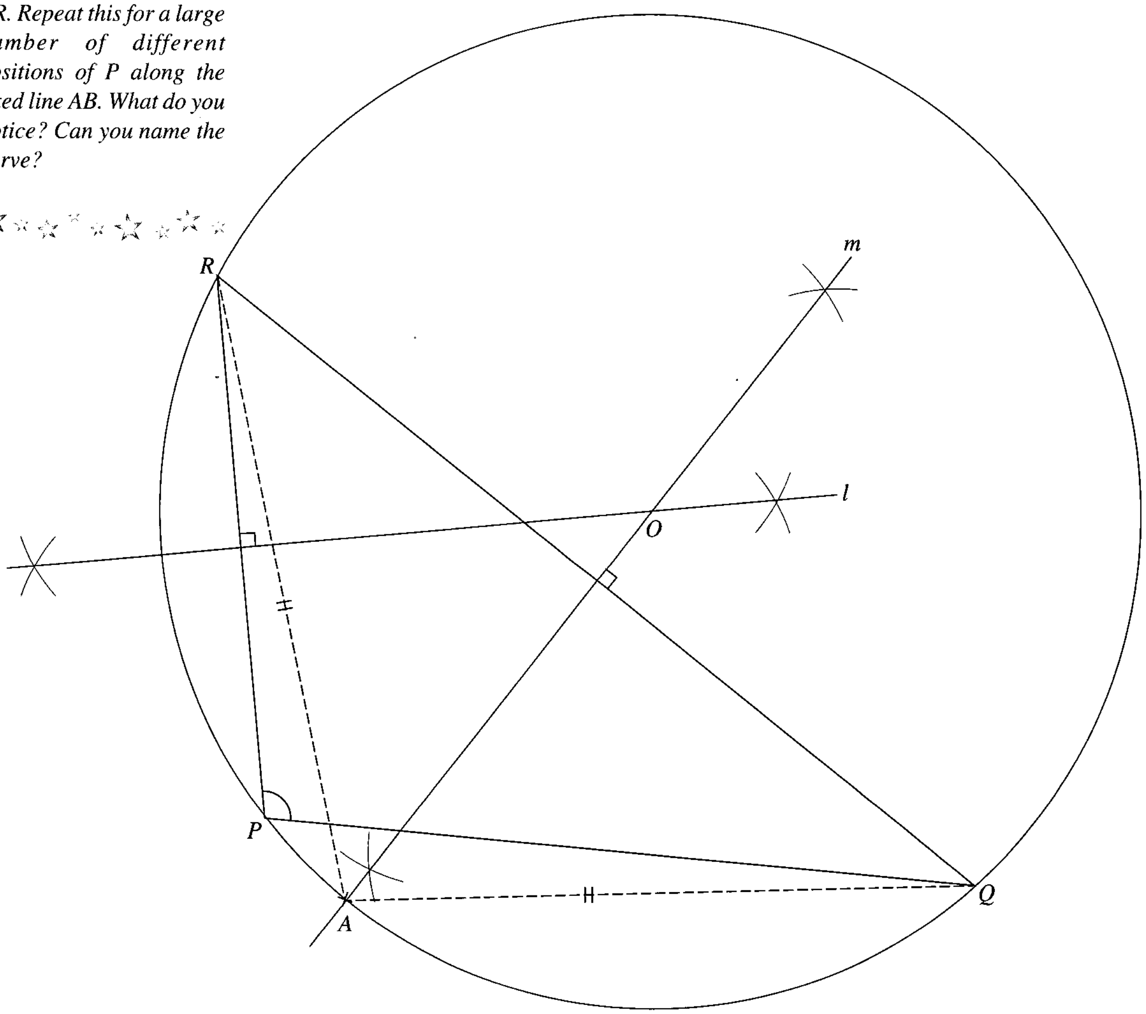
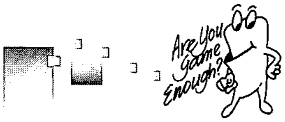


Fig. 2.29



On a sheet of drawing paper, draw a circle of radius 5 cm with the centre 12 cm from the left-hand edge of the paper. This circle will be called a base circle. Mark a fixed point A on the base circle. With the centre at any point P on the base circle and the radius PA, draw another circle. Repeat this by choosing a large number of different positions of P which spread evenly round the base circle. Do these circles touch an interesting curve?

The construction steps are as follows:

Construct $\triangle PQR$. Refer to Example 1 and Example 2.

(a) (i) Construct the perpendicular bisector, line l, of PR.
   (ii) Construct the perpendicular bisector, line m, of QR.
   (iii) l and m intersect at O which is equidistant from P, Q and R. With O as the centre and radius OP, draw the circle through P, Q and R.

(b) By measurement, $OP \approx 6.5$ cm.

(c) m cuts the circle at A so that $QA = RA$.
   $Q\hat{A}R = Q\hat{P}R = 100°$ ($\angle$s in the same segment)

Exercise 2c

1. Construct $\triangle ABC$ in which $AB = 5$ cm, $B\hat{A}C = 45°$ and $A\hat{B}C = 60°$.

Construct also the perpendicular bisector of AB and a circle with centre C and radius 4 cm. If a point P moves inside $\triangle ABC$ so that $PC < 4$ cm and $PA < PB$, shade the region in which P must lie.

2. Construct, in a single diagram,

(a) $\triangle PQR$ with side $QR = 11$ cm, $PQ = 7$ cm and $PR = 9$ cm,

(b) the locus of a point which is 5 cm from P,

(c) the locus of a point which is equidistant from PQ and PR. If a point X moves inside $\triangle PQR$ so that $PX < 5$ cm and X is nearer to PQ than to PR, indicate by suitable shading on your diagram the region in which X must lie.

3. Construct a rhombus $PQRS$ in which $PQ = 6$ cm and $Q\hat{P}S = 60°$. Measure and write down the length of PR.

The point A moves inside the rhombus so that $PA \geqslant AR$ and $PA \leqslant PQ$. Construct and show clearly by shading, the region in which A must lie.

4. A circle $ABCD$ of radius a is inscribed in a square $PQRS$. Indicate clearly the region in which the point X must lie, within the square $PQRS$, if $CX \geqslant AX$, $DX \geqslant BX$ and $OX \geqslant a$. (See Fig. 2.30.)

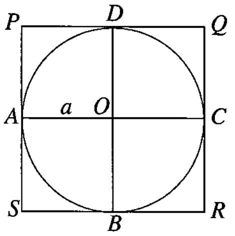


Fig. 2.30

5. Construct, in a single diagram,

(a) $\triangle ABC$, with base $AB = 8.0$ cm, $AC = 9.5$ cm and $BC = 7.5$ cm,

(b) the point X, such that $A\hat{B}X = 90°$ and X is equidistant from AB and AC. Measure XB and state this length correct to the nearest millimetre.

6. Construct the parallelogram $ABCD$ in which $AB = 10$ cm, $AD = 6$ cm and $B\hat{A}D = 50°$. Measure and write down the length of BD.

   On the same diagram, construct

   (a) the locus of a point P which moves so that it is equidistant from A and C,

   (b) the locus of a point Q which moves so that $B\hat{Q}D = 90°$.

   The position of point X, which lies inside the parallelogram, is such that $AX \leqslant XC$ and $B\hat{X}D \leqslant 90°$. Indicate clearly by shading, the region in which the point X must lie.

7. X is a point which moves inside the square $PQRS$ so that $XS > XQ$ and $XR > XP$. Make a copy of the diagram and indicate clearly, the region in which X must lie. (See Fig. 2.31.)

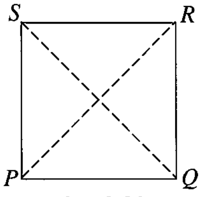


Fig. 2.31

8. $ABCD$ is a square of side 4 cm. A variable point P moves inside the square so that $PA \leqslant 4$ cm, $PC \leqslant PA$, and the area of $\triangle ABP \leqslant 6$ cm^2.

   Construct $ABCD$ accurately and shade the region in which P must lie.

9. Construct, in a single diagram,

   (a) $\triangle LMN$ with sides $LM = 7$ cm, $LN = 8$ cm and $MN = 6$ cm,

   (b) the locus of a point which is 5 cm from L,

   (c) the locus of a point which is equidistant from MN and LN.

   The position of a point P which lies outside $\triangle LMN$ is such that $LP < 5$ cm and P is nearer to MN than to LN. Indicate clearly by shading, the region in which the point P must lie.

10. Construct a parallelogram $PQRS$ in which $PQ = 9$ cm, $QR = 6$ cm and $P\hat{Q}R = 115°$. Measure and write down the length of PR. On the same diagram, construct

    (a) the locus of a point X which moves so that it is equidistant from PQ and RQ,

    (b) the locus of a point Y which moves so that $Q\hat{Y}S = 90°$.

    The position of a point Z, which lies inside the parallelogram, is such that $Q\hat{Z}S < 90°$ and Z is nearer to PQ than to RQ.

    Indicate clearly by shading, the region in which the point Z must lie.

11. Draw a fixed straight line XY, 8 cm long. Draw the locus of P if $\triangle XYP$ has an area of 8 cm$^{2\cdot}$ Find the position of P which makes $X\hat{P}Y$ a right angle.

12. Construct in a single diagram,

    (a) a circle, radius 5 cm, with diameter AC,

    (b) a point B on the circumference of the circle such that $AB = BC$,

    (c) the point D on the circumference of the circle, but on the side of AC opposite to B, such that $C\hat{A}D = 60°$.

    (d) the locus of a point equidistant from AB and AC, given that this locus cuts the circle again at P. Measure PD.

13. Construct $\triangle ABC$ in which $AB = 8$ cm, $BC = 7.5$ cm and $AC = 6$ cm. On the same diagram, construct

    (a) the locus of point P on the same side of AB as the point C such that the area of $\triangle APB$ = area of $\triangle ACB$.

    (b) **(i)** the locus of a point equidistant from A and B,

    (ii) the locus of a point equidistant from A and C,

    (iii) the circle through A, B and C.

14. Construct $\triangle PQR$ in which $PQ = 10$ cm, $QR = 9$ cm and $RP = 7$ cm. On the same diagram, draw

    (a) **(i)** the locus of points equidistant from PQ and PR,

    (ii) the locus of points equidistant from QP and QR,

    (iii) the circle touching the sides of $\triangle PQR$.

    (b) Construct the locus of a point which is equidistant from

    (i) PQ and PR,

    (ii) P and Q.

The point X moves inside the inscribed circle so that it is nearer to PQ than to PR and $PX \leqslant QX$.

Indicate clearly by suitable shading, the region in which X must lie.

15. **(a)** Construct $\triangle ABC$ where the base $BC = 10$ cm, $A\hat{B}C = 40°$ and $AC = 8.5$ cm. Measure and write down the length of AB.

(b) On your diagram, construct the locus of a point
- **(i)** 4 cm from A,
- **(ii)** equidistant from BA and BC.

(c) The point P, inside $\triangle ABC$, is more than 4 cm from A and nearer to BA than to BC. Indicate clearly by shading, the region of your diagram in which P must lie.

16. **(a)** Construct $\triangle PQR$ in which $PQ = 6$ cm, $Q\hat{P}R = 110°$ and $QR = 9$ cm.

(b) On your diagram construct the locus of a point
- **(i)** 2 cm from P,
- **(ii)** equidistant from Q and R.

(c) The point X, inside $\triangle PQR$, is less than 2 cm from P and nearer to R than to Q. Indicate clearly by shading, the region of your diagram in which X must lie.

17. **(a)** Using ruler and compasses only, construct $\triangle XYZ$ where the base $XY = 8$ cm, $YZ = 6$ cm and $XZ = 11$ cm. Measure and write down the size of $X\hat{Y}Z$.

(b) On your diagram construct the locus of a point
- **(i)** 3.8 cm from Y,
- **(ii)** equidistant from X and Z,
- **(iii)** equidistant from ZX and ZY.

(c) The point M, inside $\triangle XYZ$, is less than 3.8 cm from Y, nearer to Z than to X and nearer to ZX than to ZY. Indicate clearly by shading, the region of your diagram in which M must lie.

Summary

The **locus** of a point is defined to be a path or a region in a plane traced out by a moving object in a specific manner. In this chapter, we are only concerned with the path that is either a straight line or a circle and the region in a plane on one side of a straight line or enclosed in a circle.

Review Questions 2

1. A wooden panel is in the shape of $\triangle ABC$, where $AB = 4.5$ m, $BC = 6.5$ m and $CA = 4$ m.

(a) Using a scale of 2 cm to 1 m, construct an accurate drawing of $\triangle ABC$. Measure the largest angle.

Peter wishes to cut the largest possible circle out from this wooden panel.

(b) On the same diagram, draw the locus of a point that is equidistant from
(i) AB and AC, **(ii)** AB and BC.

(c) Mark clearly with the letter O, the centre of the circle that touches the three sides of $\triangle ABC$.

(d) Draw the circle.

(e) By measurement, find the radius of the largest circle that Peter can cut out from the wooden panel.

2. Fig. 2.32 shows the sketch of a river which flows due east between parallel banks. Two points P and Q are 50 m apart on the south bank. R is a point on the north bank such that the bearing of R from P is 042° and the bearing of Q from R is 140°.

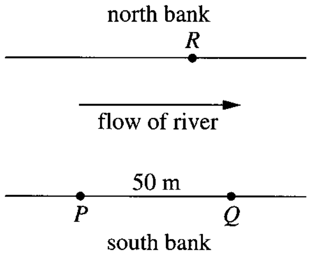


Fig. 2.32

(a) Using a scale of 2 cm to 10 m, construct $\triangle PQR$. Measure PR and find, correct to the nearest metre, the width of the river.

(b) A boat B_1 is 45 m from P and equidistant from Q and R. On your diagram, draw the locus that represents points which are
  (i) 45 m from P,
  (ii) equidistant from Q and R.
  Mark clearly the position of B_1.

(c) A second boat B_2 is 15 m from QR and equidistant from PR and QR. On your diagram, draw the locus which represents points which are
  (i) 15 m from QR,
  (ii) equidistant from PR and QR.
  Mark clearly, the position of B_2.

(d) By measurement, find the distance between the two boats correct to the nearest metre.

3. In Fig. 2.33, A represents a house which lies 135 m due west of a church C. Roads run from A to B and from B to C and a path leads from B to the point D on AC. $AB = 90$ m, $BD = 80$ m and $AD = 100$ m.

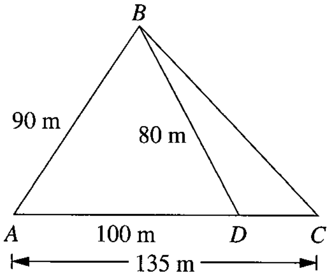


Fig. 2.33

(a) Using a scale of 1 cm to 10 m, construct an accurate drawing of the diagram. Measure and write down the bearing of B from A.

(b) A school S is equidistant from A and D. It is also equidistant from AB and AD. On your diagram, draw the locus which represents points which are equidistant from
  (i) A and D,
  (ii) AB and AD.

Mark clearly the position of the point S.

(c) A swimming pool P is 80 m from the church C, 40 m from the road AB and nearer to C than A. On your diagram, draw the locus which represents points which are
  (i) 80 m from C,
  (ii) 40 m from AB.

Mark clearly the position of the point P.

(d) By measurement, find the distance between the house A and the swimming pool P, correct to the nearest metre.

4. Draw $\triangle XYZ$ such that $XY = 9.1$ cm, $XZ = 6.8$ cm and $Y\hat{X}Z = 80°$.

(a) Measure and write down the length of YZ.

(b) On your diagram,
  (i) draw the locus of a point equidistant from X and Y,
  (ii) by making a further construction, mark clearly with the letter O the centre of the circle that passes through the three vertices of the triangle,
  (iii) draw the circle.

5. (a) Construct $\triangle ABC$ in which $AB = 11$ cm, $A\hat{B}C = 102°$ and $BC = 8$ cm.
  (i) Measure and write down the length of AC.
  (ii) Measure and write down the size of $B\hat{A}C$.

(b) On your diagram,
  (i) draw the locus of a point equidistant from A and B,
  (ii) by making a further construction, mark clearly with the letter X, the point which is equidistant from A, B and C,

- **(iii)** measure and write down the length of AX.

6. **(a)** Draw $\triangle PQR$ such that $PQ = 12.6$ cm, $QR = 10$ cm and $RP = 5$ cm. Measure and write down the size of $Q\hat{R}P$.
- **(b)** Draw the locus of a point equidistant from Q and R.
- **(c)** Draw the locus of a point A, on the same side of QR as P, such that $\triangle AQR$ and $\triangle PQR$ have the same area.
- **(d)** Hence, mark and clearly label the point X, such that $XQ = XR$ and $\triangle XQR$ and $\triangle PQR$ have the same area.

7. **(a)** Construct $\triangle XYZ$ in which $XY = 10.2$ cm, $XZ = 11$ cm and $Z\hat{X}Y = 62°$. Measure and write down the length of YZ.
- **(b)** Draw the locus of a point which is
  - **(i)** equidistant from XZ and ZY,
  - **(ii)** 6 cm from Z.
- **(c)** Measure and write down the length of PQ, given that the two loci in b(i) and b(ii) intersect at P and Q.
- **(d)** Construct, on the same diagram, the rectangle $XYRS$ equal in area to $\triangle XYZ$ with R and S on the opposite side of XY to Z.

8. **(a)** Construct a quadrilateral $PQRS$ in which $PQ = 6$ cm, $PS = 6$ cm, $SR = 7$ cm, $Q\hat{P}R = 45°$ and $P\hat{Q}R = 60°$.
- **(b)** On your diagram,
  - **(i)** draw the locus of a point which is 2 cm from S,
  - **(ii)** draw the locus of a point X such that $P\hat{X}S = 90°$,
  - **(iii)** mark clearly with the letter A, the point outside the quadrilateral which is 2 cm from S and is such that $P\hat{A}S = 90°$.
- **(c)** Measure and write down the length of PA.
- **(d)** Locate on your diagram and mark clearly with the letter B, the point on SP, such that $\triangle PQR$ and $\triangle PQB$ have the same area.
- **(e)** Measure and write down the length of PB.

✶9. A farmer's field $ABCD$ which lies on horizontal ground is bounded on two sides by straight roads which meet at right angles at the point A. The point B lies on one of the roads and the point D lies on the other. Along the other two straight sides, BC and CD, there are fences.
- **(a)** Given that $AB = 120$ m, $AD = 80$ m, $A\hat{D}C = 110°$ and $A\hat{B}C = 56°$, use a scale of 1 cm to represent 10 m to construct an accurate scale drawing of the field.
- **(b)** A tree in the field is 60 m from A and is equidistant from the fences BC and CD. On your diagram, draw
  - **(i)** the locus which represents points which are 60 m from A,
  - **(ii)** the locus which represents points which are equidistant from BC and CD. Label, with the letter T, the point representing the position of the tree.
- **(c)** A well in the field is 10 m from the road AB and equidistant from the points C and D. Represent the position of the well on your diagram by using the intersection of two loci, labelling it clearly with the letter W.
- **(d)** When the angle of elevation of the sun is 15°, the shadow of the tree just reaches the corner B. By further drawing or otherwise, find, correct to the nearest metre, the height of the tree.

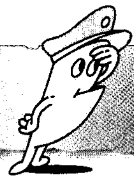
✶10. A playground is in the shape of a quadrilateral $ABCD$. It is given that $AB = 90$ m, $AD = 70$ m, $B\hat{A}D = 115°$, $A\hat{B}C = 100°$ and $A\hat{D}C = 85°$.
- **(a)** Using a scale of 1 cm to represent 10 m, make an accurate scale drawing of the playground.
- **(b)** On your diagram, draw
  - **(i)** the locus which represents points 80 m from B,
  - **(ii)** the locus which represents points equidistant from C and D.

Label the point of intersection of these loci with the letter M.

(c) A point N on the playground is equidistant from BC and CD and is such that $A\hat{N}B = 90°$. Represent this point N on your diagram by using the intersection of two loci.

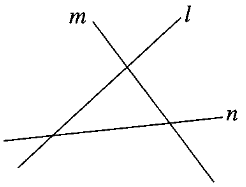
(d) Measure and write down the length MN.

(e) A square region on the playground, if represented on the scale drawing, would have an area of 5 cm^2. Calculate the length, in metres, of a side of the actual square.



Exploration

1. Draw three intersecting lines as shown in the diagram. Construct the four possible circles that will touch each of the three lines.



2. Given that the base PQ of $\triangle PQR$ is fixed and the median RS is of constant length, find the locus of R.

3. Construct $\triangle XYZ$ in which $XY = 5$ cm, $Z\hat{X}Y = 60°$ and $X\hat{Y}Z = 90°$. On the same diagram,

(a) construct the circumcircle of $\triangle XYZ$,

(b) construct, on the same side of XY as Z, the locus of the point P such that the area of $\triangle XYP$ equals half the area of $\triangle XYZ$ and the area of $\triangle XYP$ equals half the area of $\triangle XYZ$,

(c) mark, and label clearly, a point Q such that $X\hat{Q}Y = 30°$ and the area of $\triangle XYQ$ is half the area of $\triangle XYZ$.

Given that M is a point such that $X\hat{M}Y = 30°$, find the largest possible area of $\triangle XMY$.

4. A is 20 km due north of O and B is 20 km due east of O. The bearing of C from A is $\theta°$ and the bearing of C from B is $(90 - \theta)°$, where $0 < \theta < 90$. Given also that C is not more than 40 km from O, show on an accurate diagram the possible positions of C, using a scale of 1 cm to represent 2 km. Find, by measurement from your diagram, the smallest possible value of θ.

5. Draw accurately the parallelogram $PQRS$ in which $PS = 7$ cm, $SR = 12$ cm and $P\hat{S}R = 60°$. Mark the point M on PQ such that $PM = 4$ cm and draw the lines MS and PR to meet at O.

(a) State the scale factor and the centre of the enlargement which maps $\triangle PMO$ onto $\triangle RSO$.

(b) Draw a square which has one corner on OS, one corner on OR and two corners on SR.

CHAPTER 3

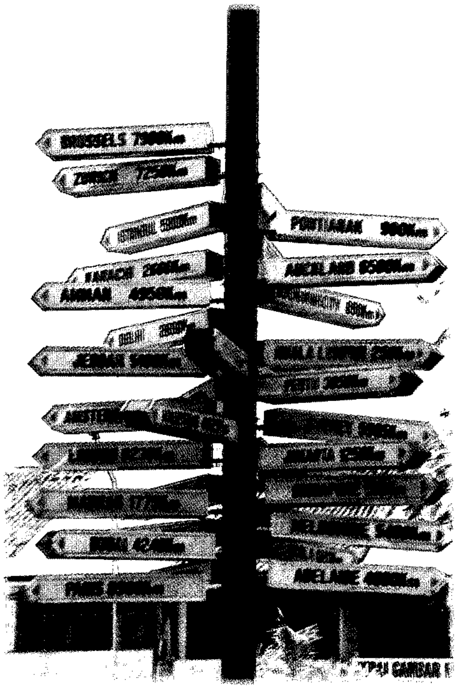
Vectors in Two Dimensions

In this chapter, you will learn how to

- △ distinguish between scalar and vector quantities;
- △ represent vector quantities by directed line segments;
- △ find the magnitude of a vector;
- △ find the sum, difference and scalar multiples of vectors and express vectors in terms of two coplanar vectors;
- △ manipulate vectors expressed in column form;
- △ express a vector in terms of position vectors;
- △ solve problems involving vectors.

Preliminary Problem

The picture shows a signpost erected outside a shopping centre in Penang, Malaysia. Each panel provides two pieces of information, the direction and the distance, of a place from the post. The two pieces of information together give us a rough idea of what a vector quantity is.



Scalar and Vector Quantities

Consider the following statements:

(a) The area of the triangle is 20 cm^2;
(b) The car takes 3 hours to complete the journey;
(c) The ship sails 8 km from P to Q on a bearing of 075°;
(d) The plane is flying due north at 750 km/h;
(e) Move the desk 5 m to the right.

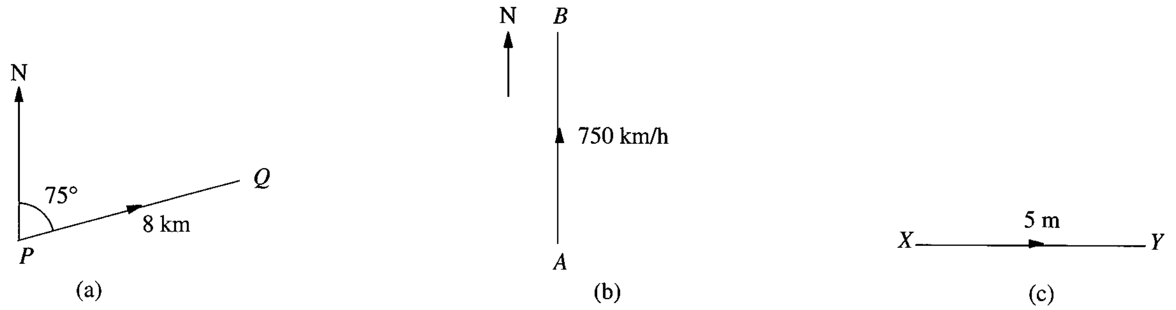
1. In what ways are the first two statements different from the last three statements?
2. In statements (a) and (b), do 20 cm^2 and 3 hours completely describe the area of the triangle and the duration of the journey, respectively?
3. Will a ship sailing 8 km on a bearing of 075° get to the same place as one sailing 8 km on a bearing of 255°?
4. Will a plane flying north at 750 km/h get to the same place as one flying east at 750 km/h?
5. Will a desk that is moved 5 m to the right end up at the same place as one that is moved 5 m to the left?
6. Can we say that the first two statements involve quantities, area and time, which are completely described by their magnitudes alone?
7. Do you agree that the last three statements involve quantities which have both magnitudes and directions?

The quantities which have only magnitudes such as area and time are called **scalar** quantities. Quantities which have both **magnitudes** and **directions** are called **vector** quantities. The movement or the *displacement* of the ship from P to Q, the *velocity* (speed and direction) of the plane and the *translation* of the desk are examples of vector quantities.

Representation of a Vector and Notation

A vector can be represented by a **directed line segment**, whose *direction* is that of the vector and whose *length* represents the magnitude of the vector.

The displacement of the ship from P to Q in statement (c) given earlier is denoted by $\overrightarrow{PQ}$ and represented by the directed line segment shown in Fig. 3.1 (a) below.



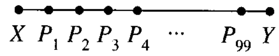
(a) (b) (c)

Fig. 3.1

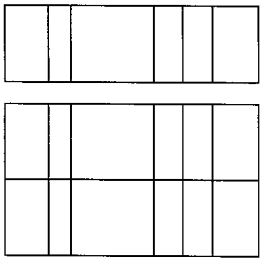
The diagram shows a line segment XY on which 99 points $P_1, P_2, \ldots, P_{99}$ are marked.

(a) Make an investigation to find the total number of possible line segments.

(b) Obtain a formula for the total number of line segments in terms of n if n points are marked on the line segment XY.



(c) Using the result above, find the total number of rectangles in each of the following diagrams.



The velocity of the plane in statement (d) is denoted by $\overrightarrow{AB}$ and represented by the directed line segment shown in Fig. 3.1(b).

The translation of the desk in statement (e) is denoted by $\overrightarrow{XY}$ and represented by the directed line segment shown in Fig. 3.1(c).

The vector $\overrightarrow{PQ}$ shown in Fig. 3.1(a) has length PQ and direction given by the arrow from P to Q. In the notation $\overrightarrow{PQ}$, the order of the letters is important. It indicates the sense of direction of $\overrightarrow{PQ}$. P is called the **initial point** and Q the **terminal point** of $\overrightarrow{PQ}$.

A vector can also be denoted by a single letter in bold typeface.

In Fig. 3.2, the vector $\overrightarrow{AB}$ is denoted by **a**. In written work, it is more convenient to use $\underline{a}$ or $\underset{\sim}{a}$.

The Magnitude of a Vector

The **magnitude** of a vector is the length of the corresponding line segment.

In Fig. 3.2, the magnitude of $\overrightarrow{AB}$, denoted by $|\overrightarrow{AB}|$, $|\underset{\sim}{a}|$ or $|\mathbf{a}|$, is the length of the line segment AB, i.e.

$$|\overrightarrow{AB}| = \text{length of } AB.$$

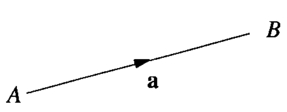


Fig. 3.2

Equal Vectors

When the desk in statement (e) is moved, every point of the desk is moved 5 m to the right. Fig. 3.3 shows the displacements of the four corners of the top of the desk.

The vectors $\overrightarrow{AA'} = \mathbf{a}$, $\overrightarrow{BB'} = \mathbf{b}$, $\overrightarrow{CC'} = \mathbf{c}$ and $\overrightarrow{DD'} = \mathbf{d}$ represent the displacements of the corners A, B, C and D of the desk.

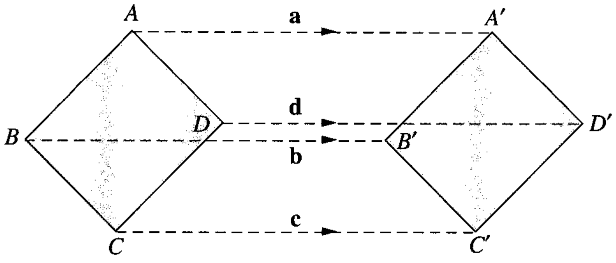


Fig. 3.3

These vectors **a**, **b**, **c** and **d** possess the same magnitude and the same direction. They are **equal** vectors, i.e.

$$\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{d}.$$

Any one of these vectors can be used to describe the displacement or the translation of the desk.

Two vectors are equal if they have the same magnitude and the same direction.

In Fig. 3.4, $\overrightarrow{AB} = \overrightarrow{CD} = \mathbf{p}$, but $\mathbf{p} \neq \mathbf{q} \neq \mathbf{r}$ although **p**, **q** and **r** have the same magnitude.

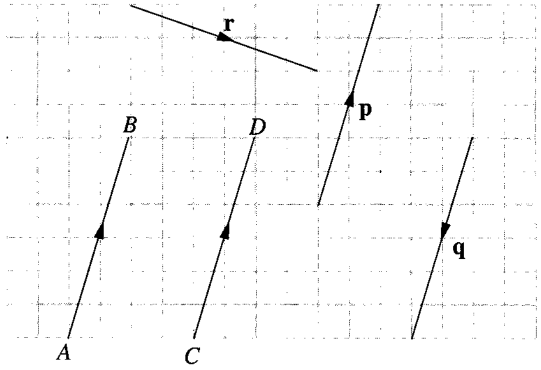


Fig. 3.4

Column Vectors

Fig. 3.5 shows the translation of $\triangle ABC$ to $\triangle A'B'C'$. The translation, represented by the vector $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, or $\overrightarrow{CC'}$, can also be achieved by moving $\triangle ABC$ 5 units to the right and 2 units up. We write

$$\overrightarrow{AA'} = \overrightarrow{BB'} = \overrightarrow{CC'} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

$\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is a **column vector**. 5 and 2 are the **components** of the vector $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

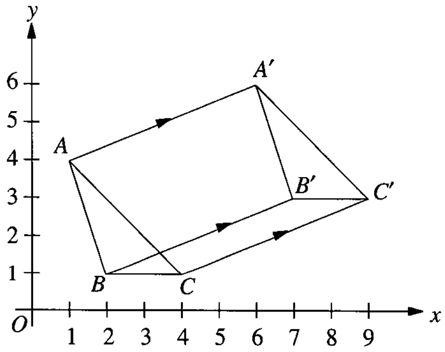
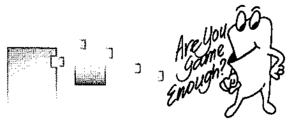


Fig. 3.5

Example 1

In Fig. 3.6, $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ *or* $\mathbf{u} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ *(2 units to the right of P followed by 5 units down to reach Q). Express each of the following as a column vector.*

(a) $\overrightarrow{RS}$ *(b)* $\overrightarrow{TU}$ *(c)* $\overrightarrow{VW}$ *(d)* $\overrightarrow{XY}$



Have you figured out how the column vectors are written down?

How do you relate the sign of the first component of each vector to the right or left movement from the initial point?

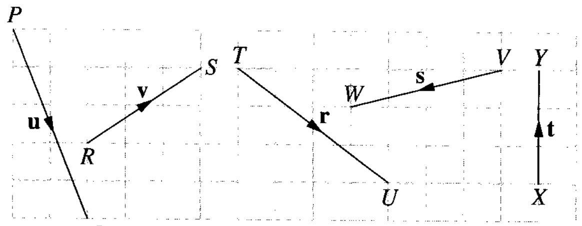


Fig. 3.6

Solution

(a) $\overrightarrow{RS} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (3 units to the right of R, 2 units up.)

(b) $\overrightarrow{TU} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (4 units to the right of T, 3 units down.)

(c) $\overrightarrow{VW} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ (4 units to the left of V, 1 unit down.)

(d) $\overrightarrow{XY} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ (3 units up.)

Example 2

Write vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$, $\overrightarrow{FG}$, $\overrightarrow{GH}$ and $\overrightarrow{HI}$ as column vectors.

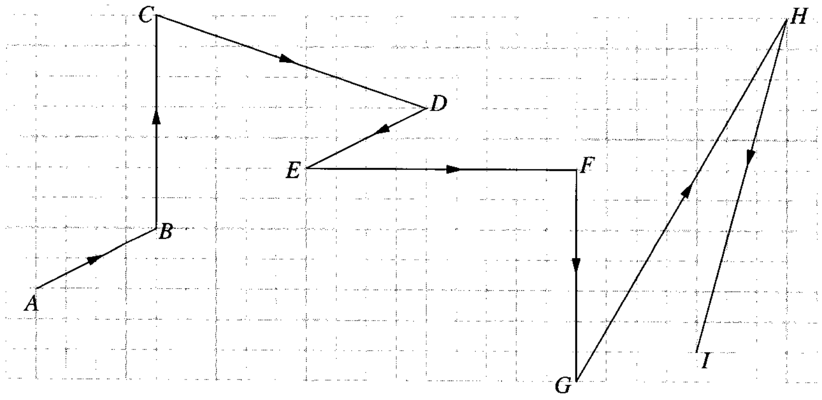


Fig. 3.7

Solution

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$ $\overrightarrow{CD} = \begin{pmatrix} 9 \\ -3 \end{pmatrix}$ $\overrightarrow{DE} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$

$\overrightarrow{EF} = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$ $\overrightarrow{FG} = \begin{pmatrix} 0 \\ -7 \end{pmatrix}$ $\overrightarrow{GH} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$ $\overrightarrow{HI} = \begin{pmatrix} -3 \\ -11 \end{pmatrix}$

Magnitude of a Column Vector

The magnitude of a column vector **a** = $\begin{pmatrix} u \\ v \end{pmatrix}$, is given by

$$|\mathbf{a}| = \sqrt{u^2 + v^2},$$

using Pythagoras' theorem. (Fig. 3.8)

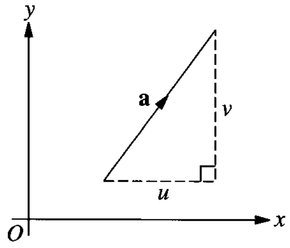


Fig. 3.8

Example 3

Express each of the vectors in the diagram as a column vector and find its magnitude.

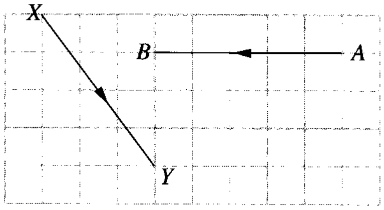


Fig. 3.9

Solution

$\overrightarrow{XY} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $|\overrightarrow{XY}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$ units

$\overrightarrow{AB} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ and $|\overrightarrow{AB}| = \sqrt{(-5)^2 + 0^2} = \sqrt{25} = 5$ units

Equality of Column Vectors

In the earlier discussion on equality of vectors, we learnt that two vectors **a** and **b** are **equal** only if they have the same magnitude and direction. We can express **a** and **b** as **equal column vectors.**

In general,

if a = $\begin{pmatrix} p \\ q \end{pmatrix}$ **and b =** $\begin{pmatrix} r \\ s \end{pmatrix}$ **and a = b,**

then $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$ **and** $p = r$ **and** $q = s$.

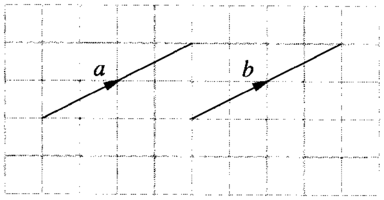


Fig. 3.10

In Fig. 3.10, **a** and **b** are equal vectors. **a** = $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and **b** = $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Example 4

The column vectors **a** *and* **b** *are defined by* $\mathbf{a} = \begin{pmatrix} 10 - p \\ 4 - q \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} p + 2 \\ q - 5 \end{pmatrix}$.

(a) Given that $\mathbf{a} = \mathbf{b}$,

(i) find the values of p and q,

(ii) show that $|\mathbf{a}| = |\mathbf{b}| = \dfrac{\sqrt{145}}{2}$.

(b) Given that $|\mathbf{a}| = |\mathbf{b}|$, *express q in terms of p.*

Solution

(a) (i) $\mathbf{a} = \mathbf{b} \Rightarrow \begin{pmatrix} 10 - p \\ 4 - q \end{pmatrix} = \begin{pmatrix} p + 2 \\ q - 5 \end{pmatrix}$

$$\therefore \quad 10 - p = p + 2 \qquad 4 - q = q - 5$$

$$p = 4 \qquad\qquad q = 4\frac{1}{2}$$

(ii) $\mathbf{a} = \mathbf{b} = \begin{pmatrix} 6 \\ -\frac{1}{2} \end{pmatrix} \Rightarrow |\mathbf{a}| = |\mathbf{b}| = \sqrt{6^2 + \left(-\frac{1}{2}\right)^2}$

$$= \sqrt{36 + \frac{1}{4}} = \frac{\sqrt{145}}{2}$$

(b) $|\mathbf{a}| = |\mathbf{b}| \Rightarrow \sqrt{(10 - p)^2 + (4 - q)^2} = \sqrt{(p + 2)^2 + (q - 5)^2}$

$$\therefore \quad 100 - 20p + p^2 + 16 - 8q + q^2 = p^2 + 4p + 4 + q^2 - 10q + 25$$

$$2q = 24p - 87$$

$$\text{i.e.} \quad q = \frac{24p - 87}{2}$$

Negative Vectors

On its return journey, the ship in statement (c) (on page 55) sails 8 km from Q to P on a bearing of 255°. The vector $\overrightarrow{QP}$ in Fig. 3.11 gives the displacement of the ship from Q to P. $\overrightarrow{QP}$ is opposite in direction to $\overrightarrow{PQ}$ but has the same magnitude. $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are **negative vectors** of each other and we write algebraically,

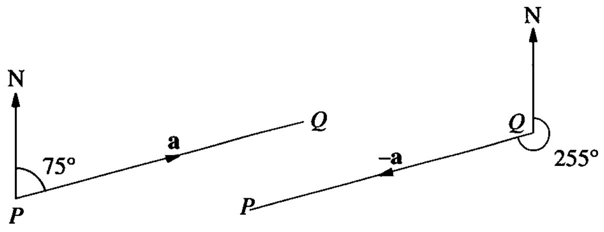


Fig. 3.11

$$\overrightarrow{PQ} = -\overrightarrow{QP} \text{ or } \overrightarrow{QP} = -\overrightarrow{PQ} \text{ or } \mathbf{a} = -(-\mathbf{a}).$$

(Take note of the change in the order of the letters P and Q accompanying the change of signs).

In Example 2, we notice that vectors $\overrightarrow{AB}$ and $\overrightarrow{DE}$ have the same length (magnitude) but are in opposite directions. Also,

$$\begin{pmatrix}4\\2\end{pmatrix} = -\begin{pmatrix}-4\\-2\end{pmatrix} \quad \text{or} \quad -\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-4\\-2\end{pmatrix}$$

$$\overrightarrow{AB} = -\overrightarrow{DE} \quad \text{or} \quad -\overrightarrow{AB} = \overrightarrow{DE}$$

i.e. $\overrightarrow{AB}$ and $\overrightarrow{DE}$ are *negative vectors* of each other.

Note that $|\overrightarrow{AB}| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

and $|\overrightarrow{DE}| = \sqrt{(-4)^2 + (-2)^2} = 2\sqrt{5}$.

Can you identify the other pair of such vectors in Example 2?

In general, a negative sign reverses the direction of a vector, i.e. a vector **a** and its negative vector **–a** have the same magnitude but are opposite in direction.

In Fig. 3.12, $\mathbf{a} = \begin{pmatrix}2\\3\end{pmatrix}$ and $-\mathbf{a} = -\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}-2\\-3\end{pmatrix}$.

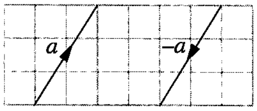


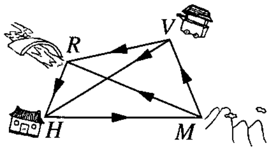
Fig. 3.12

$$|\mathbf{a}| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

$$|-\mathbf{a}| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}.$$

$$\therefore \quad |-\mathbf{a}| = |\mathbf{a}|$$

Anthony wants to go to three different places, the mountains (M), the village market (V) and the river (R) as shown in the map.



If he can visit these places in only two given sequences, i.e.

Route A : H → M → V → R → H

or Route B : H → M → R → V → H,

find out which is the shorter route for Anthony to travel.

Example 5

The figure consists of a square and four identical rhombus.

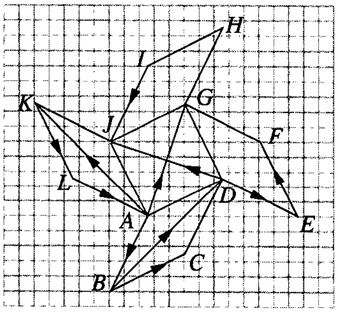


Fig. 3.13

(a) (i) Explain why $\overrightarrow{AB} = \overrightarrow{IJ}$.

(ii) Name two other vectors that are equal to $\overrightarrow{AB}$.

(b) Name all the vectors that are equal to

(i) $\overrightarrow{KL}$; (ii) $\overrightarrow{DE}$;

(iii) $\overrightarrow{BC}$; (iv) $\overrightarrow{AK}$.

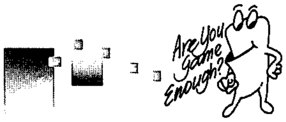
(c) Give a reason why $\overrightarrow{AG} \neq \overrightarrow{DJ}$.

(d) The line segments BD and HJ have the same length and are parallel. Explain why $\overrightarrow{BD} \neq \overrightarrow{HJ}$.

(e) Name a negative vector of

(i) $\overrightarrow{BC}$; (ii) $\overrightarrow{EF}$; (iii) $\overrightarrow{LA}$.

Solution



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Fill the nine circles in the diagram with different numbers using the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 so that each inequality involved is valid. Try to find as many solutions as you can.

(a) (i) $\overrightarrow{AB} = \overrightarrow{IJ}$ because $AB = IJ$ and $\overrightarrow{AB}$ and $\overrightarrow{IJ}$ have the same direction ($AB \mathbin{//} IJ$ and A to B is in the same sense as I to J).

(ii) $\overrightarrow{AB} = \overrightarrow{DC} = \overrightarrow{HG}$

(b) (i) $\overrightarrow{KL} = \overrightarrow{JA} = \overrightarrow{GD} = \overrightarrow{FE}$ (ii) $\overrightarrow{DE} = \overrightarrow{GF} = \overrightarrow{KJ} = \overrightarrow{LA}$

(iii) $\overrightarrow{BC} = \overrightarrow{AD} = \overrightarrow{JG} = \overrightarrow{IH}$ (iv) $\overrightarrow{AK} = \overrightarrow{EG}$

(c) $\overrightarrow{AG}$ and $\overrightarrow{DJ}$ have *different* directions (AG is not parallel to DJ).

(d) $\overrightarrow{BD}$ and $\overrightarrow{HJ}$ have *opposite* directions ($BD \mathbin{//} HJ$ but B to D is in the opposite sense as H to J).

(e) (i) $\overrightarrow{DA}$ ($\overrightarrow{BC} = -\overrightarrow{DA}$) (ii) $\overrightarrow{GD}$ ($\overrightarrow{EF} = -\overrightarrow{GD}$)

(iii) $\overrightarrow{ED}$ ($\overrightarrow{LA} = -\overrightarrow{ED}$)

Example 6

ABCD is a parallelogram. Given that $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ *and* $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$,

(a) find the value of $|\overrightarrow{AB}|$;

(b) express each of the following as a column vector.

(i) $\overrightarrow{DC}$

(ii) $\overrightarrow{DA}$

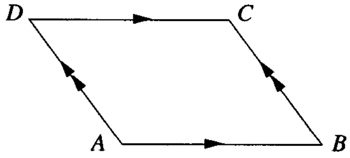


Fig. 3.14

Solution

(a) $|\overrightarrow{AB}| = \sqrt{7^2 + 0^2} = \sqrt{49} = 7$ units

(b) (i) $\overrightarrow{DC} = \overrightarrow{AB} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$

(ii) $\overrightarrow{AD} = \overrightarrow{BC} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$\therefore \overrightarrow{DA} = -\overrightarrow{AD} = -\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

Exercise 3a

1. Write down the column vectors represented by $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{CD}$, $\overrightarrow{DC}$, $\overrightarrow{EF}$ and $\overrightarrow{FE}$.

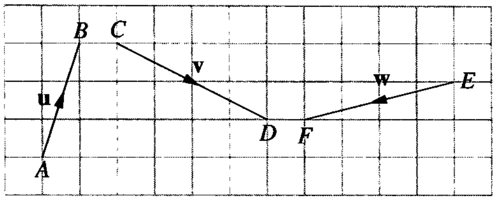


Fig. 3.15

2. Write down the vectors which are equal. In each case, express the equal vectors as a column vector.

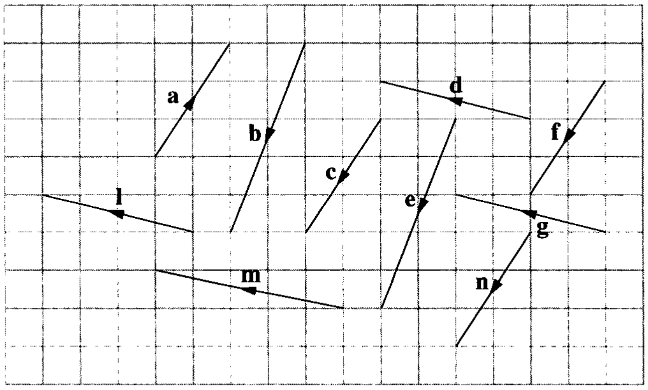


Fig. 3.16

3. On a sheet of squared paper, draw representations of the vectors:

(a) $\begin{pmatrix}3\\4\end{pmatrix}$ **(b)** $\begin{pmatrix}-2\\5\end{pmatrix}$ **(c)** $\begin{pmatrix}-1\\-2\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\3\end{pmatrix}$ **(e)** $\begin{pmatrix}-3\\0\end{pmatrix}$

4. Write down the *negatives* of the vectors:

(a) $\begin{pmatrix}4\\3\end{pmatrix}$ **(b)** $\begin{pmatrix}-2\\5\end{pmatrix}$ **(c)** $\begin{pmatrix}7\\-9\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\0\end{pmatrix}$ **(e)** $\begin{pmatrix}p\\q\end{pmatrix}$

(f) $\begin{pmatrix}3\\7\end{pmatrix}$ **(g)** $\begin{pmatrix}-5\\2\end{pmatrix}$ **(h)** $\begin{pmatrix}8\\-2\end{pmatrix}$ **(i)** $\begin{pmatrix}-1\\-3\end{pmatrix}$ **(j)** $\begin{pmatrix}x\\y\end{pmatrix}$

5. Find the magnitude of each of the following vectors, leaving the answer in square root form where necessary.

(a) $\begin{pmatrix}5\\-12\end{pmatrix}$ **(b)** $\begin{pmatrix}-6\\8\end{pmatrix}$ **(c)** $\begin{pmatrix}5\\2\end{pmatrix}$ **(d)** $\begin{pmatrix}-7\\-1\end{pmatrix}$ **(e)** $\begin{pmatrix}0\\-3\end{pmatrix}$

6. Given that $\overrightarrow{XY} = \begin{pmatrix}p\\-2\end{pmatrix}$, find the possible values of p such that $|\overrightarrow{XY}| = 5$ units.

7. Given that $\overrightarrow{AB} = \begin{pmatrix}-5\\0\end{pmatrix}$ and $\overrightarrow{PQ} = \begin{pmatrix}t\\-3\end{pmatrix}$, find

(a) $|\overrightarrow{AB}|$, **(b)** two possible values of t if $|\overrightarrow{AB}| = |\overrightarrow{PQ}|$.

8. Given that $\mathbf{a} = \begin{pmatrix}12\\5\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}s\\0\end{pmatrix}$, where s is positive, find the value of s such that $|\mathbf{a}| = |\mathbf{b}|$.

9. Given that $\overrightarrow{AB} = \begin{pmatrix}-4\\2\end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix}p\\-12\end{pmatrix}$, find

(a) $|\overrightarrow{AB}|$, giving your answer correct to 2 significant figures,

(b) the positive value of p if $|\overrightarrow{CD}| = 3|\overrightarrow{AB}|$.

10. Given that $\overrightarrow{PQ} = \begin{pmatrix} 13 \\ 0 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$, show that $|\overrightarrow{PQ}| = |\overrightarrow{RS}|$. Explain why $\overrightarrow{PQ} \neq \overrightarrow{RS}$ although $|\overrightarrow{PQ}| = |\overrightarrow{RS}|$.

11. If $\mathbf{u} = \begin{pmatrix} 5 \\ s \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} s - 2t \\ -3 \end{pmatrix}$ and $\mathbf{u} = \mathbf{v}$, find the values of s and t.

12. Fig. 3.17 shows the positions of the points P, A and B where $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(a) Express $\overrightarrow{PB}$ as a column vector.

(b) Q is a point such that $ABQP$ is a parallelogram. Express $\overrightarrow{BQ}$ as a column vector.

(c) R is a point such that $ABPR$ is a parallelogram. Express $\overrightarrow{PR}$ as a column vector.

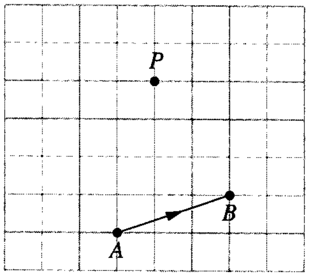


Fig. 3.17

13. **(a)** State which of the following are equal vectors.

(b) Identify pairs of vectors that are negative vectors of each other.

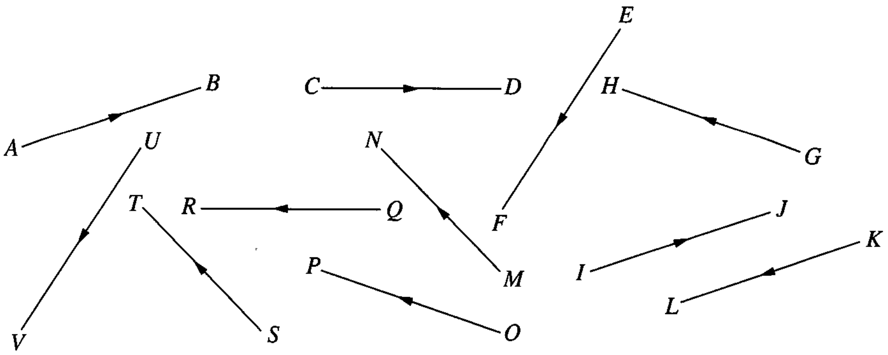


Fig. 3.18

14. **(a)** Name all the vectors in Fig. 3.19 that are equal to:

(i) $\overrightarrow{IJ}$ (ii) $\overrightarrow{AJ}$ (iii) $\overrightarrow{HI}$

(iv) $\overrightarrow{BC}$ (v) $\overrightarrow{AK}$ (vi) $\overrightarrow{LB}$

(b) Name a negative vector of

(i) $\overrightarrow{JH}$; (ii) $\overrightarrow{AB}$; (iii) $\overrightarrow{AJ}$.

(c) Explain why

(i) $\overrightarrow{AB} \neq \overrightarrow{DE}$; (ii) $\overrightarrow{AK} \neq \overrightarrow{AB}$.

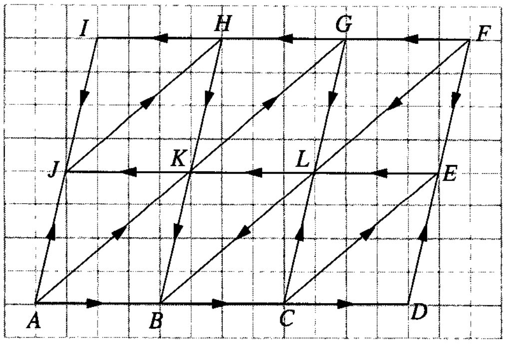


Fig. 3.19

15. Copy and complete the equalities below in each of the diagrams **(a)** – **(d)**. The first equality in **(a)** and **(b)** have been done for you.

(a)

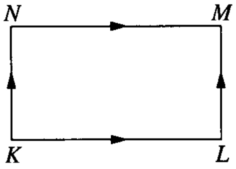


Fig. 3.20a

$\overrightarrow{KN} = \overrightarrow{LM}$

$\overrightarrow{NM} =$

(b)

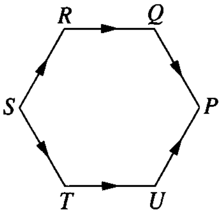


Fig. 3.20b

$\overrightarrow{SR} = \overrightarrow{UP}$

$\overrightarrow{RQ} =$

$\overrightarrow{QP} =$

(c)

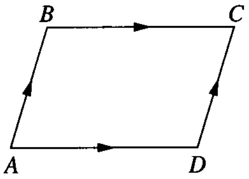


Fig. 3.20c

$\overrightarrow{AB} =$

$\overrightarrow{BC} =$

(d)

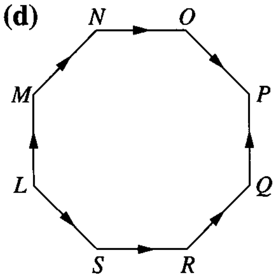


Fig. 3.20d

$\overrightarrow{LM} =$

$\overrightarrow{MN} =$

$\overrightarrow{NO} =$

$\overrightarrow{OP} =$

Addition of Vectors

Fig. 3.21 shows the translation of a body, from P to Q, followed by another translation from Q to R.

The first translation is described by the vector

$$\overrightarrow{PQ} = \mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and the second translation is described by the vector

$$\overrightarrow{QR} = \mathbf{b} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

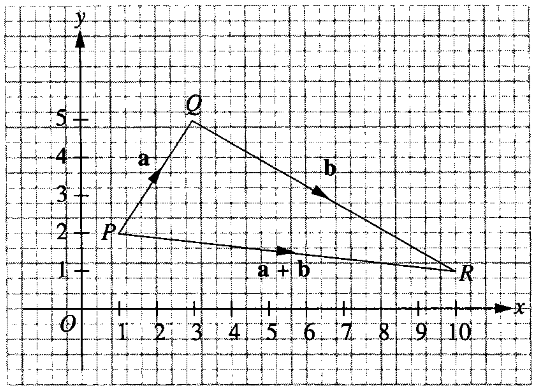


Fig. 3.21

Is there a single translation that will move the body from P to R directly? In Fig. 3.21, the translation of the body from P to R directly is described by the vector $\overrightarrow{PR}$.

We write algebraically,

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} \quad \text{or} \quad \overrightarrow{PR} = \mathbf{a} + \mathbf{b}.$$

In column vector form, we have

$$\overrightarrow{PQ} + \overrightarrow{QR} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix} = \overrightarrow{PR}.$$

Thus, the vector $\overrightarrow{PR}$ can be written as the *sum* of the two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, i.e.

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}.$$

You should take note of the *intermediate* letter Q.

Example 7

*Find the sum of the two given vectors **a** and **b**.*

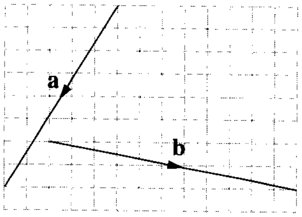


Fig. 3.22

Solution

Method 1

Copy the vector $\overrightarrow{AB}$ = **a**. From the terminal point B, draw the vector $\overrightarrow{BC}$ = **b**. Join A to C to complete the triangle. Insert an arrow which points away from A. $\overrightarrow{AC}$ represents the sum of **a** and **b**, i.e. $\overrightarrow{AC}$ = **a** + **b**. This process is known as the **Triangle Law of Vector Addition**.

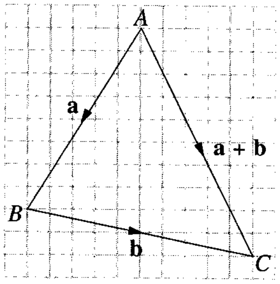


Fig. 3.23

Method 2

Copy the vector $\overrightarrow{AB}$ = **a**. From the initial point A, draw the vector $\overrightarrow{AD}$ = **b**. Complete the parallelogram $ABCD$ and then draw the diagonal AC. Put in an arrow which points away from A. $\overrightarrow{AC}$ represents the sum of **a** and **b**. This process is known as the **Parallelogram Law of Vector Addition**.

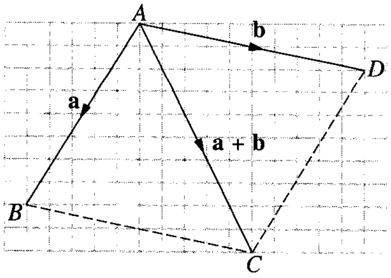


Fig. 3.24

Both methods give the same vector $\overrightarrow{AC}$ which represents the sum of **a** and **b**.

Example 8

*Copy on a sheet of squared paper the directed line segments $\overrightarrow{PQ}$, $\overrightarrow{RS}$ and $\overrightarrow{VW}$ shown in Fig. 3.25 to represent **u**, **v** and **s** respectively. Find the sum of*

*(a) the two vectors **u** and **v**;* *(b) the two vectors **u** and **s**.*

In each case, write the vector sum in column vector form.

Solution

(a) $\overrightarrow{XZ} = \mathbf{u} + \mathbf{v} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ (b) $\overrightarrow{LN} = \mathbf{u} + \mathbf{s} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$

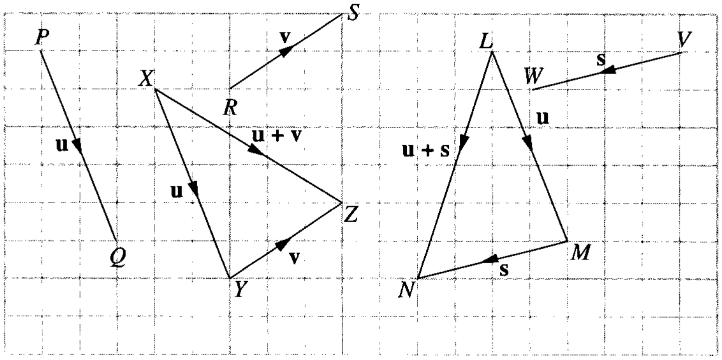


Fig. 3.25

Addition of Column Vectors

In Example 8, since $\mathbf{u} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, and $\mathbf{s} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$,

$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, and $\mathbf{u} + \mathbf{s} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$.

In general, for any two column vectors $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} r \\ s \end{pmatrix}$, $\mathbf{a} + \mathbf{b} = \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p + r \\ q + s \end{pmatrix}$,

i.e. the sum of two column vectors can be obtained by adding the corresponding components.

Example 9

Illustrate graphically the following vector sums using the vectors given in Fig. 3.26:

(a) $\boldsymbol{a} + \boldsymbol{b}$ *(b)* $\boldsymbol{b} + \boldsymbol{a}$

(c) $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$ *(d)* $\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$

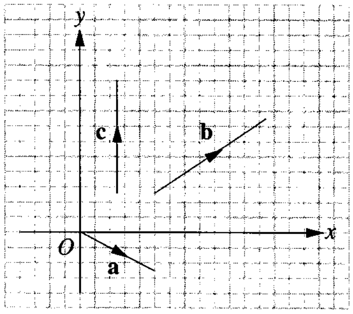


Fig. 3.26

Solution

(a)

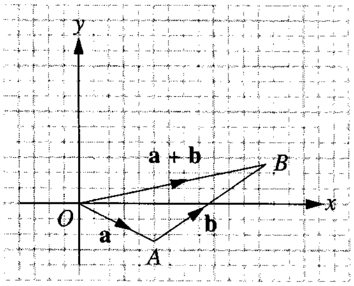


Fig. 3.27a

(b)

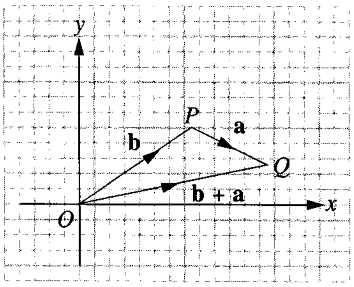


Fig. 3.27b

(c)

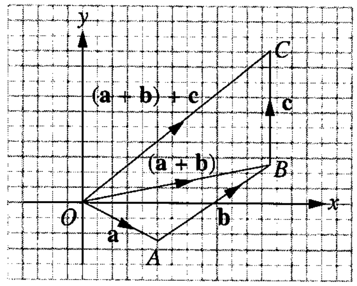


Fig. 3.27c

(d)

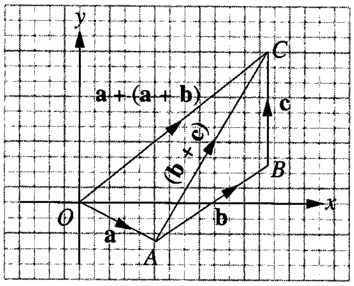


Fig. 3.27d

From (a) and (b), we have

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad \text{or} \quad \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

and we say that vector addition is **commutative.**

From (c) and (d), we have

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \quad \text{or} \quad \left[\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right] + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right] = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

and we say that vector addition is **associative.**

Example 10

ABCD is a quadrilateral. Simplify

(a) $\overrightarrow{AB} + \overrightarrow{BC}$;

(b) $\overrightarrow{AD} + \overrightarrow{DB}$;

(c) $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$.

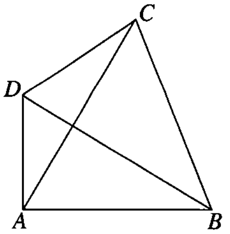


Fig. 3.28

Solution

(a) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (triangle law)

(b) $\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$ (triangle law)

(c) $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD} = (\overrightarrow{AC} + \overrightarrow{CB}) + \overrightarrow{BD}$ (associative law)

$= \overrightarrow{AB} + \overrightarrow{BD}$ (triangle law)

$= \overrightarrow{AD}$ (triangle law)

For your information

The principle of adding two vectors can be extended to any member of vectors.

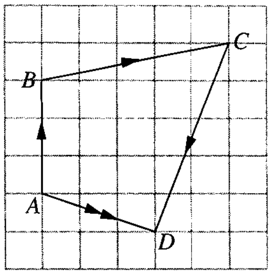
In the figure below,

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$

or $\begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix} =$

$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \overrightarrow{AD}$

i.e. $\overrightarrow{AD}$ *is the net result of the application of all three vectors.*



Zero Vectors

Recall the two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ which represent the journey of a ship from P to Q followed by the return journey from Q to P. Clearly, the result of the whole journey is a zero displacement of the ship from P. Thus

$$\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}.$$

0 is called the **zero** or **null** vector.

Example 11

Simplify the following:

(a) $\begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ 2 \end{pmatrix}$ *(b)* $\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ *(c)* $\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

Solution

(a) $\begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

In the example that you have just seen,

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the column vector form of the **zero** vector, **0**.

$\begin{pmatrix} -5 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ are *negatives* of $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ respectively.

Note: In general, for any vector **a**, there is always a negative vector –**a** such that

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0} = (-\mathbf{a}) + \mathbf{a}.$$

Difference of Two Vectors

The difference of **a** and **b**, denoted by **a** – **b**, is the sum of **a** and the negative of **b** which is –**b**, i.e.

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

To subtract vectors, see Fig. 3.29, where S is the point on QR produced, such that $QR = RS$.

In terms of column vectors,

$$\overrightarrow{PR} = \mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \overrightarrow{RQ} = \mathbf{b} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \overrightarrow{RS} = -\mathbf{b} = -\overrightarrow{RQ} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\therefore \quad \mathbf{a} - \mathbf{b} = \overrightarrow{PR} - \overrightarrow{RQ} = \overrightarrow{PR} + (-\overrightarrow{RQ})$$
$$= \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}$$

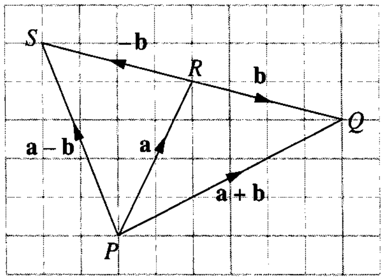


Fig. 3.29

or $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - 4 \\ 4 + 1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \overrightarrow{PS}.$

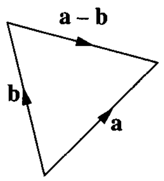


Fig. 3.30

Fig. 3.30 shows that **b** + (**a** – **b**) = **a**.

Subtraction of Column Vectors

For any two column vectors a = $\begin{pmatrix} p \\ q \end{pmatrix}$ **and b =** $\begin{pmatrix} r \\ s \end{pmatrix}$**,** $\mathbf{a} - \mathbf{b} = \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p - r \\ q - s \end{pmatrix}$,

i.e. the difference of two column vectors can be obtained by subtracting the corresponding components.

Example 12

In the diagram, PQTS is a quadrilateral and PQRS is a parallelogram. Find single vectors equivalent to

(a) $\overrightarrow{PQ} - \overrightarrow{PS}$;

(b) $\overrightarrow{PQ} + \overrightarrow{RS} + \overrightarrow{TR} - \overrightarrow{QR}$.

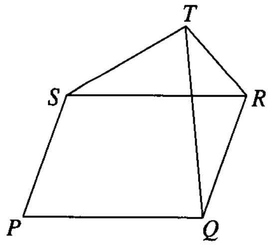


Fig. 3.31

Solution

(a) $\overrightarrow{PQ} - \overrightarrow{PS} = \overrightarrow{PQ} + (-\overrightarrow{PS}) = \overrightarrow{PQ} + \overrightarrow{SP}$ (negative vector)

$= \overrightarrow{SP} + \overrightarrow{PQ}$ (commutative law)

$= \overrightarrow{SQ}$ (triangle law)

(b) $\overrightarrow{PQ} + \overrightarrow{RS} + \overrightarrow{TR} - \overrightarrow{QR} = \overrightarrow{PQ} + \overrightarrow{QP} + \overrightarrow{TR} - \overrightarrow{QR}$ (equal vectors, $\overrightarrow{RS} = \overrightarrow{QP}$)

$= \mathbf{0} + \overrightarrow{TR} - \overrightarrow{QR}$ ($\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$)

$= \overrightarrow{TR} + \overrightarrow{RQ}$ (negative vector)

$= \overrightarrow{TQ}$ (triangle law)

Example 13

(a) Simplify $\begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -7 \\ 1 \end{pmatrix}$.

(b) If $\mathbf{u} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ *and* $\mathbf{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, *express* $-\mathbf{u} + \mathbf{v} - \mathbf{w}$ *as a column vector.*

Solution

(a)
$$\begin{aligned}\begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -7 \\ 1 \end{pmatrix} &= \left[\begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right] - \begin{pmatrix} -7 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 3 \end{pmatrix} - \begin{pmatrix} -7 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix}\end{aligned}$$

or simply
$$\begin{aligned}\begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -7 \\ 1 \end{pmatrix} &= \begin{pmatrix} -4 - 2 + 7 \\ 6 - 3 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix}\end{aligned}$$

(b)
$$\begin{aligned}-\mathbf{u} + \mathbf{v} - \mathbf{w} &= -\begin{pmatrix} 9 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -9 - 2 - 3 \\ -4 + 5 + 1 \end{pmatrix} \\ &= \begin{pmatrix} -14 \\ 2 \end{pmatrix}\end{aligned}$$

Example 14

Find the values of x and y in each of the following equations:

(a) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ *(b)* $\begin{pmatrix} 4 \\ y \end{pmatrix} + \begin{pmatrix} x \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ x \end{pmatrix}$ *(c)* $\begin{pmatrix} 2 \\ 3x \end{pmatrix} - \begin{pmatrix} x - 2y \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$

Solution

(a)
$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$
$$\begin{pmatrix} x + 5 \\ y - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$\therefore \quad x + 5 = 2$ and $y - 2 = -3$
i.e. $\quad x = -3$ and $y = -1$

(b)
$$\begin{pmatrix} 4 \\ y \end{pmatrix} + \begin{pmatrix} x \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ x \end{pmatrix}$$
$$\begin{pmatrix} 4 + x \\ y + 2 \end{pmatrix} = \begin{pmatrix} 5 \\ x \end{pmatrix}$$

$\therefore \quad 4 + x = 5$ and $y + 2 = x$
$4 + x = 5$
$x = 1$
$\therefore \quad y + 2 = 1$
$y = -1$

(c)
$$\begin{pmatrix} 2 \\ 3x \end{pmatrix} - \begin{pmatrix} x - 2y \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$$
$$\begin{pmatrix} 2 - x + 2y \\ 3x + 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$$

$\therefore \quad 3x + 1 = 10$
i.e. $\quad x = 3$
and $\quad 2 - x + 2y = 3$
or $\quad 2y = x + 1 = 4$
i.e. $\quad y = 2$

Exercise 3b

1. **(a)** Copy on a sheet of squared paper the directed line segments $\overrightarrow{PQ}$, $\overrightarrow{RS}$ and $\overrightarrow{TU}$ to represent vectors **a**, **b** and **c** respectively.

   (b) Draw a triangle ABC with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$, to show the addition of these vectors.

   (c) Draw a triangle EDF with $\overrightarrow{ED} = \mathbf{b}$ and $\overrightarrow{DF} = \mathbf{c}$, to show the addition of these vectors.

   (d) Draw a parallelogram $KMNL$ in which $\overrightarrow{MN} = \mathbf{a}$ and $\overrightarrow{MK} = \mathbf{b}$. Name the directed line segment which represents $\mathbf{a} + \mathbf{b}$.

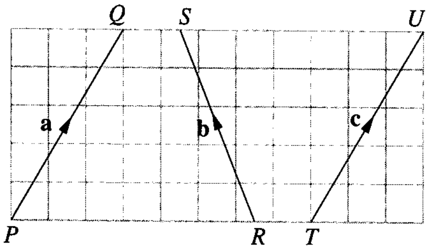


Fig. 3.32

2. **(a)** Copy on a sheet of squared paper, the directed line segments $\overrightarrow{LM}$, $\overrightarrow{XY}$ and $\overrightarrow{PQ}$ to represent vectors **u**, **v** and **r** respectively.

   (b) Draw appropriate triangles to illustrate the following vector additions:

   (i) $\mathbf{u} + \mathbf{v}$,
   (ii) $\mathbf{u} + \mathbf{r}$,
   (iii) $\mathbf{v} + \mathbf{r}$,
   (iv) $\mathbf{u} - \mathbf{v}$.

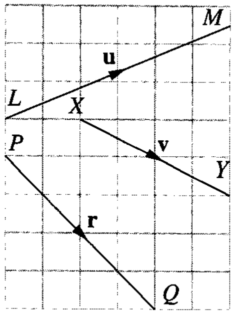


Fig. 3.33

3. In Fig. 3.34, the various line segments are taken to be representatives of vectors. Express in terms of a single vector the sum of the following vectors.

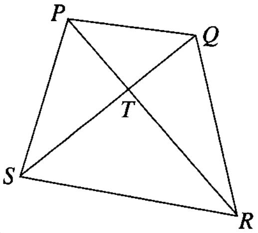


Fig. 3.34

   (a) $\overrightarrow{PT} + \overrightarrow{TR}$
   (b) $\overrightarrow{SQ} + \overrightarrow{QR}$
   (c) $\overrightarrow{ST} + \overrightarrow{TR}$
   (d) $\overrightarrow{SQ} + \overrightarrow{QT}$
   (e) $\overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR}$
   (f) $\overrightarrow{RQ} + \overrightarrow{QT} + \overrightarrow{TP} + \overrightarrow{PS}$

4. $ABCD$ is a quadrilateral. Simplify

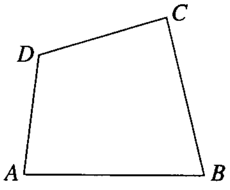


Fig. 3.35

   (a) $\overrightarrow{AD} + \overrightarrow{DC}$;
   (b) $\overrightarrow{AB} + \overrightarrow{BD}$;
   (c) $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$;
   (d) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$;
   (e) $\overrightarrow{AC} - \overrightarrow{DC}$;
   (f) $\overrightarrow{CB} + \overrightarrow{BD} - \overrightarrow{AD}$.

5. $PQRS$ is a parallelogram. O is the point of intersection of its diagonals.

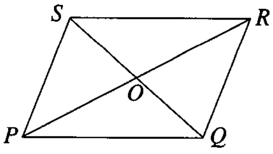


Fig. 3.36

   (a) Simplify

   (i) $\overrightarrow{PQ} + \overrightarrow{PS}$;
   (ii) $\overrightarrow{RO} - \overrightarrow{QO}$;
   (iii) $\overrightarrow{PR} - \overrightarrow{SR} + \overrightarrow{SQ}$.

   (b) Find a vector which can replace **x** in each of the following equations.

   (i) $\overrightarrow{SO} + \mathbf{x} = \overrightarrow{SP}$
   (ii) $\overrightarrow{PO} + \mathbf{x} = \overrightarrow{PR}$
   (iii) $\mathbf{x} + \overrightarrow{SQ} = \overrightarrow{RQ}$
   (iv) $\overrightarrow{PR} + \mathbf{x} = \mathbf{0}$
   (v) $\overrightarrow{PQ} + \mathbf{x} + \overrightarrow{RS} = \overrightarrow{PS}$
   (vi) $\overrightarrow{QR} + \overrightarrow{RS} + \mathbf{x} = \overrightarrow{PS}$

(c) If $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PS} = \mathbf{b}$, find in terms of **a** and/or **b**:

(i) $\overrightarrow{SR}$; **(ii)** $\overrightarrow{PR}$; **(iii)** $\overrightarrow{SQ}$.

6. In Fig. 3.37, the diagonals of $PQRS$ intersect at K. Find, for each of the following equations, a vector which can replace **u**.

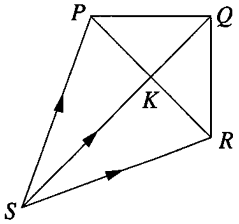


Fig. 3.37

(a) $\overrightarrow{SK} + \mathbf{u} = \mathbf{0}$

(b) $\overrightarrow{SP} + \overrightarrow{PQ} = \mathbf{u} = \mathbf{0}$

(c) $\overrightarrow{PS} + \overrightarrow{SK} + \overrightarrow{KR} = \mathbf{u}$

(d) $\overrightarrow{PK} + (-\overrightarrow{SK}) = \mathbf{u}$

(e) $\overrightarrow{PS} + (-\overrightarrow{RS}) = \mathbf{u}$

(f) $\overrightarrow{PQ} + \overrightarrow{QR} + (-\overrightarrow{PR}) = \mathbf{u}$

7. In Fig. 3.38, ABC is a triangle. Find a vector which can replace **v** in each of the following equations.

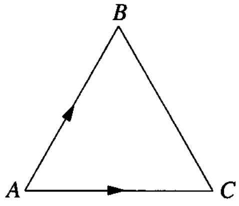


Fig. 3.38

(a) $\overrightarrow{AB} - \overrightarrow{AC} = \mathbf{v}$

(b) $\overrightarrow{AC} - \overrightarrow{AB} = \mathbf{v}$

(c) $\overrightarrow{CA} - \overrightarrow{CB} = \mathbf{v}$

8. Find **x** in each of the following cases. (Refer to Fig. 3.39.)

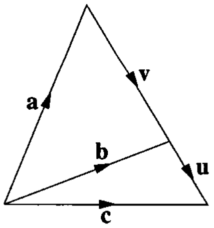


Fig. 3.39

(a) $\mathbf{x} + \mathbf{u} = \mathbf{c}$

(b) $\mathbf{x} + \mathbf{a} = \mathbf{b}$

(c) $\mathbf{x} + \mathbf{c} = \mathbf{b}$

(d) $\mathbf{x} - \mathbf{b} = \mathbf{u}$

9. Simplify the following:

(a) $\begin{pmatrix} -3 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -5 \\ -2 \end{pmatrix}$

(c) $\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ **(d)** $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$

10. Find the values of a and b in each of the following equations:

(a) $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$

(c) $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

11. If $\mathbf{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$, express in column vector form:

(a) $\mathbf{u} + \mathbf{v}$ **(b)** $\mathbf{v} + \mathbf{u}$

(c) $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ **(d)** $\mathbf{u} + (\mathbf{v} + \mathbf{w})$

Which two laws do the answers illustrate?

12. If $\mathbf{a} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$, express in column vector forms:

(a) $\mathbf{a} + \mathbf{b}$ **(b)** $\mathbf{b} + \mathbf{a}$

(c) $(\mathbf{a} + \mathbf{b}) + \mathbf{c}$ **(d)** $\mathbf{a} + (\mathbf{b} + \mathbf{c})$

13. Given that $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$, if $\mathbf{u} = \mathbf{v}$, what can be said about a, b, c and d?

14. Find the values of x and y in the following equations:

(a) $\begin{pmatrix} x \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ y \end{pmatrix}$

(b) $\begin{pmatrix} 3 \\ y \end{pmatrix} + \begin{pmatrix} x \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

(c) $\begin{pmatrix} x \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

(d) $\begin{pmatrix} 2x \\ 5 \end{pmatrix} - \begin{pmatrix} y - 3 \\ 3y \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \end{pmatrix}$

15. Copy and complete the following:

(a) $\begin{pmatrix} 3 \\ \ \end{pmatrix} + \begin{pmatrix} \ \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} 5 \\ -7 \end{pmatrix} + \begin{pmatrix} \ \\ \ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} x \\ -y \end{pmatrix} + \begin{pmatrix} \ \\ \ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

16. Simplify:

(a) $\begin{pmatrix}-1\\5\end{pmatrix}-\begin{pmatrix}4\\2\end{pmatrix}$ **(b)** $\begin{pmatrix}3\\7\end{pmatrix}-\begin{pmatrix}4\\3\end{pmatrix}$

(c) $\begin{pmatrix}8\\-2\end{pmatrix}-\begin{pmatrix}-1\\-2\end{pmatrix}$ **(d)** $\begin{pmatrix}4\\1\end{pmatrix}-\begin{pmatrix}-5\\2\end{pmatrix}$

17. Find **u** in column vector form in each of the following:

(a) $\mathbf{u}+\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}7\\1\end{pmatrix}$

(b) $\mathbf{u}-\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}$

(c) $\begin{pmatrix}1\\2\end{pmatrix}+\mathbf{u}=\begin{pmatrix}-1\\-3\end{pmatrix}$

(d) $\begin{pmatrix}-1\\3\end{pmatrix}-\mathbf{u}=\begin{pmatrix}4\\1\end{pmatrix}$

18. Simplify:

(a) $\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}-\begin{pmatrix}3\\5\end{pmatrix}$

(b) $\begin{pmatrix}-1\\2\end{pmatrix}-\begin{pmatrix}4\\5\end{pmatrix}+\begin{pmatrix}3\\-2\end{pmatrix}$

(c) $\begin{pmatrix}4\\3\end{pmatrix}-\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}-3\\5\end{pmatrix}$

19. If $\mathbf{u}=\begin{pmatrix}7\\2\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}-1\\4\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}-2\\-3\end{pmatrix}$, express in column vector forms:

(a) $\mathbf{u}+\mathbf{v}-\mathbf{w}$ **(b)** $\mathbf{u}-\mathbf{v}+\mathbf{w}$

(c) $\mathbf{u}-\mathbf{v}-\mathbf{w}$

20. If $\mathbf{a}=\begin{pmatrix}4\\3\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\1\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}-3\\1\end{pmatrix}$, express as column vectors:

(a) $\mathbf{a}-\mathbf{b}$ **(b)** $\mathbf{b}-\mathbf{c}$

(c) $\mathbf{a}+\mathbf{b}-\mathbf{c}$ **(d)** $\mathbf{a}-\mathbf{b}+\mathbf{c}$

(e) $\mathbf{a}-(\mathbf{b}+\mathbf{c})$ **(f)** $\mathbf{a}-(\mathbf{b}-\mathbf{c})$

21. In Fig. 3.40, *ABCD* is a parallelogram in which $\overrightarrow{AB}=\begin{pmatrix}5\\12\end{pmatrix}$, $\overrightarrow{AD}=\begin{pmatrix}15\\0\end{pmatrix}$.

(a) Find the value of the following:

(i) $|\overrightarrow{AD}|$ **(ii)** $|\overrightarrow{DC}|$

(b) Express each of the following as a column vector:

(i) $\overrightarrow{AC}$

(ii) $\overrightarrow{DB}$

(iii) $\overrightarrow{BA}$

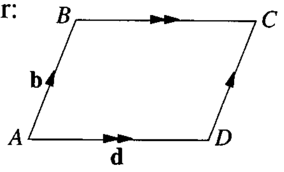


Fig. 3.40

Scalar Multiple of a Vector

Referring to Fig. 3.41,

(1) Is vector **u** parallel to vectors **a** and **b**?

(2) What can you say about the direction and magnitude of **a** when compared to those of **u**?

(3) Can we write **a** = **u** + **u** + **u** = 3**u**?

(4) What about the direction and magnitude of **b** when compared to those of **u**?

(5) Do you agree that **b** = (–**u**) + (–**u**) = –2**u**?
3**u** and –2**u** are said to be scalar multiples of **u**.
In terms of column vectors, we have

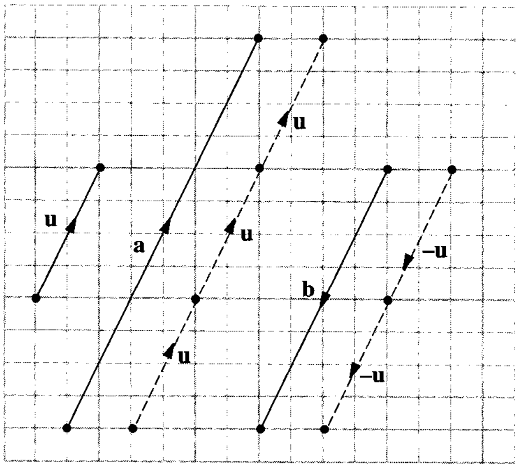


Fig. 3.41

$$\mathbf{u}=\begin{pmatrix}2\\4\end{pmatrix},\ \mathbf{a}=\begin{pmatrix}6\\12\end{pmatrix}=3\begin{pmatrix}2\\4\end{pmatrix}=3\mathbf{u}\quad\text{and}\quad\mathbf{b}=\begin{pmatrix}-4\\-8\end{pmatrix}=-2\begin{pmatrix}2\\4\end{pmatrix}=-2\mathbf{u}.$$

Note: In general,

(a) the magnitude of $k\mathbf{a}$ is $|k|$ times that of $\mathbf{a}$ or $|k\mathbf{a}| = |k|\,|\mathbf{a}|$,

(b) when
   (i) $k > 0$, $k\mathbf{a}$ is in the same direction as $\mathbf{a}$,
   (ii) $k < 0$, $k\mathbf{a}$ is in the opposite direction as $\mathbf{a}$,
   (iii) $k \neq 0$, $\mathbf{a} \neq 0$, $k\mathbf{a}$ is called a scalar multiple of $\mathbf{a}$.

Parallel Vectors

If $\mathbf{b} = k\mathbf{a}$, where k is a scalar, then

(i) $\mathbf{b}$ is parallel to $\mathbf{a}$,
(ii) $\mathbf{b}$ has a magnitude $|k|$ times that of $\mathbf{a}$.

In Fig. 3.42

$$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2\mathbf{a}$$

$$\text{and} \quad \mathbf{c} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} = -3\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -3\mathbf{a}.$$

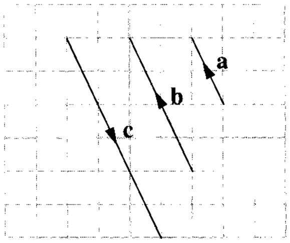


Fig. 3.42

It is shown that **a**, **b** and **c** are parallel vectors.

b and **a** are in the same direction with the magnitude of **b** being twice that of **a**.

c and **a** are in the opposite directions. The magnitude of **c** is $|-3| = 3$ times that of **a**.

Example 15

Given that $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ *and* $\begin{pmatrix} 12 \\ p \end{pmatrix}$ *are parallel vectors, find the value of* p.

Solution

Since $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ p \end{pmatrix}$ are parallel, let $\begin{pmatrix} 12 \\ p \end{pmatrix} = k\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 4k \\ -3k \end{pmatrix}$, where k is a scalar.

$\therefore \quad 4k = 12$ or $k = 3$
and $\quad p = -3k = -3(3) = -9$.

Alternatively,

If $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ p \end{pmatrix} = 3\begin{pmatrix} 4 \\ \frac{p}{3} \end{pmatrix}$ are parallel, then $-3 = \frac{p}{3}$ and $p = -9$.

Example 16

It is given that $\mathbf{u} = \begin{pmatrix} -15 \\ 8 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. *If* $\mathbf{u} = k\mathbf{v}$ *where k is a positive constant and* $|\mathbf{v}| = 51$, *find the values of k, a and b.*

Solution

$$\mathbf{u} = k\mathbf{v} \quad \Rightarrow \quad |\mathbf{u}| = k|\mathbf{v}| \quad (k > 0)$$

i.e. $|\mathbf{u}| = \sqrt{(-15)^2 + 8^2}$

$= \sqrt{289}$

$= 17$

$|\mathbf{v}| = 51$

$|\mathbf{u}| = k|\mathbf{v}|$

$17 = 51k$

$k = \frac{17}{51} = \frac{1}{3}$

$\therefore \quad \mathbf{u} = \frac{1}{3}\mathbf{v}$

$$\begin{pmatrix} -15 \\ 8 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{3} \\ \frac{b}{3} \end{pmatrix}$$

$\frac{a}{3} = -15$ and $\frac{b}{3} = 8$

$\therefore \quad a = -45$ and $b = 24$

Example 17

Vectors $\mathbf{p}$ *and* $\mathbf{q}$ *both have lengths of 6 km and directions on a bearing of 060° and 120° respectively. Find*

(a) the bearing of $\frac{1}{3}\mathbf{p}$;

(b) (i) the length of $-3\mathbf{p}$, *(ii) the bearing of* $-3\mathbf{p}$;

(c) (i) the bearing of $(\mathbf{p} + \mathbf{q})$, *(ii) the length of* $(\mathbf{p} + \mathbf{q})$.

Solution

(a) $\frac{1}{3}\mathbf{p}$ is parallel to **p** and has the same direction as **p**.

$\therefore$ the bearing of $\frac{1}{3}\mathbf{p}$ is 060°.

(b) (i) The length of $-3\mathbf{p}$ is $|-3| = 3$ times the length of **p**.

$\therefore$ the length of $-3\mathbf{p} = 3 \times 6 = 18$ km

(ii) $-3\mathbf{p}$ is parallel to **p** but opposite in direction to **p**.

$\therefore$ the bearing of $-3\mathbf{p}$ is 060° + 180° = 240°. (See Fig. 3.43.)

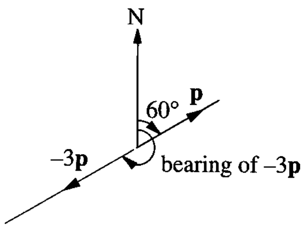


Fig. 3.43

(c) Fig. 3.44 shows the vectors $\mathbf{p} = \overrightarrow{OA}$ and $\mathbf{q} = \overrightarrow{OB}$. $\mathbf{p} + \mathbf{q}$ is represented by $\overrightarrow{OC}$, where $OBCA$ is a rhombus.

(i) $A\hat{O}C = \frac{1}{2}A\hat{O}B = 30°$ and $N\hat{O}C = 60° + 30° = 90°$

$\therefore$ the bearing of $\mathbf{p} + \mathbf{q}$ is 090°.

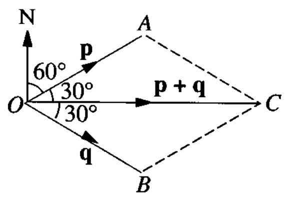


Fig. 3.44

(ii) In Fig. 3.45, $\triangle AOC$ is isosceles.
AM is perpendicular to OC and M is the mid-point of OC.
$OM = 6 \cos 30° = 5.196$ km
$\therefore$ the length of $(\mathbf{p} + \mathbf{q}) = OC = 2 \times 5.196$
$= 10.39$ km

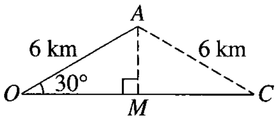


Fig. 3.45

Expression of a Given Vector in Terms of Two Vectors

We know that the result of adding two vectors is a vector. Now, our question is, can we always express a vector as the addition of two vectors? Let us look at the following example.

Example 18

*Fig. 3.46 shows 2 non-parallel vectors **u** and **v**.*

*Express $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of **u** and **v**.*

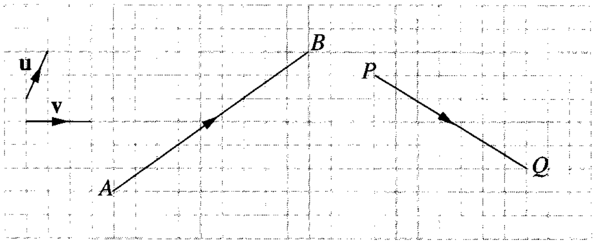


Fig. 3.46

Solution

Parallelogram $ADBC$, with the given vector $\overrightarrow{AB}$ as diagonal and with $\overrightarrow{AC}$ parallel to **u** and $\overrightarrow{AD}$ parallel to **v**, is completed as shown in Fig. 3.47.

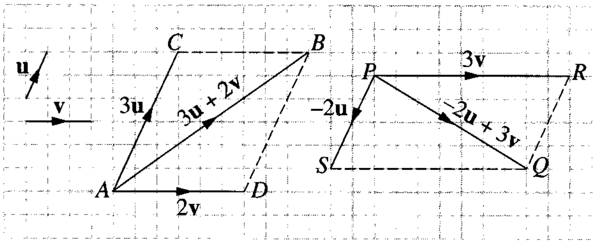


Fig. 3.47

By the parallelogram law of vector addition,

$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{AD} = 3\mathbf{u} + 2\mathbf{v}.$$

Similarly, parallelogram $PRQS$ is completed and we have

$$\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{PS} = 3\mathbf{v} - 2\mathbf{u}.$$

Can we form more than one parallelogram with the given vector $\overrightarrow{AB}$ or $\overrightarrow{PQ}$ as diagonal and its sides parallel to **u** and **v**?

Note: In general, a given vector **a** can be expressed **uniquely** in terms of two non-parallel vectors **u** and **v** (lying in the same plane) in the form $\mathbf{a} = p\mathbf{u} + q\mathbf{v}$, where p and q are scalars, by drawing the parallelogram having **a** as diagonal and its sides parallel to **u** and **v**.

The uniqueness of the expression implies that if

$$p\mathbf{u} + q\mathbf{v} = r\mathbf{u} + s\mathbf{v} \ (= \mathbf{a})$$

then $p = r$ and $q = s$.

Example 19

*Given that (13 – h)**x** + 9**y** = (3h + 5)**x** + (2k + 3)**y** where x and y are two non-parallel vectors, find the values of h and k.*

Solution

$(13 - h)\mathbf{x} + 9\mathbf{y} = (3h + 5)\mathbf{x} + (2k + 3)\mathbf{y}$

then $\quad 13 - h = 3h + 5 \quad$ and $\quad 9 = 2k + 3$

$\therefore h = 2 \quad$ and $\quad k = 3$

Exercise 3c

1. State which of the following pairs of vectors are parallel:

(a) $\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}9\\3\end{pmatrix}$ **(b)** $\begin{pmatrix}2\\5\end{pmatrix}, \begin{pmatrix}-2\\-5\end{pmatrix}$

(c) $\begin{pmatrix}10\\4\end{pmatrix}, \begin{pmatrix}5\\4\end{pmatrix}$ **(d)** $\begin{pmatrix}12\\9\end{pmatrix}, \begin{pmatrix}8\\6\end{pmatrix}$

(e) $\begin{pmatrix}-3\\-21\end{pmatrix}, \begin{pmatrix}1\\7\end{pmatrix}$ **(f)** $\begin{pmatrix}4\\-9\end{pmatrix}, \begin{pmatrix}-4\\9\end{pmatrix}$

2. For each of the following vectors, write down 2 other vectors that are *parallel* to it, one in the *same direction*, and one in the *opposite direction.*

(a) $\begin{pmatrix}2\\1\end{pmatrix}$ **(b)** $\begin{pmatrix}-3\\-2\end{pmatrix}$ **(c)** $\begin{pmatrix}8\\-5\end{pmatrix}$

(d) $\begin{pmatrix}-7\\4\end{pmatrix}$ **(e)** $\begin{pmatrix}-\frac{1}{2}\\2\end{pmatrix}$ **(f)** $\begin{pmatrix}3\\\frac{2}{3}\end{pmatrix}$

(g) $\begin{pmatrix}-2\\-9\end{pmatrix}$ **(h)** $\begin{pmatrix}0.5\\0.8\end{pmatrix}$

3. If $\mathbf{u} = \begin{pmatrix}-2\\3\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}4\\1\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}6\\2\end{pmatrix}$, find a single column vector for each of the following:

(a) $\mathbf{u} - 2\mathbf{v}$ **(b)** $3\mathbf{u} + \mathbf{v}$

(c) $2\mathbf{u} - \mathbf{v} + 3\mathbf{w}$ **(d)** $-2\mathbf{u} + 3\mathbf{v} - \mathbf{w}$

(e) $-5\mathbf{u} - 2\mathbf{v} + \frac{1}{2}\mathbf{w}$

(f) $-4\mathbf{u} + 7\mathbf{v} - 2\mathbf{w}$

4. If $\mathbf{a} = \begin{pmatrix}5\\5\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}4\\-5\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}-4\\5\end{pmatrix}$, find the column vector for each of the following:

(a) $3\mathbf{a} - 2\mathbf{c}$ **(b)** $2\mathbf{a} - \frac{1}{3}\mathbf{c}$

(c) $2\mathbf{a} + \mathbf{b} - 3\mathbf{c}$ **(d)** $2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$

5. If $\overrightarrow{PQ} = \begin{pmatrix}6\\12\end{pmatrix}$, $\overrightarrow{RS} = \begin{pmatrix}-12\\3\end{pmatrix}$ and $\overrightarrow{TU} = \begin{pmatrix}5\\-15\end{pmatrix}$, evaluate the following:

(a) $\frac{1}{3}\overrightarrow{PQ}$ **(b)** $\overrightarrow{PQ} + 2\overrightarrow{RS}$

(c) $3\overrightarrow{RS} + 2\overrightarrow{TU} - \overrightarrow{PQ}$

(d) $\frac{2}{3}\overrightarrow{PQ} - \frac{1}{3}\overrightarrow{RS} - \frac{3}{5}\overrightarrow{TU}$

6. Study the diagram and draw vectors (on a sheet of squared paper) equal to

(a) $3\mathbf{a}$ **(b)** $1\frac{2}{3}\mathbf{b}$ **(c)** $-2\mathbf{a}$

(d) $-\frac{1}{2}\mathbf{c}$ **(e)** $-1\frac{3}{4}\mathbf{d}$ **(f)** $-\frac{1}{3}\mathbf{e}$

(g) $1\frac{1}{2}\mathbf{c}$ **(h)** $1\frac{1}{3}\mathbf{e}$

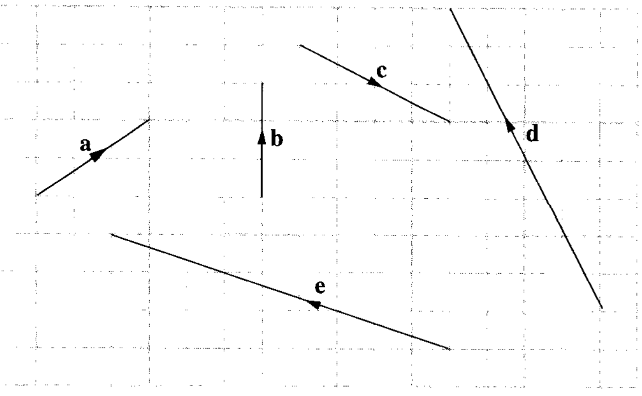


Fig. 3.48

7. Given that $\mathbf{a} = \begin{pmatrix}-1\\1\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\2\end{pmatrix}$, illustrate each of the following on a sheet of squared paper:

(a) $2\mathbf{a} + \mathbf{b}$ **(b)** $3\mathbf{a} + 2\mathbf{b}$

(c) $\mathbf{a} - 2\mathbf{b}$

8. Given that $\begin{pmatrix}8\\-6\end{pmatrix}$ and $\begin{pmatrix}2\\p\end{pmatrix}$ are parallel vectors, find the value of p.

9. Given that $\begin{pmatrix}u\\12\end{pmatrix}$ and $\begin{pmatrix}3\\-9\end{pmatrix}$ are parallel vectors, find the value of u.

10. If $\begin{pmatrix}a\\b\end{pmatrix}$ and $\begin{pmatrix}c\\d\end{pmatrix}$ are two parallel vectors, show that $ad = bc$.

11. Simplify the following:

(a) $2(\mathbf{a} + \mathbf{b}) + 3\mathbf{a} - \mathbf{b}$

(b) $\frac{1}{2}(\mathbf{u} - \mathbf{v}) + \frac{1}{3}(\mathbf{u} + \mathbf{v})$

(c) $3\left(\mathbf{u} + \frac{1}{6}\mathbf{v}\right) + 2\left(\frac{1}{2}\mathbf{u} - \frac{1}{4}\mathbf{v}\right)$

12. For each of the following, find the values of x and y.

(a) $\mathbf{a} = \begin{pmatrix}x\\y\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}3\\4\end{pmatrix}$ and $\mathbf{a} + 2\mathbf{b} = \begin{pmatrix}8\\9\end{pmatrix}$.

(b) $\mathbf{u} = \begin{pmatrix}2\\y\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}x\\2\end{pmatrix}$ and $4\mathbf{u} + \mathbf{v} = 2\begin{pmatrix}\frac{1}{2}\\9\end{pmatrix}$.

(c) $\mathbf{p} = \begin{pmatrix}x\\5\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}6\\y\end{pmatrix}$ and

$5\mathbf{p} + 2\mathbf{q} = \begin{pmatrix}3\\23\end{pmatrix}$.

13. Vectors **a**, **b** and **c** are such that $\mathbf{a} = \begin{pmatrix}8\\7\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}-12\\5\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}x\\y\end{pmatrix}$.

(a) Express $3\mathbf{a} + 2\mathbf{b}$ as a column vector.

(b) Find $\left|\mathbf{a} + \frac{1}{2}\mathbf{b}\right|$, giving your answer correct to the nearest whole number.

(c) Given that $\mathbf{a} - 3\mathbf{b} = 2\mathbf{c}$, find the value of x and of y.

14. *PQRS* is a parallelogram, where $\overrightarrow{PQ} = \begin{pmatrix}8\\0\end{pmatrix}$ and $\overrightarrow{PS} = \begin{pmatrix}2\\5\end{pmatrix}$.

(a) Find the value of $|\overrightarrow{PS}|$, giving your answer correct to two significant figures.

(b) Express $\overrightarrow{RQ}$ as a column vector.

(c) Given that $\overrightarrow{RS} = \begin{pmatrix}3-2p\\3q-2\end{pmatrix}$, find the values of p and q.

15. $\mathbf{p} = \begin{pmatrix}5\\-12\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}2\\3\end{pmatrix}$, $\mathbf{r} = \begin{pmatrix}20\\m\end{pmatrix}$.

(a) Express $2\mathbf{p} + 3\mathbf{q}$ as a column vector.

(b) Find

(i) $|\mathbf{p}|$,

(ii) $|-\mathbf{p} + 2\mathbf{q}|$, giving your answers correct to the nearest whole number.

(c) Given that **r** is parallel to **p**, write down the value of m.

16. Given that $\overrightarrow{AB} = \begin{pmatrix}-3\\5\end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix}1\\4\end{pmatrix}$ and $\overrightarrow{EF} = \begin{pmatrix}k\\7.5\end{pmatrix}$,

(a) express as a column vector $2\overrightarrow{AB} + 5\overrightarrow{CD}$;

(b) find the value of k if $\overrightarrow{EF}$ is parallel to $\overrightarrow{AB}$;

(c) find $|\overrightarrow{AB}|$, giving your answer correct to one decimal place.

17. The vector **a** has a length of 5 km and a direction on a bearing of 045°. Find

(a) **(i)** the bearing of $\frac{1}{5}\mathbf{a}$,

(ii) the length of $\frac{3}{5}\mathbf{a}$;

(b) **(i)** the bearing of $-3\mathbf{a}$,

(ii) the length of $-3\mathbf{a}$.

18. The vector **u** has a length of 5 km and a direction on a bearing of 045°. The vector **v**

has a length of 12 km and a direction on a bearing of 135°. Find

(a) **(i)** the length of ($\mathbf{u} + \mathbf{v}$),
 (ii) the bearing of ($\mathbf{u} + \mathbf{v}$);

(b) **(i)** the length of ($\mathbf{v} - \mathbf{u}$),
 (ii) the bearing of ($\mathbf{v} - \mathbf{u}$).

19. It is given that $\mathbf{u} = \begin{pmatrix} -12 \\ 5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} s \\ t \end{pmatrix}$. If $\mathbf{u} = k\mathbf{v}$ where k is a positive constant and $|\mathbf{v}| = 32.5$, find the values of k, s and t.

20. It is given that $\mathbf{p} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} a \\ b \end{pmatrix}$. If $\mathbf{p} = t\mathbf{q}$ where t is a positive constant and $|\mathbf{q}| = 20$, find the values of t, a and b.

21. Given that $(s - 2)\mathbf{x} + (3t - 4)\mathbf{y} = \left(\frac{s}{5} + 4\right)\mathbf{x} + (t + 10)\mathbf{y}$, find the values of s and t.

22. Given that $(7a - 8)\mathbf{u} + (4b + 1)\mathbf{v} = (12 - 3a)\mathbf{u} + (8 - 3b)\mathbf{v}$, find the values of a and b.

Solving Problems Involving Column Vectors

Let us look at some examples of problem solving that involve column vectors.

Example 20

If $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $\overrightarrow{RS} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$ *and* $\overrightarrow{TU} = \begin{pmatrix} 13 \\ -4 \end{pmatrix}$, *evaluate*

(a) $2\overrightarrow{PQ}$; *(b)* $\frac{1}{2}\overrightarrow{RS} + \overrightarrow{TU}$; *(c)* $3\overrightarrow{PQ} + 2\overrightarrow{RS} - \overrightarrow{TU}$.

Solution

(a) $2\overrightarrow{PQ} = 2\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix}$

(b) $$\begin{aligned} \frac{1}{2}\overrightarrow{RS} + \overrightarrow{TU} &= \frac{1}{2}\begin{pmatrix} -6 \\ 2 \end{pmatrix} + \begin{pmatrix} 13 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 13 \\ -4 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} \end{aligned}$$

(c) $$\begin{aligned} 3\overrightarrow{PQ} + 2\overrightarrow{RS} - \overrightarrow{TU} &= 3\begin{pmatrix} 3 \\ 7 \end{pmatrix} + 2\begin{pmatrix} -6 \\ 2 \end{pmatrix} - \begin{pmatrix} 13 \\ -4 \end{pmatrix} = \begin{pmatrix} 9 \\ 21 \end{pmatrix} + \begin{pmatrix} -12 \\ 4 \end{pmatrix} - \begin{pmatrix} 13 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 9 - 12 - 13 \\ 21 + 4 + 4 \end{pmatrix} = \begin{pmatrix} -16 \\ 29 \end{pmatrix} \end{aligned}$$

Example 21

Given that $\mathbf{p} = 3\mathbf{u} - 2\mathbf{v}$ *and* $\mathbf{q} = 5\mathbf{u} + 4\mathbf{v}$, *calculate in terms of* $\mathbf{u}$ *and* $\mathbf{v}$,

(a) $\mathbf{p} + \mathbf{q}$, *(b)* $3\mathbf{p} - 2\mathbf{q}$.

Solution

(a) $$\begin{aligned} \mathbf{p} + \mathbf{q} &= 3\mathbf{u} - 2\mathbf{v} + 5\mathbf{u} + 4\mathbf{v} \\ &= 8\mathbf{u} + 2\mathbf{v} \end{aligned}$$

(b) $$\begin{aligned} 3\mathbf{p} - 2\mathbf{q} &= 3(3\mathbf{u} - 2\mathbf{v}) - 2(5\mathbf{u} + 4\mathbf{v}) \\ &= 9\mathbf{u} - 6\mathbf{v} - 10\mathbf{u} - 8\mathbf{v} \\ &= -\mathbf{u} - 14\mathbf{v} \end{aligned}$$

Example 22

Given that $\mathbf{a} = \begin{pmatrix} x \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ y \end{pmatrix}$ *and* $3\mathbf{a} - 2\mathbf{b} = 4\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, *find the values of x and y.*

Solution

$$3\mathbf{a} - 2\mathbf{b} = 4\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$3\begin{pmatrix} x \\ 4 \end{pmatrix} - 2\begin{pmatrix} 3 \\ y \end{pmatrix} = 4\begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} 3x - 6 \\ 12 - 2y \end{pmatrix} = \begin{pmatrix} -4 \\ 12 \end{pmatrix}$$

$$\therefore \quad 3x - 6 = -4 \quad \text{and} \quad 12 - 2y = 12$$

$$x = \frac{2}{3} \quad \text{and} \quad y = 0$$

Example 23

In the diagram, $\overrightarrow{DX} = 2\overrightarrow{XB}$, $\overrightarrow{AD} = \mathbf{a}$, $\overrightarrow{BC} = 3\mathbf{a}$ *and* $\overrightarrow{BA} = \mathbf{b}$. *Find the following in terms of* **a** *and* **b**.

(a) $\overrightarrow{BD}$ *(b)* $\overrightarrow{XD}$ *(c)* $\overrightarrow{CD}$

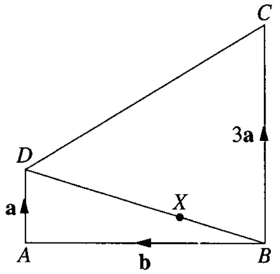


Fig. 3.50

Solution

(a) $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$
$= \mathbf{b} + \mathbf{a}$ or $\mathbf{a} + \mathbf{b}$

(b) $\overrightarrow{XD} = \frac{2}{3}\overrightarrow{BD}$
$= \frac{2}{3}(\mathbf{a} + \mathbf{b})$

(c) $\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD}$
$= -3\mathbf{a} + \mathbf{a} + \mathbf{b}$
$= \mathbf{b} - 2\mathbf{a}$

Example 24

In the diagram, SPQR is a parallelogram. The point U, on SR, is such that $SU = \frac{2}{5}SR$. *The lines PS and QU, when produced, meet at T. Given that* $\overrightarrow{PQ} = 10\mathbf{a}$ *and* $\overrightarrow{PS} = 5\mathbf{b}$, *express the following in terms of* **a** *and/or* **b**.

(a) $\overrightarrow{PR}$ *(b)* $\overrightarrow{UR}$

(c) $\overrightarrow{TU}$

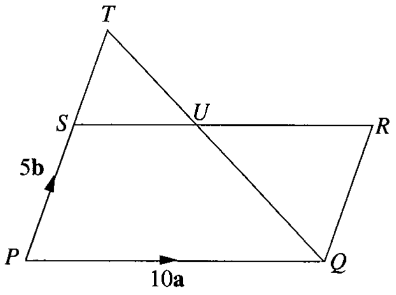


Fig. 3.51

Solution

(a) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PQ} + \overrightarrow{PS}$

$= 10\mathbf{a} + 5\mathbf{b} = 5(2\mathbf{a} + \mathbf{b})$

(b) $SU = \frac{2}{5}SR \Rightarrow \overrightarrow{UR} = \frac{3}{5}\overrightarrow{SR}$

i.e. $\overrightarrow{UR} = \frac{3}{5}\overrightarrow{PQ} = \frac{3}{5}(10\mathbf{a}) = 6\mathbf{a}$

$\overrightarrow{PQ} = \overrightarrow{SR}$

$PQ = SR$

(c) $\triangle TSU$ and $\triangle TPQ$ are similar.

$\frac{TU}{TQ} = \frac{SU}{PQ} = \frac{2}{5}$

$\therefore \quad TU = \frac{2}{3}\overrightarrow{UQ} = \frac{2}{3}(\overrightarrow{UR} + \overrightarrow{RQ})$

$= \frac{2}{3}(6\mathbf{a} - 5\mathbf{b})$

Example 25

PQRS is a parallelogram and M is the midpoint of SR. PS is produced to T so that PS = ST. PM is produced to N and RT = 3RN. $\overrightarrow{PQ} = 2\boldsymbol{a}$ and $\overrightarrow{QR} = 2\boldsymbol{b}$.

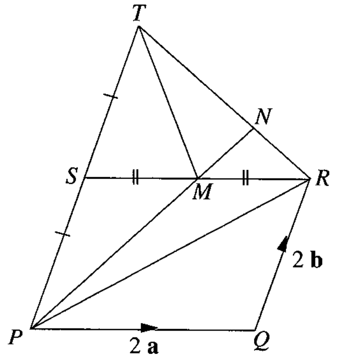


Fig. 3.52

*(a) Express, as simply as possible, in terms of **a** and/or **b**,*

(i) $\overrightarrow{ST}$; *(ii)* $\overrightarrow{TR}$;

(iii) $\overrightarrow{TN}$.

(b) Show that $\overrightarrow{PN} = \frac{4}{3}(\boldsymbol{a} + 2\boldsymbol{b})$.

(c) Express $\overrightarrow{PM}$ *as simply as possible, in terms of **a** and **b**.*

(d) Calculate the value of *(i)* $\frac{PM}{PN}$; *(ii)* $\frac{\text{the area of } \triangle MTP}{\text{the area of } \triangle MTN}$.

Solution

(a) (i) $\overrightarrow{ST} = \overrightarrow{QR} = 2\mathbf{b}$

(ii) $\overrightarrow{SR} = \overrightarrow{PQ} = 2\mathbf{a}$

$\overrightarrow{TR} = \overrightarrow{TS} + \overrightarrow{SR}$

$= -2\mathbf{b} + 2\mathbf{a}$

$= 2(\mathbf{a} - \mathbf{b})$

(iii) $\overrightarrow{TN} = \frac{2}{3}\overrightarrow{TR}$

$= \frac{4}{3}(\mathbf{a} - \mathbf{b})$

(b) $\overrightarrow{PT} = 2\,\overrightarrow{ST} = 2(2\mathbf{b}) = 4\mathbf{b}$

$\overrightarrow{PN} = \overrightarrow{PT} + \overrightarrow{TN} = 4\mathbf{b} + \frac{4}{3}\mathbf{a} - \frac{4}{3}\mathbf{b} = \frac{4}{3}\mathbf{a} + \frac{8}{3}\mathbf{b} = \frac{4}{3}(\mathbf{a} + 2\mathbf{b})$

(c) $\overrightarrow{PM} = \overrightarrow{PS} + \overrightarrow{SM} = 2\mathbf{b} + \mathbf{a} = \mathbf{a} + 2\mathbf{b}$

(d) (i) $\overrightarrow{PN} = \frac{4}{3}\overrightarrow{PM}$

$\therefore \quad \frac{PM}{PN} = \frac{3}{4}$

(ii) $\frac{PM}{PN} = \frac{3}{4} \Rightarrow \frac{PM}{MN} = \frac{3}{1} = 3$

$\frac{\text{the area of } \triangle MTP}{\text{the area of } \triangle MTN} = \frac{\frac{1}{2}PM \times ht}{\frac{1}{2}MN \times ht} = \frac{PM}{MN} = 3$

Exercise 3d

1. In the diagram, if $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, and M and N are the mid-points of AB and AC respectively, find in terms of $\mathbf{u}$ and $\mathbf{v}$,

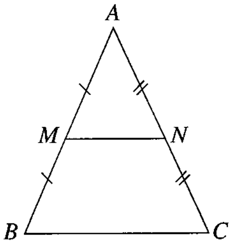


Fig. 3.53

(a) $\overrightarrow{BC}$, (b) $\overrightarrow{AM}$,

(c) $\overrightarrow{AN}$, (d) $\overrightarrow{MN}$.

What can you say about $\overrightarrow{BC}$ and $\overrightarrow{MN}$?

2. Fig. 3.54 shows the vectors $\mathbf{a}$ and $\mathbf{b}$. Express each of the following vectors in terms of $\mathbf{a}$ and/or $\mathbf{b}$.

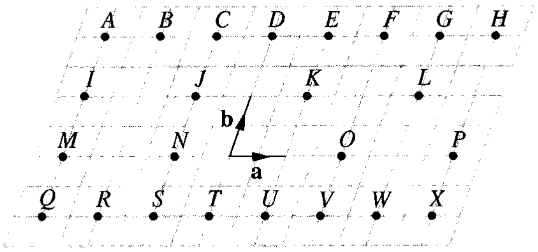


Fig. 3.54

(a) $\overrightarrow{KL}$ (b) $\overrightarrow{KE}$ (c) $\overrightarrow{LW}$

(d) $\overrightarrow{DA}$ (e) $\overrightarrow{OL}$ (f) $\overrightarrow{MJ}$

(g) $\overrightarrow{CK}$ (h) $\overrightarrow{MC}$ (i) $\overrightarrow{OE}$

(j) $\overrightarrow{MT}$ (k) $\overrightarrow{SO}$ (l) $\overrightarrow{HV}$

(m) $\overrightarrow{SF}$ (n) $\overrightarrow{KS}$ (o) $\overrightarrow{NH}$

3. In the diagram, D is a point on BC such that $BD = 3DC$. Given that $\overrightarrow{BA} = \mathbf{p}$ and $\overrightarrow{BD} = \mathbf{q}$, express in terms of $\mathbf{p}$ and/or $\mathbf{q}$,

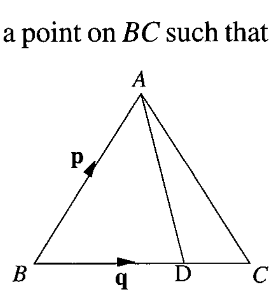


Fig. 3.55

(a) $\overrightarrow{BC}$; (b) $\overrightarrow{AD}$; (c) $\overrightarrow{CA}$.

4. In the diagram, $PQRS$ is a parallelogram. M is the mid-point of PQ and N is on SR such that $SR = 3SN$. Given that $\overrightarrow{PS} = \mathbf{a}$ and $\overrightarrow{PM} = 2\mathbf{b}$, express in terms of $\mathbf{a}$ and $\mathbf{b}$,

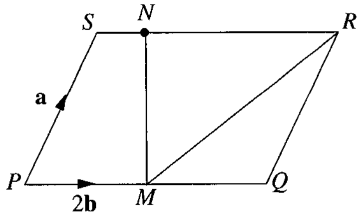


Fig. 3.56

(a) $\overrightarrow{MR}$; (b) $\overrightarrow{RN}$; (c) $\overrightarrow{NM}$.

5. $ABCD$ is a parallelogram with M as the mid-point of BC. If $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AD} = \mathbf{q}$, express in terms of $\mathbf{p}$ and/or $\mathbf{q}$,

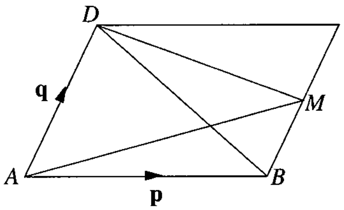


Fig. 3.57

(a) $\overrightarrow{CM}$; (b) $\overrightarrow{DB}$; (c) $\overrightarrow{AM}$;

(d) $\overrightarrow{MD}$.

6. In the diagram, $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, $\overrightarrow{CD} = \frac{3}{2}\mathbf{u}$, and $\overrightarrow{BE} = \frac{2}{5}\overrightarrow{BC}$. Express in terms of $\mathbf{u}$ and $\mathbf{v}$,

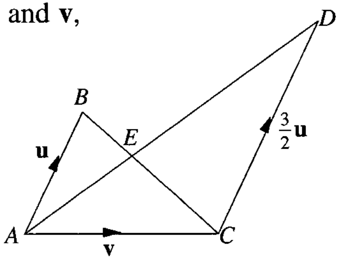


Fig. 3.58

(a) $\overrightarrow{BC}$; (b) $\overrightarrow{BE}$; (c) $\overrightarrow{AD}$;

(d) $\overrightarrow{AE}$; (e) $\overrightarrow{BD}$.

7. In the diagram, $ABCD$ is a trapezium with $AD \parallel BC$ and $AD = \frac{2}{3}BC$. P and Q are points on AB and DC respectively such that P is the mid-point of AB and $DQ = \frac{1}{4}\overrightarrow{DC}$. Given that $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{AD} = \mathbf{v}$, express in terms of **a** and **b**,

(a) $\overrightarrow{AC}$;

(b) $\overrightarrow{DC}$;

(c) $\overrightarrow{AQ}$;

(d) $\overrightarrow{PQ}$.

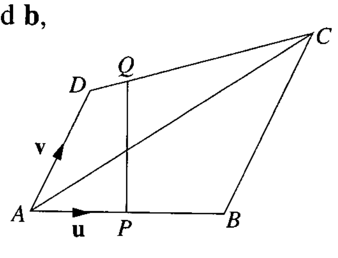


Fig. 3.59

8. $PQRS$ is a parallelogram. $\overrightarrow{BQ} = 2\overrightarrow{RB}$, $\overrightarrow{AR} = \frac{1}{3}\overrightarrow{SR}$, $\overrightarrow{PS} = \mathbf{a}$ and $\overrightarrow{PQ} = \mathbf{b}$.

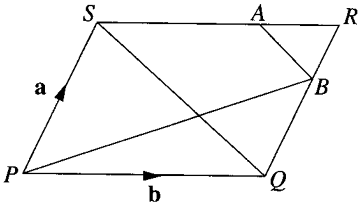


Fig. 3.60

(a) Express in terms of **a** and/or **b**,

(i) $\overrightarrow{SA}$; **(ii)** $\overrightarrow{QB}$; **(iii)** $\overrightarrow{PB}$;

(iv) $\overrightarrow{QS}$; **(v)** $\overrightarrow{BA}$.

(b) Calculate the value of

(i) $\frac{BA}{QS}$; **(ii)** $\frac{\text{the area of } \triangle ABR}{\text{the area of } PQRS}$.

9. In the diagram, T is the point of intersection of the diagonals of the quadrilateral $PQRS$. $\overrightarrow{PR} = 3\overrightarrow{PT}$, $\overrightarrow{PS} = 5\mathbf{b}$, $\overrightarrow{PQ} = 4\mathbf{a} + \mathbf{b}$ and $\overrightarrow{PR} = 3\mathbf{a} + 12\mathbf{b}$.

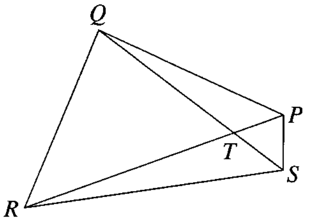


Fig. 3.61

(a) Express, as simply as possible, in terms of **a** and **b**,

(i) $\overrightarrow{RS}$; **(ii)** $\overrightarrow{RT}$; **(iii)** $\overrightarrow{RQ}$.

(b) Show that $\overrightarrow{QT} = 3(\mathbf{b} - \mathbf{a})$.

(c) Express $\overrightarrow{QS}$ as simply as possible, in terms of **a** and **b**.

(d) Calculate the value of

(i) $\frac{QT}{QS}$; **(ii)** $\frac{\text{the area of } \triangle PQT}{\text{the area of } \triangle PQS}$;

(iii) $\frac{\text{the area of } \triangle PQT}{\text{the area of } \triangle PQR}$.

10. In triangle PQR, the point N on PR is such that $PN = \frac{2}{3}PR$. M is the mid-point of PQ, L is the mid-point of MN, and PL produced meets RQ at K. $4\overrightarrow{RK} = 3\overrightarrow{KQ}$, $\overrightarrow{PL} = \frac{7}{12}\overrightarrow{PK}$, $\overrightarrow{PN} = 2\mathbf{a}$ and $\overrightarrow{PM} = 2\mathbf{b}$.

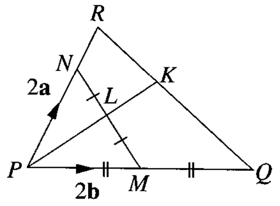


Fig. 3.62

(a) Express, as simply as possible, in terms of **a** and/or **b**,

(i) $\overrightarrow{NM}$; **(ii)** $\overrightarrow{NL}$; **(iii)** $\overrightarrow{PK}$;

(iv) $\overrightarrow{PR}$; **(v)** $\overrightarrow{PQ}$.

(b) Express $\overrightarrow{RQ}$ as simply as possible, in terms of **a** and **b**.

(c) Calculate the value of $\frac{KR}{QR}$.

(d) Show that $\overrightarrow{KR} = \frac{3}{7}(3\mathbf{a} - 4\mathbf{b})$.

(e) Calculate the value of

(i) $\frac{\text{the area of } \triangle PKR}{\text{the area of } \triangle PQR}$;

(ii) $\frac{\text{the area of } \triangle PKN}{\text{the area of } \triangle PQR}$.

Position Vectors

When we are considering the location of points in a plane, it is useful to refer to their positions relative to a fixed point O.

In Fig. 3.63, the displacement vector $\overrightarrow{OA}$ describes the position of A relative to the origin O. $\overrightarrow{OA}$ is called the **position vector** of A relative to O.

Similarly, $\overrightarrow{OB}$ is the position vector of B relative to O.

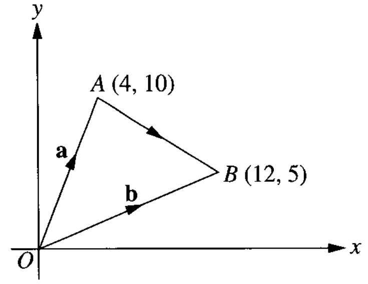


Fig. 3.63

For your information

In general, in the cartesian plane, we take the position vector of any point as being relative to the origin O. Also, if P is the point with coordinates (x, y), then $\overrightarrow{OP} = \mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$; *conversely, if* $\overrightarrow{OP} = \mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$, *then the coordinates of the point P is (x, y).*

We often denote $\overrightarrow{OA}$ by **a**, $\overrightarrow{OB}$ by **b** and so on.

The vector $\overrightarrow{AB}$ can be expressed in terms of position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as follows:

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \quad \text{(See Fig. 3.63)}$$

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$$

In Fig. 3.63, the coordinates of A and B are (4, 10) and (12, 5) respectively.

$$\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix} 4 \\ 10 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \mathbf{b} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

Example 26

In Fig. 3.64, the coordinates of P, Q and R are (1, 2), (7, 3) and (4, 7) respectively. Find the coordinates of S if PQSR is a parallelogram.

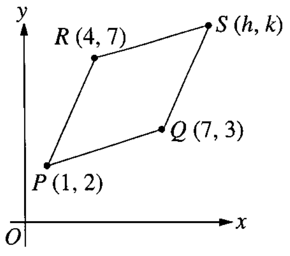


Fig. 3.64

Solution

Let the coordinates of S be (h, k).

$$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}, \overrightarrow{OR} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \text{ and } \overrightarrow{OS} = \begin{pmatrix} h \\ k \end{pmatrix}.$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}7\\3\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}6\\1\end{pmatrix}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = \begin{pmatrix}h\\k\end{pmatrix} - \begin{pmatrix}4\\7\end{pmatrix} = \begin{pmatrix}h-4\\k-7\end{pmatrix}.$$

Since $PQRS$ is a parallelogram, $\overrightarrow{PQ} = \overrightarrow{RS}$.

$$\therefore \quad \begin{pmatrix}h-4\\k-7\end{pmatrix} = \begin{pmatrix}6\\1\end{pmatrix}$$

i.e. $h - 4 = 6$ and $k - 7 = 1$
or $h = 10$ and $k = 8$

$$\overrightarrow{OS} = \begin{pmatrix}10\\8\end{pmatrix}$$

$\therefore$ the coordinates of S are (10, 8).

Example 27

$\overrightarrow{PQ} = \begin{pmatrix}-12\\16\end{pmatrix}$ *and* $\overrightarrow{RS} = \frac{5}{4}\overrightarrow{PQ}$.

(a) *Express* $\overrightarrow{RS}$ *as a column vector.*
(b) *Given that P is the point (10, –8), find the coordinates of the point Q.*
(c) *Given that S is the point (2, 5), find the coordinates of the point R.*

Solution

(a) $\overrightarrow{RS} = \frac{5}{4}\begin{pmatrix}-12\\16\end{pmatrix} = \begin{pmatrix}-15\\20\end{pmatrix}$

(b) P is the point (10, –8).

$$\therefore \quad \overrightarrow{OP} = \begin{pmatrix}10\\-8\end{pmatrix}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\overrightarrow{OQ} = \overrightarrow{PQ} + \overrightarrow{OP} = \begin{pmatrix}-12\\16\end{pmatrix} + \begin{pmatrix}10\\-8\end{pmatrix} = \begin{pmatrix}-2\\8\end{pmatrix}$$

$\therefore$ the coordinates of the point Q are (–2, 8).

(c) S is the point (2, 5).

$$\therefore \quad \overrightarrow{OS} = \begin{pmatrix}2\\5\end{pmatrix}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$\overrightarrow{OR} = \overrightarrow{OS} - \overrightarrow{RS} = \begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}-15\\20\end{pmatrix} = \begin{pmatrix}17\\-15\end{pmatrix}$$

$\therefore$ the coordinates of the point R are (17, –15).

Exercise 3e

1. Write down the position vectors of the following points as column vectors.

(a) $A(4, 7)$ **(b)** $B(-2, 5)$
(c) $C(6, -1)$ **(d)** $D(-4, -9)$

2. If P, Q and R are the points $(3, -2)$, $(2, -4)$ and $(2, 3)$ respectively, express the following as column vectors.

(a) $\overrightarrow{PQ}$ **(b)** $\overrightarrow{QR}$ **(c)** $\overrightarrow{RP}$

3. If L, M and N are the points $(2, 2)$, $(4, 7)$ and $(8, 1)$ respectively, express the following as column vectors.

(a) $\overrightarrow{NM}$ **(b)** $\overrightarrow{LM}$ **(c)** $\overrightarrow{LN}$

4. P is the point $(-1, 3)$, $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$.

(a) Find the coordinates of Q and R.
(b) Find the gradient of QR and the vector $\overrightarrow{QR}$.

5. A, B, C and D are four points such that A is $(-5, 3)$, C is $(7, 4)$, $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$ and $\overrightarrow{AD} = \begin{pmatrix} 8 \\ -9 \end{pmatrix}$. Find

(a) the coordinates of B and D,
(b) the vectors $\overrightarrow{BC}$ and $\overrightarrow{CD}$.

6. P, Q, R and S are four points such that R is $(-3, -5)$, $\overrightarrow{RS} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$, $\overrightarrow{RP} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ and $\overrightarrow{RQ} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$. Find the coordinates of P, Q and S.

7. The coordinates of P, Q and R are $(1, 0)$, $(4, 2)$ and $(5, 4)$ respectively. Use a vector method to determine the coordinates of S if

(a) $PQRS$ is a parallelogram,
(b) $PRQS$ is a parallelogram.

8. Relative to an origin O which is not shown in the diagram, P is the point $(1, 11)$, Q is the point $(2, 8)$, R is the point $(-1, 7)$, S is the point $(-2, 8)$ and T is the point $(-4, 6)$. The lines PQ and SR produced intersect at U.

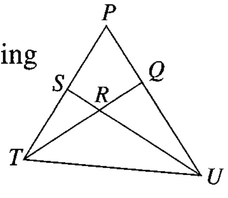


Fig. 3.65

(a) Express the following as column vectors.
  (i) $\overrightarrow{PQ}$ **(ii)** $\overrightarrow{SR}$
  (iii) $\overrightarrow{RQ}$ **(iv)** $\overrightarrow{TQ}$
(b) Find the numerical value of the ratio $\frac{RQ}{TQ}$.

9. $\overrightarrow{AB} = \begin{pmatrix} 9 \\ -15 \end{pmatrix}$ and $\overrightarrow{CD} = \frac{2}{3}\overrightarrow{AB}$.

(a) Express $\overrightarrow{CD}$ as a column vector.
(b) Given that A is the point $(-2, 7)$, find the coordinates of the point B.
(c) Given that D is the point $(8, -5)$, find the coordinates of the point C.

10. Two points A and B have position vectors **a** and **b** respectively, relative to the origin O. Given that A is the point $(7, 4)$ and $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find

(a) **b**,
(b) the coordinates of the point C, such that $\overrightarrow{OC} = \overrightarrow{BA}$.

11. L is the point $(-3, 2)$ and M is $(t, 6)$.

(a) Express $\overrightarrow{LM}$ as a column vector.
(b) If $\overrightarrow{LM}$ is parallel to $\mathbf{p} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$, find the value of t.
(c) If instead, $|\overrightarrow{LM}| = |\mathbf{p}|$, find the two possible values of t.

12. Given that A is the point $(1, 2)$, $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and that M is the mid-point of $\overrightarrow{BC}$, find

(a) $\overrightarrow{BC}$; **(b)** $\overrightarrow{AM}$;
(c) the coordinates of the point D such that $ABDC$ is a parallelogram.

Solving Problems Involving Position Vectors

Let us look at some examples of problem solving that involve position vector.

Example 28

*The point X in Fig. 3.66 lies on the straight line AC such that $AX = \frac{1}{4}AC$. With respect to the origin O, the position vector of A is 16**p** and the position vector of C is 16**q**. Express the following in terms of **p** and **q**.*

(a) $\overrightarrow{AC}$ *(b)* $\overrightarrow{AX}$ *(c)* $\overrightarrow{OX}$

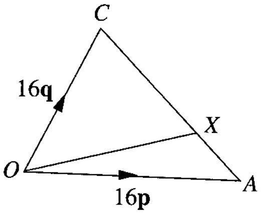


Fig. 3.66

Solution

(a) $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= 16\mathbf{q} - 16\mathbf{p}$
$= 16(\mathbf{q} - \mathbf{p})$

(b) $AX = \frac{1}{4}AC \Rightarrow \overrightarrow{AX} = \frac{1}{4}\overrightarrow{AC}$
$\therefore \quad \overrightarrow{AX} = \frac{1}{4} \times 16(\mathbf{q} - \mathbf{p})$
$= 4(\mathbf{q} - \mathbf{p})$

(c) $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$
$= 16\mathbf{p} + 4\mathbf{q} - 4\mathbf{p}$
$= 12\mathbf{p} + 4\mathbf{q}$
$= 4(3\mathbf{p} + \mathbf{q})$

Example 29

*In Fig. 3.67, OABC is a parallelogram. X is a point on AB such that AX : XB = 1 : 2 and Y is the mid-point of BC. $\overrightarrow{OA}$ = 6**p** and $\overrightarrow{OC}$ = 6**q**. Express the following in terms of **p** and/or **q**.*

(a) $\overrightarrow{AX}$ *(b)* $\overrightarrow{OX}$ *(c)* $\overrightarrow{OY}$
(d) $\overrightarrow{XY}$ *(e)* $\overrightarrow{AY}$

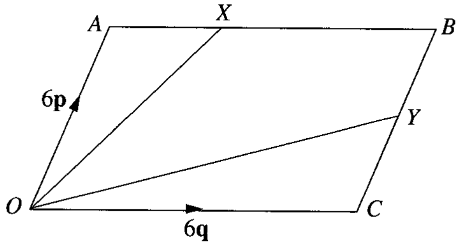


Fig. 3.67

Solution

(a) $AX : XB = 1 : 2 \Rightarrow AX = \frac{1}{3}AB = \frac{1}{3}OC$ (*OABC* is a parallelogram)
$\therefore \quad \overrightarrow{AX} = \frac{1}{3}\overrightarrow{OC}$
$= \frac{1}{3}(6\mathbf{q})$
$= 2\mathbf{q}$

(b) $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = 6\mathbf{p} + 2\mathbf{q}$

$= 2(3\mathbf{p} + \mathbf{q})$

(c) $\overrightarrow{OY} = \overrightarrow{OC} + \overrightarrow{CY} = 6\mathbf{q} + \frac{1}{2}(6\mathbf{p})$ $\left(\overrightarrow{CY} = \frac{1}{2}\overrightarrow{CB} = \frac{1}{2}\overrightarrow{OA}\right)$

$= 6\mathbf{q} + 3\mathbf{p}$

$= 3(2\mathbf{q} + \mathbf{p})$

(d) $\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX}$

$= 6\mathbf{q} + 3\mathbf{p} - 6\mathbf{p} - 2\mathbf{q}$

$= 4\mathbf{q} - 3\mathbf{p}$

(e) $\overrightarrow{AY} = \overrightarrow{OY} - \overrightarrow{OA}$

$= 6\mathbf{q} + 3\mathbf{p} - 6\mathbf{p}$

$= 6\mathbf{q} - 3\mathbf{p}$

$= 3(2\mathbf{q} - \mathbf{p})$

Exercise 3f

1. In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and M is the mid-point of OA. Write $\overrightarrow{BM}$ in terms of **a** and **b**.

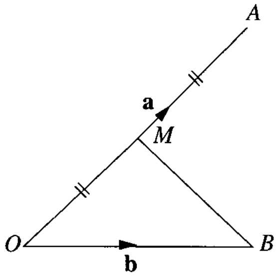


Fig. 3.68

2. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, and $\overrightarrow{AC} = \frac{2}{3}\overrightarrow{CB}$. Find $\overrightarrow{OC}$ in terms of **a** and **b**.

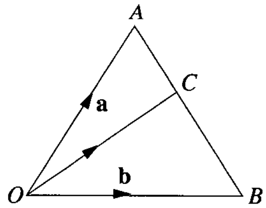


Fig. 3.69

3. The position vectors of three points A, B and C, relative to the origin O, are $7\mathbf{p} - 3\mathbf{q}$, $-2\mathbf{p} + \mathbf{q}$ and $4\mathbf{p} - 5\mathbf{q}$ respectively.

Express in terms of **p** and **q**,

(a) $\overrightarrow{AB}$; **(b)** $\overrightarrow{BC}$; **(c)** $\overrightarrow{CA}$.

4. In the diagram, M is the mid-point of OA and $BP = 3PM$. Given that the position vectors of A and B relative to O are **a** and **b**, find the position vector of P relative to O.

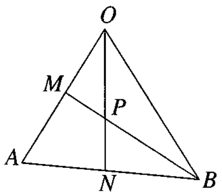


Fig. 3.70

✶**5.** In Fig. 3.71, $\overrightarrow{OU}$ and $\overrightarrow{OV}$ represent the vectors $15\mathbf{u}$ and $15\mathbf{v}$ respectively. $\overrightarrow{OA} = \frac{1}{3}\overrightarrow{OU}$ and $\overrightarrow{OB} = \frac{1}{3}\overrightarrow{OV}$. Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{UV}$ in terms of **u** and **v**. Given that $\overrightarrow{AX} = \frac{1}{4}\overrightarrow{AV}$, express the vectors $\overrightarrow{VA}$, $\overrightarrow{UX}$ and $\overrightarrow{XB}$ in terms of **u** and **v**.

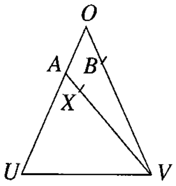


Fig. 3.71

✶**6.** $OPQR$ is a parallelogram. The point A on PR is such that $\overrightarrow{AR} = \frac{3}{4}\overrightarrow{PR}$. The point B on PQ is such that $\overrightarrow{PB} = \frac{1}{3}\overrightarrow{PQ}$. Given that $\overrightarrow{OP} = 15\mathbf{a}$ and $\overrightarrow{OR} = 15\mathbf{b}$, express the following vectors in terms of **a** and **b**.

(a) $\overrightarrow{PR}$

(b) $\overrightarrow{PA}$

(c) $\overrightarrow{OA}$

(d) $\overrightarrow{OB}$

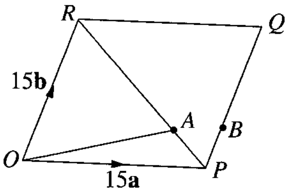


Fig. 3.72

★7. $OPQR$ is a parallelogram and X is the mid-point of QR. OR is produced to S so that $\overrightarrow{OR} = \frac{1}{2}\overrightarrow{RS}$. Given that $\overrightarrow{OP} = 4\mathbf{a}$ and $\overrightarrow{PQ} = 4\mathbf{b}$, express the following vectors in terms of **a** and **b**, giving your answers in the simplest form.

(a) $\overrightarrow{OQ}$

(b) $\overrightarrow{OX}$

(c) $\overrightarrow{QS}$

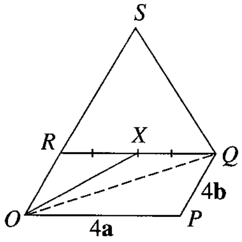


Fig. 3.73

★8. OPA and OQC are straight lines and PC intersects QA at B. Given that $\overrightarrow{OQ} = \frac{2}{3}\overrightarrow{QC}$, $\frac{PB}{BC} = \frac{1}{3}$, $\overrightarrow{OP} = 8\mathbf{p}$ and $\overrightarrow{OQ} = 8\mathbf{q}$, express the following vectors as simply as possible in terms of **p** and **q**.

(a) $\overrightarrow{PC}$

(b) $\overrightarrow{PB}$

(c) $\overrightarrow{OB}$

(d) $\overrightarrow{QB}$

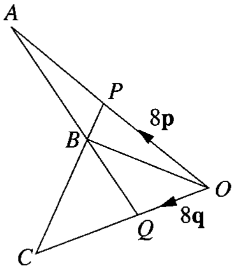


Fig. 3.74

9. $OABC$ is a parallelogram and ACT is a straight line. OC is produced to meet BT at R. $BT = 4BR$, $\overrightarrow{OA} = \mathbf{p}$, $\overrightarrow{OC} = \mathbf{q}$ and $\overrightarrow{TC} = 3(\mathbf{p} - \mathbf{q})$.

(a) Express, as simply as possible, in terms of **p** and **q**,

(i) $\overrightarrow{OT}$;

(ii) $\overrightarrow{AT}$;

(iii) $\overrightarrow{OB}$;

(iv) $\overrightarrow{BT}$;

(v) $\overrightarrow{TR}$.

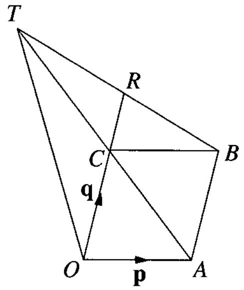


Fig. 3.75

(b) Show that $\overrightarrow{CR} = \frac{3}{4}\mathbf{q}$.

(c) Find the value of

(i) $\frac{CR}{OC}$; (ii) $\frac{\text{the area of } \triangle TCR}{\text{the area of } \triangle TAB}$.

10. In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OP} = 2\mathbf{a}$, $OQ : QB = 2 : 1$, and M is the mid-point of AB.

(a) Express, as simply as possible, in terms of **a** and/or **b**,

(i) $\overrightarrow{OQ}$;

(ii) $\overrightarrow{PQ}$;

(iii) $\overrightarrow{OM}$;

(iv) $\overrightarrow{QM}$.

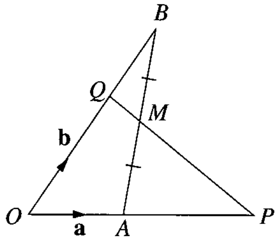


Fig. 3.76

(b) Find the value of $\frac{PM}{MQ}$.

Summary

1. A scalar has magnitude alone whereas a vector possesses both **magnitude** and **direction**.
2. Two vectors are equal when they have the same direction (parallel) and magnitude.
3. The sum of two vectors, **a** and **b**, can be determined by drawing, using the triangle law or parallelogram law.

4. The magnitude of a column vector $\mathbf{a} = \begin{pmatrix} u \\ v \end{pmatrix}$ is given by

$$|\mathbf{a}| = \sqrt{u^2 + v^2}.$$

5. For any two column vectors $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} r \\ s \end{pmatrix}$,

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p + r \\ q + s \end{pmatrix},$$

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p - r \\ q - s \end{pmatrix}.$$

6. $\mathbf{a} = k\mathbf{u}$, where k is a scalar, is called the **scalar multiple** of **u**.
   a and **u** are parallel vectors and $|\mathbf{a}| = |k|\,|\mathbf{u}|$.

7. A given vector **a** can be expressed uniquely in terms of two **non-parallel vectors u** and **v** (lying in the same plane) in the form

$$\mathbf{a} = k\mathbf{u} + h\mathbf{v}, \text{ where } k \text{ and } h \text{ are scalars.}$$

8. A **position vector** defines the position of one point with respect to an origin. In the cartesian plane, we take the origin to be the point O (0, 0).

Review Questions 3

1. Copy the vectors **a** and **b** which are represented by $\overrightarrow{AB}$ and $\overrightarrow{XY}$ respectively in the given figure. Construct on a sheet of squared paper representations of
   - (a) $\mathbf{a} + \mathbf{b}$,
   - (b) $2(\mathbf{a} + \mathbf{b})$,
   - (c) $\mathbf{a} - \mathbf{b}$,
   - (d) $\frac{2}{3}\mathbf{a} + 2\mathbf{b}$,
   - (e) $-(\mathbf{a} + \mathbf{b})$,
   - (f) $\mathbf{a} + 2\mathbf{b}$.

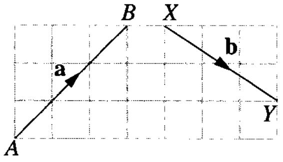


Fig. 3.77

   Express your answers as column vectors.

2. (a) Refer to Fig. 3.78 and write down the column vector representing the following.
   - (i) $\overrightarrow{PQ}$
   - (ii) $\overrightarrow{PR}$
   - (iii) $\overrightarrow{RQ}$

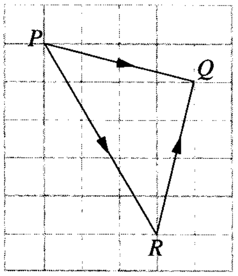


Fig. 3.78

   (b) If S is a point such that $PQRS$ is a square, write down $\overrightarrow{SQ}$ in column vector form.

   (c) Find the length of
   - (i) a side of the square $PQRS$;
   - (ii) the diagonal of the square $PQRS$.

3. Given that $\overrightarrow{OA} = \begin{pmatrix} 8 \\ a \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} a \\ 2 \end{pmatrix}$, where $a \neq 0$, find the possible values of a in each of the following cases.
   - (a) $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are parallel
   - (b) $|\overrightarrow{OB}| = |\overrightarrow{AB}|$

★4. It is given that $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$.
   (a) Express as a column vector and find the magnitude of each of the following.
   (i) $\overrightarrow{AB}$ (ii) $\overrightarrow{AC}$ (iii) $\overrightarrow{BC}$

(b) Hence state the special property of $\triangle ABC$ and find its area.

✶5. Given three points A, B and C with position vectors $\overrightarrow{OA} = 2\mathbf{p} + \mathbf{q}$, $\overrightarrow{OB} = 2k\mathbf{p} + \mathbf{q}$ and $\overrightarrow{OC} = 12\mathbf{p} + 4\mathbf{q}$, where k is a constant, express the following in terms of $\mathbf{p}$, $\mathbf{q}$ and/or k.

(a) $\overrightarrow{AB}$ **(b)** $\overrightarrow{AC}$

✶6. The position vectors of P, Q and R relative to the origin O are $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively. Given that $\mathbf{p} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$, find $\mathbf{q}$ and $|PQ|$. Given also that $\mathbf{r} = \begin{pmatrix} 22 \\ -11 \end{pmatrix}$ and $\lambda\mathbf{p} + \mu\mathbf{q} = \mathbf{r}$, write down two simultaneous equations in λ and μ and solve them.

✶7. Given that $\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $|\overrightarrow{AB}| = 10$, find the value of a and of b where $b > 0$ such that $\overrightarrow{AB}$ is parallel to $\overrightarrow{PQ} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

✶8. In the figure, $\overrightarrow{OA} = 3\mathbf{a}$, $\overrightarrow{OC} = \mathbf{b}$ and $\overrightarrow{CB} = 3\mathbf{b}$. D is a point on AC such that $\frac{CD}{CA} = \frac{1}{3}$ and E is a point on AB such that $\frac{AE}{AB} = \frac{1}{3}$. Express the following in terms of $\mathbf{a}$ and $\mathbf{b}$.

(a) $\overrightarrow{AB}$

(b) $\overrightarrow{AC}$

(c) $\overrightarrow{OD}$

(d) $\overrightarrow{OE}$

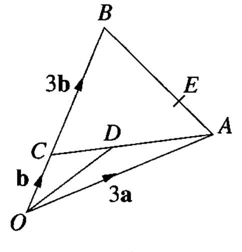


Fig. 3.79

9. In Fig. 3.80, $OABC$ is a parallelogram whose diagonals meet at D. M is the mid-point of BC. Given that $\overrightarrow{OA} = 5\mathbf{p} - \mathbf{q}$ and $\overrightarrow{OC} = \mathbf{p} + 3\mathbf{q}$, express the following in terms of $\mathbf{p}$ and $\mathbf{q}$.

(a) $\overrightarrow{OD}$

(b) $\overrightarrow{AC}$

(c) $\overrightarrow{AM}$

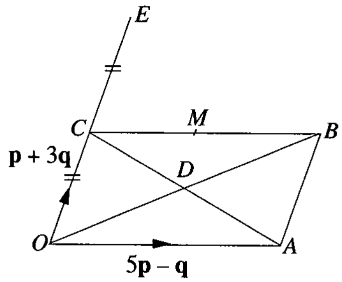


Fig. 3.80

(d) Express $\overrightarrow{AE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$ if OC is produced to E such that $OC = CE$.

10. (a) It is given that $\mathbf{p} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} a \\ b \end{pmatrix}$.

(i) Find $|\mathbf{q}|$.

(ii) Express $2\mathbf{p} + 3\mathbf{q}$ as a column vector.

(iii) Given that $\mathbf{p} - \mathbf{q} = 2\mathbf{r}$, find the value of a and the value of b.

(b) $ABCD$ is a quadrilateral in which $\overrightarrow{AB} = 2\mathbf{s}$, $\overrightarrow{DC} = 6\mathbf{s}$ and $\overrightarrow{DA} = 2\mathbf{t}$.

(i) Sketch the quadrilateral $ABCD$.

(ii) Consider the pair of sides AB and DC and write down two important facts about this pair of sides.

(iii) Express $\overrightarrow{BC}$, as simply as possible, in terms of $\mathbf{s}$ and/or $\mathbf{t}$.

(iv) The sides DA and CB, when produced, meet at X. Express $\overrightarrow{XA}$, as simply as possible, in terms of $\mathbf{s}$ and/or $\mathbf{t}$. (*C*)

11. In Fig. 3.81, P is the point (3, 4) and O is the origin.

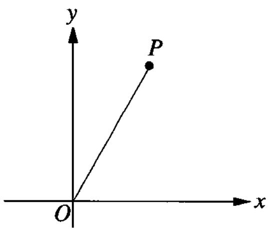


Fig. 3.81

(a) The point P_1 lies on OP produced. Given that $\overrightarrow{OP_1} = 2\overrightarrow{OP}$, express $\overrightarrow{OP_1}$ as a column vector.

(b) P_2 is the reflection of P in the y-axis. Express $\overrightarrow{OP_2}$ as a column vector.

(c) OP is rotated in a clockwise direction about O so that P is mapped onto P_3, where P_3 is a point on the positive x-axis. Express $\overrightarrow{OP_3}$ as a column vector. *(C)*

★12. In Fig. 3.82, $ABCD$ is a parallelogram. The point E, on DC, is such that $DE = \frac{1}{3}DC$. The lines AD and BE, when produced, meet at F. Given that $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $\overrightarrow{AD} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$,

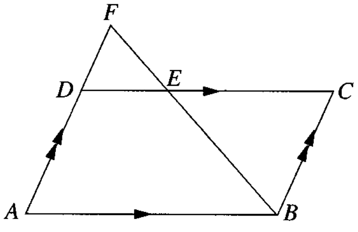


Fig. 3.82

(a) Find the value of $|\overrightarrow{AB}|$,

(b) express each of the following as a column vector.

(i) $\overrightarrow{CB}$ **(ii)** $\overrightarrow{EC}$ **(iii)** $\overrightarrow{FE}$

13. Given that $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$, $\overrightarrow{QR} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{RS} = \begin{pmatrix} h \\ 10.5 \end{pmatrix}$.

(a) Express as a column vector $\overrightarrow{PQ} + 3\overrightarrow{QR}$.

(b) Given that $\overrightarrow{RS}$ is parallel to $\overrightarrow{PQ}$, find the value of h.

(c) Find $|\overrightarrow{PQ}|$, giving your answer correct to the nearest whole number. *(C)*

★14. In Fig. 3.83, the lines ST, PQ and OR are parallel. OPS, OQT and RQS are straight lines and $\overrightarrow{OR} = 3\mathbf{a}$, $\overrightarrow{OP} = \mathbf{b}$, $\overrightarrow{PS} = 2\mathbf{b}$ and $\overrightarrow{OQ} = 2\mathbf{a} + \mathbf{b}$.

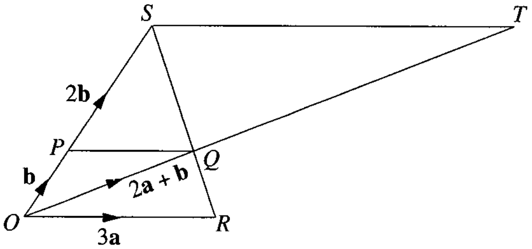


Fig. 3.83

(a) Express in terms of **a** and/or **b**,

(i) $\overrightarrow{PQ}$, **(ii)** $\overrightarrow{QR}$,

(iii) $\overrightarrow{QT}$, **(iv)** $\overrightarrow{ST}$.

(b) Find the numerical value of

(i) $\dfrac{\text{the area of } \triangle OPQ}{\text{the area of } \triangle SPQ}$;

(ii) $\dfrac{\text{the area of } \triangle OPQ}{\text{the area of } \triangle OST}$;

(iii) $\dfrac{\text{the area of } \triangle OPQ}{\text{the area of } \triangle ORQ}$. *(C)*

★15. In Fig. 3.84, $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AC} = \mathbf{q}$ and X is the point on CB such that $CX = \frac{1}{3}CB$.

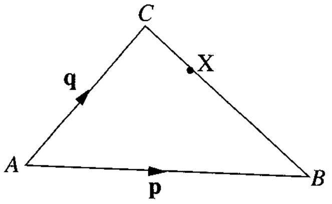
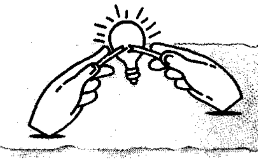


Fig. 3.84

(a) Express the following as simply as possible in terms of **p** and/or **q**.

(i) $\overrightarrow{CB}$ **(ii)** $\overrightarrow{AX}$

(b) Given further that $\overrightarrow{AX} = h\mathbf{p} + k\mathbf{q}$, and that T is the point such that $\overrightarrow{AT} = h\mathbf{p}$, mark and label the point T on Fig. 3.84 in the answer space. *(C)*



Problem Solving

1. In Fig. 3.85, *ABCD* is a quadrilateral and *P* and *Q* are the mid-points of *AC* and *BD* respectively. Show that the sum of the vectors $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$ and $\overrightarrow{AD}$ is $4\overrightarrow{PQ}$.

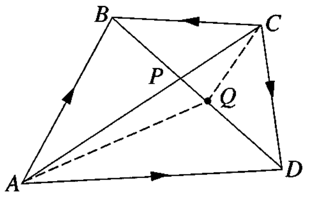


Fig. 3.85

2. In Fig. 3.86, *A*, *B* and *C* are the mid-points of *PQ*, *QR* and *RP* respectively. *G* is the point of intersection of *AR*, *BP*, and *CQ*. Given that the position vectors of *P*, *Q* and *R* relative to an origin *O*, (which is not shown in the diagram), are **p**, **q** and **r** respectively, prove that $\mathbf{p} + \mathbf{q} + \mathbf{r} = 3\mathbf{g}$, where **g** is the position vector of *G* relative to the origin *O*.

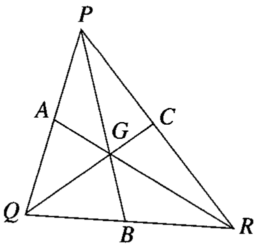


Fig. 3.86

3. In Fig. 3.87, $\overrightarrow{OM} = 2\mathbf{p}$ and $\overrightarrow{ON} = 5\mathbf{q}$.

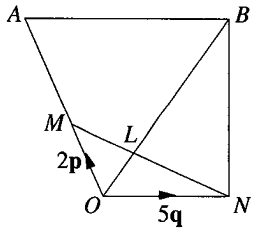


Fig. 3.87

(a) Given that $3\overrightarrow{OM} = \overrightarrow{MA}$ and $\overrightarrow{AB} = 2\overrightarrow{ON}$, express, in terms of **p** and **q**,

(i) $\overrightarrow{MN}$, (ii) $\overrightarrow{OB}$.

(b) Given that $\overrightarrow{LN} = s\overrightarrow{MN}$ and $\overrightarrow{LB} = t\overrightarrow{OB}$, show that $\overrightarrow{OL} = 2s\mathbf{p} + 5(1 - s)\mathbf{q}$ and that $\overrightarrow{OL} = 2(1 - t)(4\mathbf{p} + 5\mathbf{q})$.

(c) Hence, prove that $4t + s = 4$ and $2t - s = 1$.

(d) By solving the equations in (c) for *s* and *t*, find the numerical values of the ratios

(i) $\frac{ML}{LN}$, (ii) $\frac{BL}{LO}$.

4. In Fig. 3.88, *M* is the mid-point of *PR*, $\overrightarrow{OP} = 2\mathbf{b} - \mathbf{a}$, $\overrightarrow{OQ} = 3\mathbf{a} - \mathbf{b}$ and $\overrightarrow{OR} = 5\mathbf{a} + k\mathbf{b}$, where *k* is a constant.

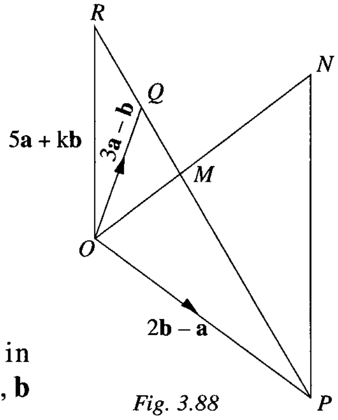


Fig. 3.88

(a) Express in terms of **a**, **b** and/or *k*,

(i) $\overrightarrow{PQ}$; (ii) $\overrightarrow{QR}$; (iii) $\overrightarrow{PR}$.

(b) By solving the equation $\overrightarrow{PQ} = \lambda\overrightarrow{QR}$, find the values of *k* and λ and hence the numerical value of the ratio $\frac{PQ}{PR}$.

(c) Express $\overrightarrow{OM}$ in terms of **a** and **b**.

(d) *OM* is produced to a point N such that $\overrightarrow{PN} = h\overrightarrow{OR}$. Express $\overrightarrow{ON}$ in terms of **a**, **b** and *h*.

(e) Given that $\frac{ON}{OM} = l$, express $\overrightarrow{ON}$ in terms of **a**, **b** and *l*.

(f) Using the expression for $\overrightarrow{ON}$ in **(d)** and **(e)**, form an equation connecting *h*, *l*, **a** and **b**. Use this equation to find the values of *h* and *l* and hence find the numerical values of the following ratios

(i) $\frac{PN}{OR}$, (ii) $\frac{OM}{MN}$.

CHAPTER

4

Geometrical Transformations

In this chapter, you will learn how to

- reflect an object and find the line of reflection by construction;
- rotate an object, and find the centre of rotation and angle of rotation by construction;
- translate an object.

Preliminary Problem

The picture shows an escalator in a shopping centre for moving people from one floor to another. Translation is illustrated by the motion of escalators in shopping centres, the movement of lifts in high-rise buildings, etc.

What other examples can you think of?

In Book 2, we studied simple cases of geometrical transformations of reflection, rotation, translation and enlargement. In this chapter, we shall look at three of these transformations in greater detail.

Reflection

Fig. 4.1 shows the triangle ABC undergoing a reflection in the line $x = 3$ to produce the image $A'B'C'$.

Recall that a reflection is defined by its **axis of reflection** or the **axis of symmetry** or the **line of reflection** or simply, the **mirror line**. In Fig. 4.1, this mirror line is $x = 3$.

(1) Under reflection, the shape and size of an image is exactly the same as its original figure. We call this rigid transformation an isometric transformation.

(2) In Fig. 4.1, $\triangle ABC$ (read in anticlockwise direction) under reflection, becomes $\triangle A'B'C'$ (read in clockwise direction), i.e. the sense is reversed. We say that reflection does not preserve orientation.

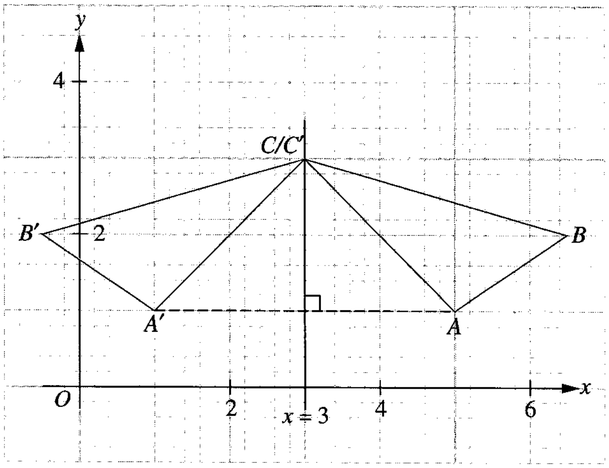


Fig. 4.1

(3) The line of reflection is the perpendicular bisector of the line joining any point and its image (e.g. AA' in Fig. 4.1).

(4) All points on the mirror line undergo no change. We say that these points are invariant. In the case of Fig. 4.1, C is the only invariant point, a point that does not undergo any change in a transformation.

Find the Axis of Reflection and its Equation

Now let us find the axis of reflection and its equation of a line segment and its image.

Fig. 4.2 shows the line segment AB and its image $A'B'$, where A, B, A' and B' are the points (1, 3), (2, 6), (3, 1) and (6, 2) respectively.

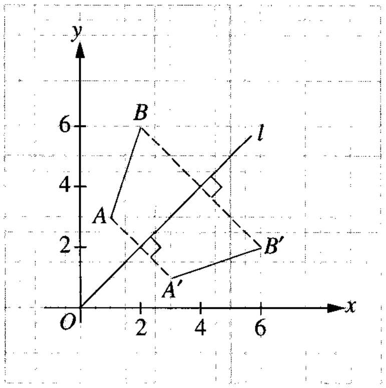


Fig. 4.2

Steps

1. Join A to A' or B to B' and construct the perpendicular bisector l of AA' or BB'. The line l is the axis of reflection.

To find the equation of l:

2. Midpoint of AA', i.e. $\left(\frac{1+3}{2}, \frac{3+1}{2}\right) = (2, 2)$.

   Midpoint of BB', i.e. $\left(\frac{2+6}{2}, \frac{6+2}{2}\right) = (4, 4)$.

3. Gradient of $l = \frac{4-2}{4-2} = 1$. (See Fig. 4.2.)

$\therefore$ equation of l is $y = x + c$ where c is a constant.

Since (2, 2) lies on l, $2 = 2 + c$, i.e. $c = 0$.

Hence the equation of the axis of reflection in Fig. 4.2 is $y = x$

Fig. 4.3 shows a point A under reflection in two lines l and m. We represent the reflection in line l by M_l and that in line m by M_m. Hence $M_mM_l(A)$ represents a reflection of point A in line l followed by line m whereas $M_lM_m(A)$ represents the reverse.

In most instances, we use symbols to represent transformations in order to simplify statements. For example, if we represent the enlargement with centre at origin and scale factor 2 as E and a reflection about the x-axis as H, then HE represents the transformation of an enlargement followed by a reflection and EH represents a reflection followed by an enlargement, HH (normally written as H^2) represents a reflection followed by another reflection, while EE(E^2) is an enlargement followed by another enlargement.

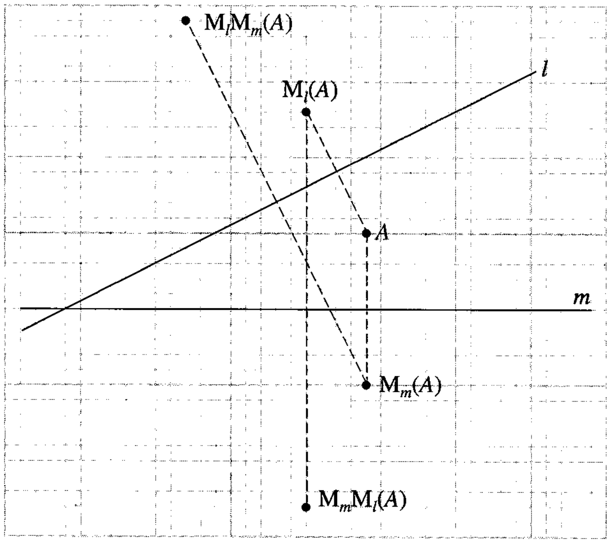


Fig. 4.3

If M represents the transformation that maps point A onto point B, then the transformation N that maps point B onto point A is called the inverse transformation of M. We shall use M^{-1} to represent the inverse transformation that will map B onto A.

From Fig. 4.3, we observe that the images of $M_mM_l(A)$ and $M_lM_m(A)$ are not the same. Can we conclude that combination of reflections is not commutative?

A palindrome reads the same either from the right or from the left. For example,

(i) 'Level' is a palindromic word and
(ii) 54145 is a palindromic number.

Example 7

The coordinates of A and B are (–2, 2) and (1, 4) respectively. The line joining A and B is reflected in the x-axis to A′B′. Find the coordinates of A′ and B′. A′B′ is then reflected in the line x = 3 to give A″ B″. Find the coordinates of A″ and B″.

Solution

From the graph, the coordinates of A' are (–2, –2) and those of B' are (1, –4). A'' is the point (8, –2) and B'' is the point (5, –4).

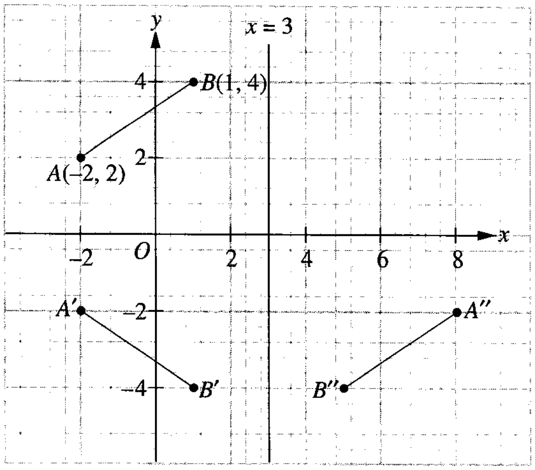


Fig. 4.4

Example 2

Find the equation of the image of the line 2y = x + 4 under a reflection in

(a) the x-axis, *(b) the y-axis,*
(c) the line x = 1.

Solution

(a) Let AB represent the line $2y = x + 4$. Fig. 4.5 shows a reflection of the line AB in the x-axis. The point A is mapped onto $A'(0, -2)$ and the point B is invariant.

The gradient of $A'B'$ is $-\frac{1}{2}$ and it cuts the y-axis at $(0, -2)$.

Hence, the equation of $A'B'$ is $y = -\frac{1}{2}x - 2$.
i.e. $2y = -x - 4$ or $2y + x + 4 = 0$.

(b) Under reflection in the y-axis, the point A is invariant and the point B is mapped onto B'' $(4, 0)$. The gradient of $A''B''$ is $-\frac{1}{2}$ and the y-intercept is 2.

Hence, the equation of $A''B''$ is $y = -\frac{1}{2}x + 2$, i.e. $2y + x = 4$.

(c) The images of A and B under reflection in the line $x = 1$ are $A'''(2, 2)$ and $B'''(6, 0)$ respectively. The gradient of $A'''B'''$ is $-\frac{1}{2}$ and the line $A'''B'''$ cuts the y-axis at $(0, 3)$.

Hence, the equation of $A'''B'''$ is $y = -\frac{1}{2}x + 3$, i.e. $2y + x = 6$.

We can obtain a palindromic number by following the rules below:

1. *Take any number.*
2. *Reverse its digits and add the number formed to the original number.*
3. *Repeat step 2 until a palindromic number is obtained.*

Examples:

Take the numbers 32, 57 and 164.

(a)
$$\begin{array}{r} 32 \\ +\ 23 \\ \hline 55 \end{array} \quad \textit{(1 step needed)}$$

(b)
$$\begin{array}{r} 57 \\ +\ 75 \\ \hline 132 \\ +\ 231 \\ \hline 363 \end{array} \quad \textit{(2 steps needed)}$$

(c)
$$\begin{array}{r} 164 \\ +\ 461 \\ \hline 625 \\ +\ 526 \\ \hline 1151 \\ +\ 1511 \\ \hline 2662 \end{array} \quad \textit{(3 steps needed)}$$

Do the following exercises:

1. *For each number from 50 to 98, find the number of steps needed before we can obtain a palindromic number. Which number requires the most number of steps? Note that 55, 66, 77, 88 are palindromic numbers and therefore no working is needed.*
2. *Which year in the 20th century is a palindromic number?*
3. *How many years in the 21st century are palindromic numbers?*

* * *

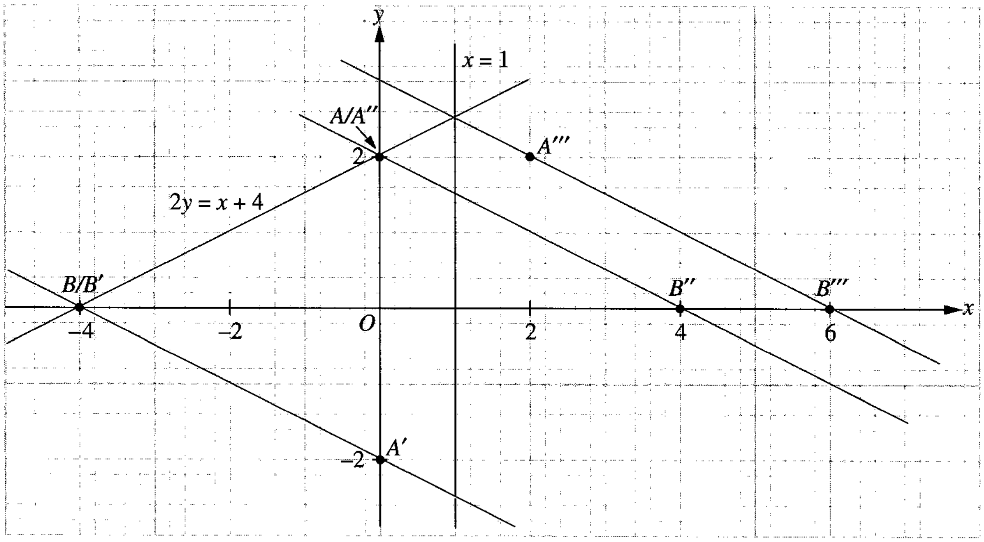


Fig. 4.5

Exercise 4a

1. Write down the coordinates of the reflections of each of the following points in
   (a) the x-axis, **(b)** the y-axis, and
   (c) the line $y = x$.
   (i) (3, 4) **(ii)** (–1, 3) **(iii)** (3, 3)
   (iv) (–3, –4) **(v)** (3, –2) **(vi)** (p, q)

2. Reflect the point $A(-1, 3)$ in the x-axis and then in the line $y = 4$. What are the coordinates of the final image?

3. Reflect the point (–1, 3) in the line $y = 4$ and then in the x-axis. What are the coordinates of the final image? Is your answer the same as that obtained in question 2?

4. State the coordinates of the reflection of the point (3, 2) in the line $x = 2$.

5. Draw the line $y = x$ on a sheet of graph paper. If the reflection of the point (3, –1) in this line is (p, q), find the value of p and of q.

6. Draw the line $y = x - 2$ on a sheet of graph paper. The reflection of the origin in the line $y = x - 2$ is the point O'. Find the coordinates of O'.

7. The point $P(2, 1)$ is transformed by M_1, a reflection in the y-axis and M_2, a reflection in the line $x = 4$. Give the coordinates of $M_1(P)$, $M_2(P)$, $M_1M_2(P)$ and $M_2M_1(P)$.

8. Under a reflection, the point $A(3, 5)$ is mapped onto (5, 3). Draw on a sheet of graph paper the axis of reflection and find its equation. If (5, 3) is transformed onto (–5, 3), find the mirror line and the equation of the line.

9. The point A and its image A' under a reflection are given below. Plot the points A and A' on a sheet of graph paper, construct the line of reflection and find its equation in each case.
   (a) $A(1, 1), A'(3, 1)$ **(b)** $A(1, -1), A'(1, 9)$
   (c) $A(2, 1), A'(0, 3)$ **(d)** $A(0, 1), A'(1, 2)$
   (e) $A(0, -1), A'(2, 1)$
   (f) $A(-1, 1), A'(3, -1)$

10. The point $A(3, 4)$ is reflected in the line $x = 2$ and then reflected in the line $y = 1$. Find the coordinates of the image of A under these two reflections. State the coordinates of the point which remains invariant under these two reflections.

11. Find the coordinates of the image of the point $A(2, 3)$ under a reflection in the line $x = 6$ followed by a reflection in the line $y = x$.

12. Find the equation of the line onto which the line $y = 3x + 2$ is mapped under a reflection in
    (a) the x-axis; **(b)** the y-axis;
    (c) the line $x = 2$.

13. Given that the coordinates of the point A is $(3, -2)$. Find the coordinates of the points B and C such that

(a) B is the reflection of A in the line $y = 4$,
(b) C is the reflection of A in the line $x + y = 6$.

14. Draw the line $y = x + 2$. The image of the origin under a reflection in the line $y = x + 2$ is A. Find the coordinates of the point A.

★15. Find the equation of the image of the line $y = x + 3$ under a reflection in the

(a) x-axis;	**(b)** y-axis;
(c) line $x = 3$;	**(d)** line $y = 1$.

16. The point $A(1, 4)$ is reflected in the line $y = x$ and then reflected in the line $x + y = 6$. Find the coordinates of the image of A under these two reflections. If the point $A(1, 4)$ is first reflected in the line $x + y = 6$ and then reflected in the line $y = x$, what is the final image of A? Would your result be the same as that obtained earlier? State the coordinates of the invariant point under these two reflections.

★17. The point $A(1, 2)$ is reflected in the line $x + y = 6$ followed by another reflection in the line $x = 4$. Find the coordinates of the final image of A. If the point $A(1, 2)$ is reflected in the line $x = 4$ and then in the line $x + y = 6$, what is the final image of A? Is the reflection commutative in this case? State the coordinates of the invariant point under these two reflections.

★18. In the diagram, the line segment PQ is reflected in the line AB so that P' is the image of P? Copy the diagram and

(a) construct accurately the line AB,
(b) mark accurately the point Q', the image of Q.

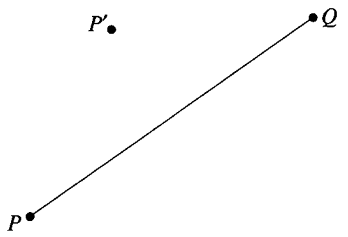


Fig. 4.6

Rotation

(a) Fig. 4.7 shows that the flag $ABCD$ is rotated through 90° anticlockwise about the origin. The image of $ABCD$ is $A'B'C'D'$.

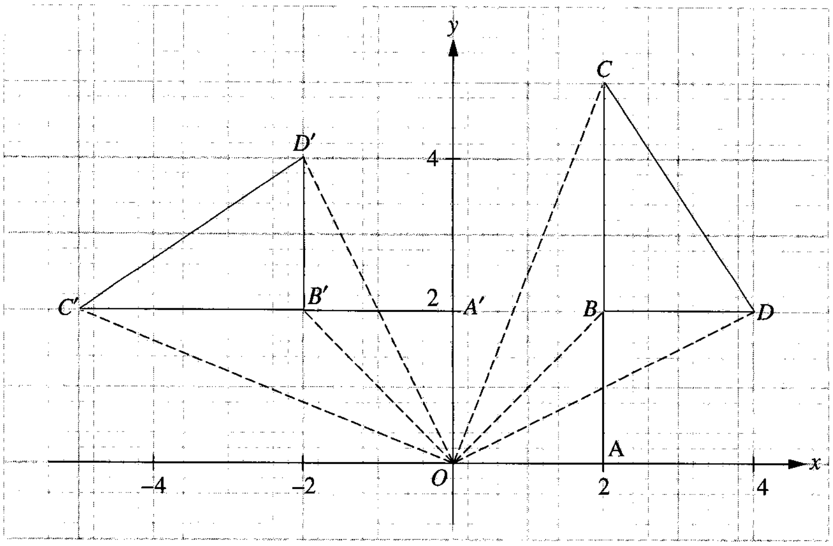


Fig. 4.7

In Fig. 4.7, we use the same scale for both the x- and y-axes to draw the original figure and its image. What will happen if the scale for the x-axis is different from that for the y-axis?

(b) The kite $PQRS$ in Fig. 4.8 is rotated through 180° about $P(0, 2)$. The image is $P'Q'R'S'$. Notice that P is invariant under the rotation. Can this transformation also be a reflection? Explain your answer.

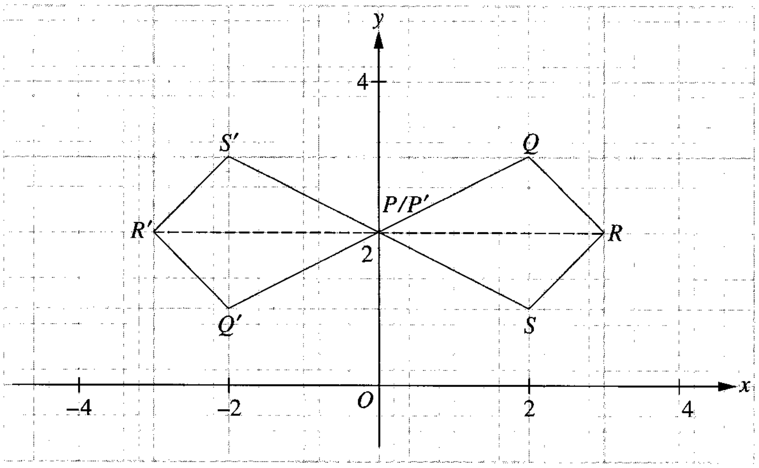


Fig. 4.8

Find the Centre and Angle of Rotation

Fig. 4.9 shows the construction of the centre of rotation. The steps taken are:

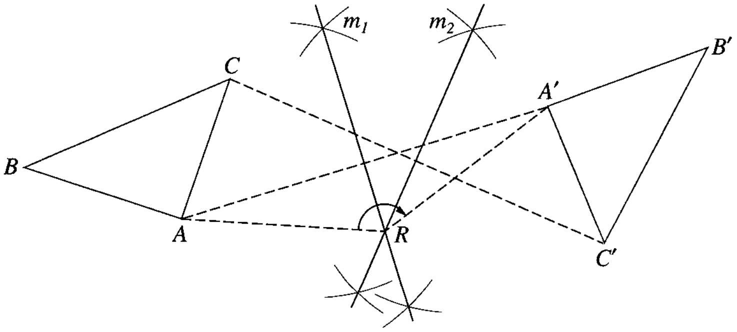


Fig. 4.9

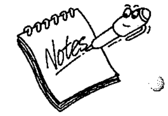
1. Join A to A' and construct the perpendicular bisector m_1.

2. Join C to C' and construct the perpendicular bisector m_2. (You may join B to B' and do the same.)

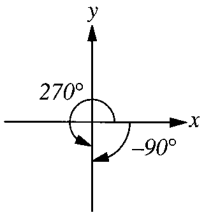
3. The point of intersection of m_1 and m_2, R, is known as the centre of rotation.

4. To find the angle of rotation, join AR and $A'R$. $A\hat{R}A'$ is the angle of rotation in a clockwise direction.

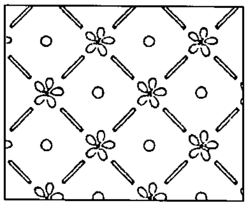
Can you explain why R lies on the intersection of the perpendicular bisector of AA' and CC'?



1. *Rotation can either be clockwise or anti-clockwise. A 90° rotation represents an anticlockwise rotation and a –90° rotation represents a clockwise rotation. Hence, a 270° rotation is equivalent to a –90° rotation.*
2. *A 180° rotation is sometimes referred to as a half turn.*
3. *In a rotation, every point of the original figure is rotated through the same angle about the centre of rotation. If the centre of rotation lies on the figure, then it is the invariant point.*
4. *Rotation preserves orientation, size and shape and it is an* ***isometric transformation****. (See Fig. 4.8)*



A tessellation is a pattern of repeated geometric shapes that all fit together without gaps and overlaps and fill an area of the plane. Transformations such as translation, rotation and reflection, are used to give matching edges so that the shapes can tessellate.



Example 3

Using a scale of 1 cm to represent 1 unit on both axes, draw $\triangle ABC$ and $\triangle PQR$ with vertices $A(2, 2)$, $B(3, 3)$, $C(1, 4)$, $P(4, 2)$, $Q(5, 1)$ and $R(6, 3)$.

(a) $\triangle ABC$ is mapped onto $\triangle LMN$ by a 90° clockwise rotation with centre of rotation at (0, 3). Draw $\triangle LMN$ and label the vertices clearly.

(b) $\triangle ABC$ is the image of $\triangle PQR$ under a rotation. Find the centre of rotation by construction and state the angle of rotation.

Solution

(a) The vertices of $\triangle LMN$ are $L(-1, 1)$, $M(0, 0)$ and $N(1, 2)$.

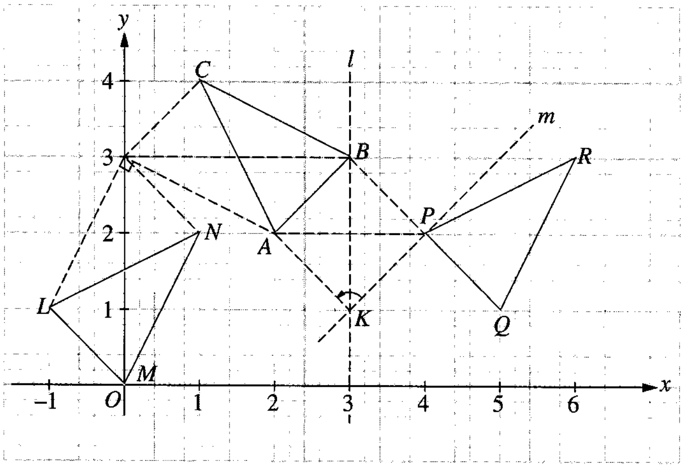


Fig. 4.10

(b) Line l is the perpendicular bisector of AP while line m is the perpendicular bisector of BQ. The point of intersection of these two perpendicular bisectors $K(3, 1)$ gives the centre of rotation. Join AK and PK. The angle of rotation is 90° anticlockwise.

Exercise 4b

1. Draw and label $\triangle ABC$ whose vertices are $A(3, 1)$, $B(4, 1)$ and $C(4, 5)$. $\triangle ABC$ is mapped onto $\triangle PQR$ by a 90° clockwise rotation about the point (2, 1). Draw and label $\triangle PQR$.

2. Draw and label $\triangle PQR$ whose vertices are $P(3, 0)$, $Q(5, 0)$ and $R(5, 3)$. $\triangle PQR$ is rotated through 90° anticlockwise to $\triangle LMN$ with the centre of rotation at (0, 1). Draw and label the vertices of $\triangle LMN$ on the same diagram.

3. Given that P is the point (2, 4), Q is the point (4, –1) and R is the point (–1, 0), find

(a) the image of P under a clockwise rotation of 90° about R,

(b) the image of Q under an anticlockwise rotation of 90° about P,

(c) the image of R under a 180° rotation about Q.

4. Under a rotation, the image of A is A' while the image of B is on the line segment $A'X$. Copy Fig. 4.11 and construct accurately

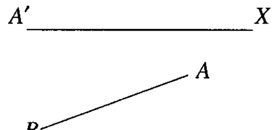


Fig. 4.11

(a) the point B',
(b) C, the centre of rotation.

5. The coordinates of $\triangle ABC$ are $A(4, 1)$, $B(6, 1)$ and $C(4, 6)$ while the coordinates of its image $\triangle A'B'C'$ under a rotation are $A'(0, -1)$, $B'(-2, -1)$ and $C'(0, -6)$. Draw $\triangle ABC$ and $\triangle A'B'C'$ on a sheet of graph paper and find the centre of rotation. State the angle of rotation.

6. **(a)** Triangle A can be mapped onto triangle B by a rotation.
Find **(i)** the coordinates of its centre, **(ii)** the angle of rotation.
(b) Triangle C can be mapped onto triangle A by a rotation.
Find **(i)** the coordinates of the centre, **(ii)** the angle of rotation.
(c) Triangle B is rotated through 90° clockwise about the point (4, 6). Find the coordinates of the image of triangle B.

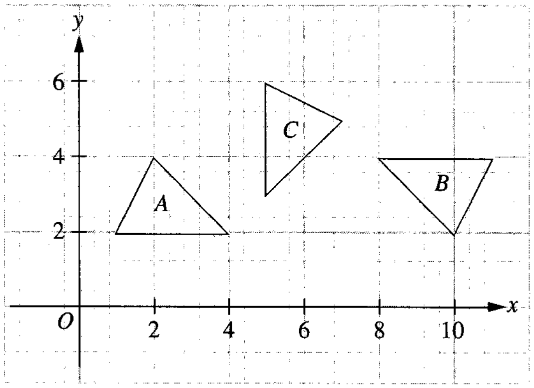


Fig. 4.12

7. Under a rotation, the line $P'Q'$ is the image of the line PQ. Given that their coordinates are $P(1, 1)$, $Q(1, 4)$, $P'(3, 1)$ and $Q'(k, 1)$, where $k > 0$, find:

(a) the value of k,

(b) the image of the point $\left(1, 2\frac{1}{2}\right)$,

(c) the coordinates of the point whose image is $\left(5\frac{1}{2}, 1\right)$.

8. The triangle $A'B'C'$ is the image of the triangle ABC under a clockwise rotation of 25° about C. Calculate, giving your answer correct to the nearest $\frac{1}{2}^{\circ}$,

(a) $C\hat{A}A'$,
(b) $A\hat{C}B'$.

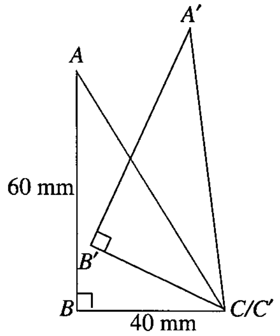


Fig. 4.13

9. $\triangle ABC$ has vertices $A(0, 0)$, $B(3, 0)$ and $C(0, 4)$. $\triangle PQR$ has vertices $P(10, 2)$, $Q(10, 5)$ and $R(6, 2)$. Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and label the vertices clearly. Given that $\triangle ABC$ is the image of $\triangle PQR$ under a rotation, find the centre of rotation by construction and state the angle of rotation.

10. Find the coordinates of the image of the point (1, 4) under a clockwise rotation of

(a) 90° about the centre (4, 2),
(b) 180° about the centre (4, 2).

11. If R represents an anticlockwise rotation of 240° about the origin, describe R^2 and R^4.

12. Find the equation of the image of the line $y = x + 2$ under a clockwise rotation of 90° about the origin.

13. Find the equation of the image of the line $x + y = 4$ under an anticlockwise rotation of 90° about the point (0, 2).

Translation

Fig. 4.14 shows $\triangle ABC$ being translated to $\triangle A'B'C'$.

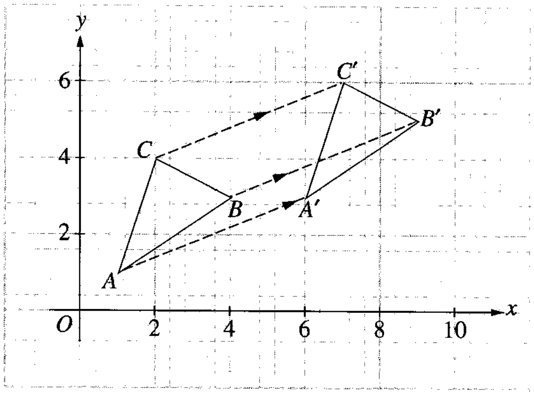


Fig. 4.14

A translation is isometric and it preserves orientation.

Can there be invariant points under a translation?

A translation can be represented by a **column vector** $\begin{pmatrix} a \\ b \end{pmatrix}$ where a and b is the number of units of moves along the x- and y-axes respectively. In Fig. 4.14, the vector equation representing a translation is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ where $\begin{pmatrix} a \\ b \end{pmatrix}$ is the translation vector and $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is the image of $\begin{pmatrix} x \\ y \end{pmatrix}$.

Example 4

Find the coordinates of the images of the quadrilateral A(1, 1), B(2, 6), C(6, 4) and D(5, 2) under the translation T_1 represented by $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$. What would be the image of the new quadrilateral if it underwent another translation T_2 represented by $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$?

Solution

The image of the quadrilateral *ABCD* is obtained as shown below.

$$T_1(A) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$T_1(B) = \begin{pmatrix} 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$T_1(C) = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$$

$$T_1(D) = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$$

Therefore, the coordinates of the images are $A'(5, -2)$, $B'(6, 3)$, $C'(10, 1)$ and $D'(9, -1)$. The image of the quadrilateral $A'B'C'D'$ is plotted in Fig. 4.15. The image of the new quadrilateral $A''B''C''D''$ under T_2 is obtained similarly.

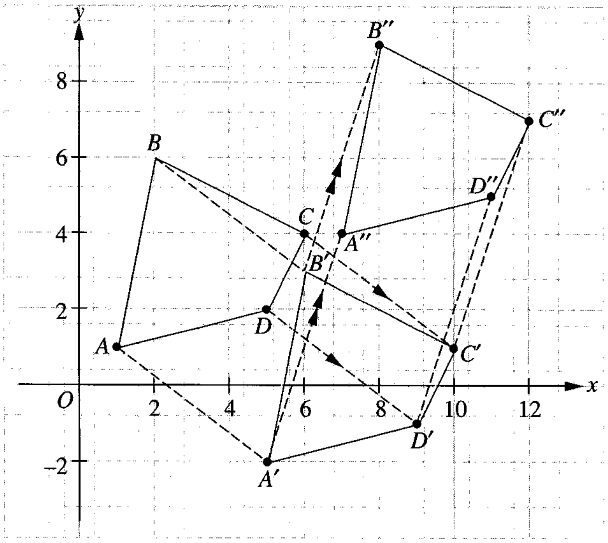


Fig. 4.15

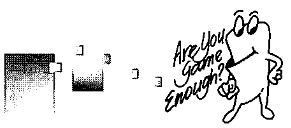
$T_2(A) = \begin{pmatrix}5\\-2\end{pmatrix} + \begin{pmatrix}2\\6\end{pmatrix} = \begin{pmatrix}7\\4\end{pmatrix}$ $\quad T_2(B) = \begin{pmatrix}6\\3\end{pmatrix} + \begin{pmatrix}2\\6\end{pmatrix} = \begin{pmatrix}8\\9\end{pmatrix}$

$T_2(C) = \begin{pmatrix}10\\1\end{pmatrix} + \begin{pmatrix}2\\6\end{pmatrix} = \begin{pmatrix}12\\7\end{pmatrix}$ $\quad T_2(D) = \begin{pmatrix}9\\-1\end{pmatrix} + \begin{pmatrix}2\\6\end{pmatrix} = \begin{pmatrix}11\\5\end{pmatrix}$

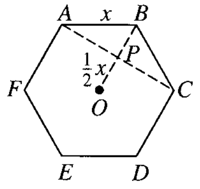
i.e. $A''(7, 4)$, $B''(8, 9)$, $C''(12, 7)$ and $D''(11, 5)$ are the images. These points are also plotted as shown in Fig. 4.15.

Would the result be the same if T_2 is performed first? Can you give a single vector that would produce the same result as the two successive transformations?

Find out about Escher's tessellations and the transformations he used in his tessellations.



The figure below shows a regular hexagon of side x cm. AC cuts OB at P where OP = $\frac{1}{2}$ x cm.



Using a pencil and ruler, mark out line segments whose lengths are $\frac{1}{3}$ x cm, $\frac{1}{4}$ x cm and $\frac{1}{5}$ x cm.

Example 5

T_1 is the translation $\begin{pmatrix}2\\3\end{pmatrix}$ and T_2 is a translation that will move the point (1, 2) to (–2, 5).

(a) Find the image of the point A(7, 4) under T_1.
(b) Find the translation vector represented by T_2.
(c) What is the image of the point B(8, –4) under T_2?

Solution

(a) $A' = T_1(A)$

$$= \begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}7\\4\end{pmatrix} = \begin{pmatrix}9\\7\end{pmatrix}$$

$\therefore$ image of A under T_1 is (9, 7).

(b) Let the translation vector of T_2 be $\begin{pmatrix}a\\b\end{pmatrix}$.

$$\begin{pmatrix}-2\\5\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}$$

$$\therefore \begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}-2\\5\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-3\\3\end{pmatrix}$$

$\therefore$ the translation vector of T_2 is $\begin{pmatrix}-3\\3\end{pmatrix}$.

(c) $B' = T_2(B)$

$$= \begin{pmatrix}-3\\3\end{pmatrix} + \begin{pmatrix}8\\-4\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix}.$$

$\therefore$ the image of B under T_2 is (5, –1).

The CD, Maths Probe, has interesting interactive activities for exploring reflections and rotations, for example, reflection in a line and rotation about a point through a given angle. Besides the CD, you can use the open tool, Geometer's Sketch Pad, to explore the different transformations we are learning.

Exercise 4c

1. The vertices of $\triangle PQR$ are $P(1, 3)$, $Q(7, 5)$ and $R(2, 0)$. Find the new coordinates of the vertices of $\triangle PQR$ under a translation T represented by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

2. A translation T maps the point (6, 2) onto the point (2, 7) and the point (–1, –5) onto the point *P*. Find the coordinates of the point *P* and the column vector of the translation T.

3. A translation T maps the point (9, 1) onto the point (2, –3) and the point *Q* onto the point (–5, 6). Find the column vector representing the translation T and the coordinates of the point *Q*.

4. T is the translation $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, *A* is the point (2, 4), *B* is (p, q) and *C* is (h, k).
   (a) Find the coordinates of the image of the point *A* under T.
   (b) Given that $T(B) = A$, find the value of p and of q.
   (c) Given that $T^2(A) = C$, find the value of h and of k.
   (d) Find the coordinates of the point *D* such that $T^2(D) = A$.

5. Under a translation T, the point (1, 3) is mapped onto the point (9, 7). Given that T maps the point (–2, –3) onto the point (x, y), find the value of x and of y. If T^2 maps the point (h, k) onto the point (4, 6), find the value of h and of k.

6. Under a translation T_1, the image of the point (5, –1) is (2, 3). Under a translation T_2, the image of the point (–2, 5) is (4, –5). Find the image of the point (7, 6) under the following transformations.
   (a) T_1 **(b)** T_2 **(c)** T_1T_2 **(d)** T_2T_1 **(e)** T_1^2

7. Fig. 4.16 shows triangles *P*, *Q*, *R*, *S* and *K*.

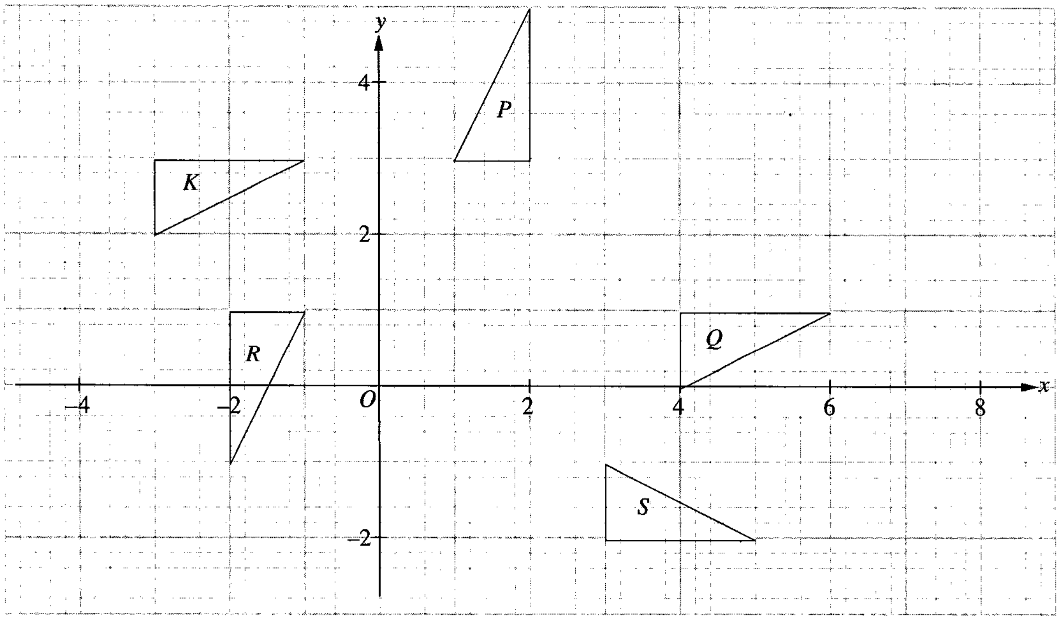


Fig. 4.16

(a) ΔP is mapped onto ΔQ by a reflection. Draw the line of reflection and determine its equation.
(b) ΔR is the image of ΔP under a rotation. Find the centre of rotation and state the angle of rotation.
(c) Describe a single transformation which maps ΔK onto ΔQ.
(d) A single transformation maps ΔS onto ΔP. Describe this transformation.

Summary

We have studied the following three isometric transformations.

1. A **translation** moves all the points of an object on a plane the same distance and in the same direction. It is isometric and preserves orientation. It has no invariant points.
2. Under a **reflection**, a figure and its image are symmetrical about the mirror line. A reflection does not preserve orientation. Points on the mirror line are invariant.
3. A **rotation** is properly defined by its centre, angle and direction (clockwise or anticlockwise) of rotation. The centre of rotation is the only invariant point.

Review Questions 4

1. Given that P is the point (3, 4), find the coordinates of the image of P under
   (a) an anticlockwise rotation of 90° about the point (2, 0),
   (b) a reflection in the line $x + 2 = 0$.

2. A translation maps the point (5, 7) onto the point (2, 9) and the point A onto the point (–3, –5). Find the coordinates of A.

3. In Fig. 4.17, A is the point (1, 3) and B is the point (6, 1).
   (a) The line AB is rotated through 90° clockwise about B. Find the equation of the image of the line AB.
   (b) The line $y = 1$ is the line of symmetry of ΔABC. Find the coordinates of the point C.

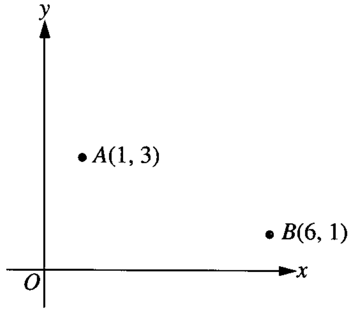


Fig. 4.17

4. Fig. 4.18 shows two squares $ABCD$ and $PQRS$. In each of the following cases, describe completely the single transformation that will map
   (a) $PQRS$ onto $ABCD$,
   (b) $ABCD$ onto $QPSR$,
   (c) $PQRS$ onto $CDAB$,
   (d) $ABCD$ onto $SPQR$,
   (e) $PQRS$ onto $DABC$.

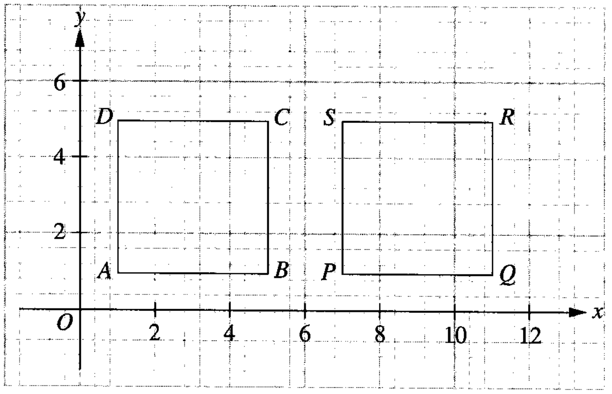


Fig. 4.18

5. M is a reflection in the line $y = -x$. Find the coordinates of the image of the point (5, 2) under
   (a) M, (b) M^2, (c) M^3.

6. Find the equation of the image of the line $y = x$ under an anticlockwise rotation of 90° about the point (2, 0).

7. Find the equation of the image of the line $y = x + 4$ under
   (a) a reflection in the line $x = 2$,
   (b) a translation represented by the column vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$,
   (c) a 90° clockwise rotation about the origin O.

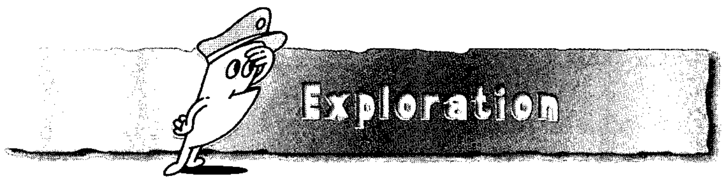
8. Find the image of the point $P(2, 1)$ under
   (a) a reflection in the line $x + y = 4$,
   (b) a 180° rotation about the point (–5, 3),
   (c) a translation represented by $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$,
   (d) a reflection in the line $y = x + 2$.

9. Given that the line $y = 2x + 3$ is mapped onto the line $y = mx + c$ by a translation represented by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, find the values of m and c.

★10. $\triangle ABC$ is an equilateral triangle of sides 3 cm, the corners being lettered anticlockwise in alphabetical order. $\triangle ABC$ is mapped onto $\triangle A_1B_1C_1$ by an anticlockwise rotation of 120° about A. $\triangle A_1B_1C_1$ is then mapped onto $\triangle A_2B_2C_2$ by an anticlockwise rotation of 120° about B. Finally $\triangle A_2B_2C_2$ is mapped onto $\triangle A_3B_3C_3$ by an anticlockwise rotation of 120° about C. Construct the complete figure accurately and state clearly a single geometrical transformation which maps $\triangle ABC$ onto $\triangle A_3B_3C_3$.

11. M is a reflection in the line $y = x$.
   (a) Find the coordinates of the image of (1, 3) under M.
   (b) Find the coordinates of the image of (2, 5) under M^{-1}.
   (c) Find the coordinates of the image of (3, 7) under M^5.

12. R is an anticlockwise rotation of 90° about the origin.
   (a) Find the coordinates of the image of
      (i) (3, 5), (ii) (7, –4) under R.
   (b) Find the coordinates of the image of
      (i) (3, 4), (ii) (–2, –3) under R^{-1}.
   (c) Find the coordinates of the image of (2, 5) under
      (i) R^5, (ii) R^8.



Exploration

1. $\triangle ABC$ is transformed to $\triangle PQR$ by a reflection in the y-axis followed by a 90° clockwise rotation about the origin. Describe a single transformation that will map $\triangle ABC$ onto $\triangle PQR$.

2. $\triangle ABC$ is transformed onto $\triangle LMN$ by a 90° clockwise rotation about the origin followed by a reflection in the y-axis. Describe a single transformation that will map $\triangle ABC$ onto $\triangle LMN$.

3. By using a translation, prove that the sum of the angles of a triangle is equal to 180°.

CHAPTER 5

Further Geometrical Transformations

In this chapter, you will learn how to

- enlarge a figure with a negative scale factor;
- find the centre and scale factor of enlargement given the original figure and its enlarged image;
- draw and find the image figure under a stretch and a shear by construction;
- find the image figure of an object under a combination of transformations.

Preliminary Problem

The picture below shows an enlarger which is an important instrument in a photograph-developing laboratory. It is used to produce enlarged photographs using their negatives. By moving the film relative to the lens, it is possible to change the size of the photograph produced.

In the previons chapter, we learnt three isometric transformations: *reflection*, *rotation* and *translation*. Now we shall study three non-isometric transformations, namely, *enlargement*, *stretch* and *shear*.

Enlargement

In Chapter 11 of Book 2, we learnt how to enlarge a figure with positive scale factors. We shall now look at enlargement with negative scale factors.

Revision

Fig. 5.1 shows a quadrilateral $ABCD$ being enlarged to quadrilateral $A'B'C'D'$ with E as the centre of enlargement and scale factor 2.

We can also say that quadrilateral $A'B'C'D'$ is being "enlarged" to quadrilateral $ABCD$ with E as the centre of enlargement and scale factor $\frac{1}{2}$, even though the image $ABCD$ is smaller than the original figure $A'B'C'D'$.

The term **enlargement** in mathematics may thus refer to the enlarging or diminishing of a figure depending on the scale factor involved.

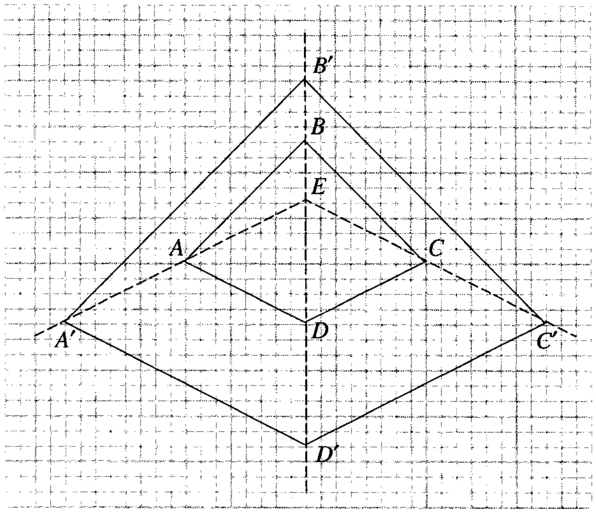


Fig. 5.1

Fig. 5.2 shows $\triangle ABC$ being enlarged with a scale factor of –2 and E as the centre of enlargement.

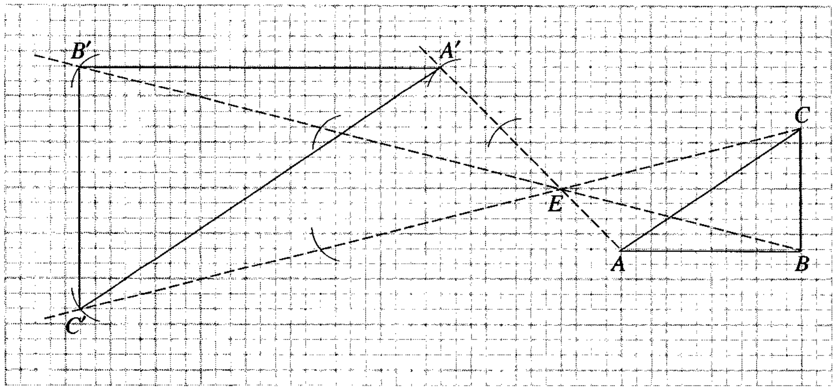


Fig. 5.2

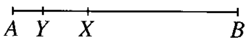
Steps for construction are:

(1) Join A to E and produce in the direction of AE.
(2) With E as the centre and radius equal to $2AE$, mark off the image A' on AE produced.
(3) Repeat (1) and (2) for points B and C to get the images B' and C'.

Note: **1.** Unlike translation, reflection and rotation, enlargement is non-isometric. However, it preserves angle and orientation. The images formed under enlargement yield similar figures. When do we get an invariant point under an enlargement?

2. If the scale factor is greater than 1, the image is enlarged. If it is between 0 and 1, the image is diminished. For a negative scale factor, what can you say about the corresponding points of the image and the original figure (compare the positions of A and A' in Fig. 5.1 and 5.2)?

Points X and Y lie on a straight line AB.



Given that AX : XB = 1 : 2 and AY : YX = 2 : 3, write down the ratio AY : XB.

Finding the Centre of Enlargement

Fig. 5.3 shows how to find the centre of enlargement and the scale factor, with $ABCD$ as the original figure and $A_1B_1C_1D_1$ as the image.

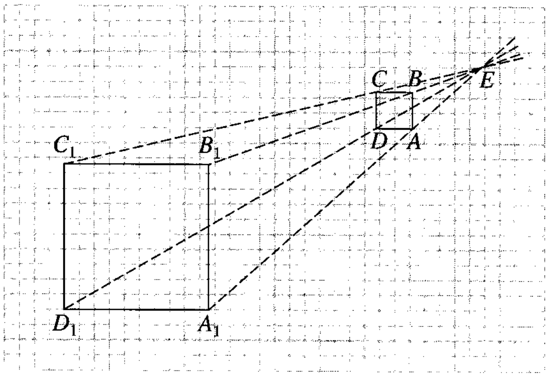


Fig. 5.3

The Babylonians (around 2000 BC) were the first to divide a circle into 360 parts called "degrees". This was the consequence of the Babylonian estimation of 360 days in a year. Moreover, the Babylonians used a sexagesimal system (base 60). Thus angles at a point equal 360°, one degree equals 60 minutes and one minute equals 60 seconds.

Steps for construction are:

(1) Join any two corresponding points from the original figure and the image (e.g. A_1A and B_1B).

(2) Extend A_1A and B_1B.

(3) The point of intersection of these lines yields the centre of enlargement E.

(4) The scale factor, $k = \frac{A_1B_1}{AB} = \frac{B_1C_1}{BC} = \frac{C_1D_1}{CD} = \frac{D_1A_1}{DA}$. Alternatively,

$$k = \frac{A_1E}{AE} = \frac{B_1E}{BE} = \frac{C_1E}{CE} = \frac{D_1E}{DE}.$$

Use the open tool, Geometer's Sketch Pad, to explore enlargement where you can alter the centre and scale factor of enlargement.

Area of Enlarged Figures

Fig. 5.4 shows $\triangle ABC$ being enlarged to $\triangle APQ$ with the centre of enlargement at A and scale factor k.

Let $AB = x$ cm and $AC = y$ cm, then $AP = kx$ cm and $AQ = ky$ cm.

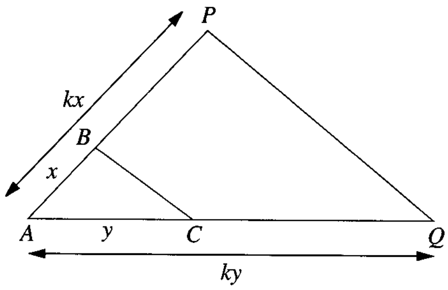


Fig. 5.4

If the radius of a sphere is doubled, calculate the factor by which its surface area and its volume are increased.

Now, area of $\triangle ABC = \frac{1}{2} \times x \times y \times \sin A$ and

$$\text{area of } \triangle APQ = \frac{1}{2} \times kx \times ky \times \sin A$$

$$\therefore \quad \frac{\text{area of } \triangle APQ}{\text{area of } \triangle ABC} = \frac{\frac{1}{2}(kx)(ky)\sin A}{\frac{1}{2}xy \sin A} = k^2$$

i.e. in general,

area of image = k^2 × (area of the original figure)

Therefore, the area of the enlarged figure is k^2 times the area of the original figure.

The centre of enlargement is invariant.

Example 7

$\triangle PQR$ has vertices $P(1, 2)$, $Q(5, 1)$ and $R(4, 4)$. An enlargement maps $\triangle PQR$ onto $\triangle PAB$. Given that the coordinates of A are $(m, 0)$, find

(a) the centre of enlargement,
(b) the value of m,
(c) the scale factor of the enlargement,
(d) the coordinates of the point B.

Express the numbers from 1 to 20 using only four 4's. You are allowed to use any mathematical symbols you know. (Note: 4! reads 4 factorial = 4 × 3 × 2 × 1)

Solution

(a) From the graph plotted, we see that P is the invariant point and hence, the centre of enlargement is (1, 2).

(b) PQ is produced to meet A on the x-axis. A is the point (9, 0).

$\therefore \quad m = 9$

(c) $PQ = QA$.

$$\therefore \quad \text{scale factor} = \frac{PA}{PQ} = \frac{2}{1} = 2$$

(d) Produce PR and mark the point B such that $PB = 2PR$. The coordinates of B are (7, 6).

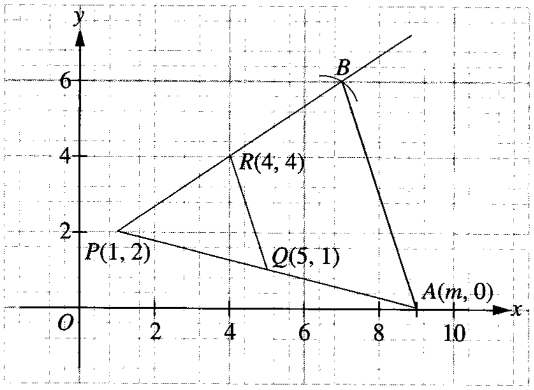


Fig. 5.5

Example 2

The image of △ABC under an enlargement is △PQR. Given that the coordinates of the triangles are A(2, 1), B(5, 1), C(3, 3), P(3, 1), Q(9, 1) and R(5, 5), draw the two triangles on a sheet of graph paper and find

(a) the coordinates of the centre of enlargement, *(b) the scale factor,*
(c) the area of △ABC and △PQR.

Solution

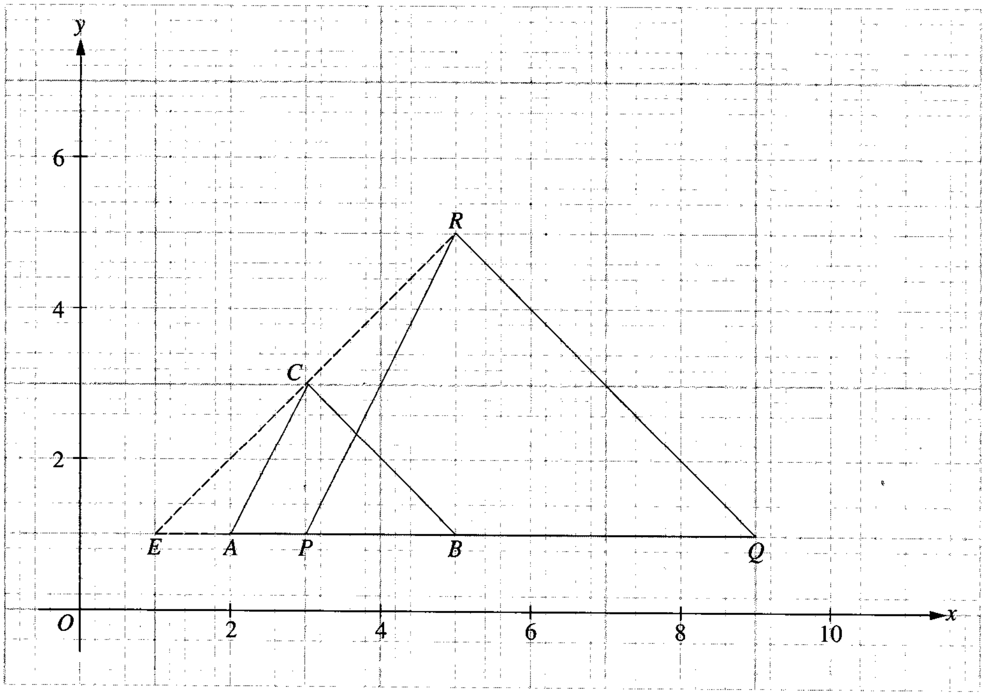


Fig. 5.6

(a) Join RC and produce; join PA and produce. RC andPA produced meet at point $E(1, 1)$. Hence, the coordinates of the centre of enlargement is (1, 1).

(b) The scale factor = $\frac{PQ}{AB} = \frac{6}{3} = 2$.

(c) Length of $AB = 3$ units, height of $\triangle ABC = 2$ units

$\therefore$ the area of $\triangle ABC = \frac{1}{2}(3)(2) = 3$ units^2

Area of $\triangle PQR = 2^2 \times (3 \text{ units}^2)$
$= 12$ units^2

Exercise 5a

1. The coordinates of the quadrilateral $ABCD$ are $A(2, 3)$, $B(6, 2)$, $C(10, 5)$ and $D(8, 8)$. Find the image of the point
   - **(a)** A under an enlargement centre at (0, 2) and scale factor 2,
   - **(b)** B under an enlargement centre at (4, 0) and scale factor 3,
   - **(c)** C under an enlargement centre at (8, 4) and scale factor –2,
   - **(d)** D under an enlargement centre at (1, 2) and scale factor $\frac{1}{2}$.
2. Find the centre of enlargement and scale factor for each of the following enlargements which map ABC onto $A'B'C'$.

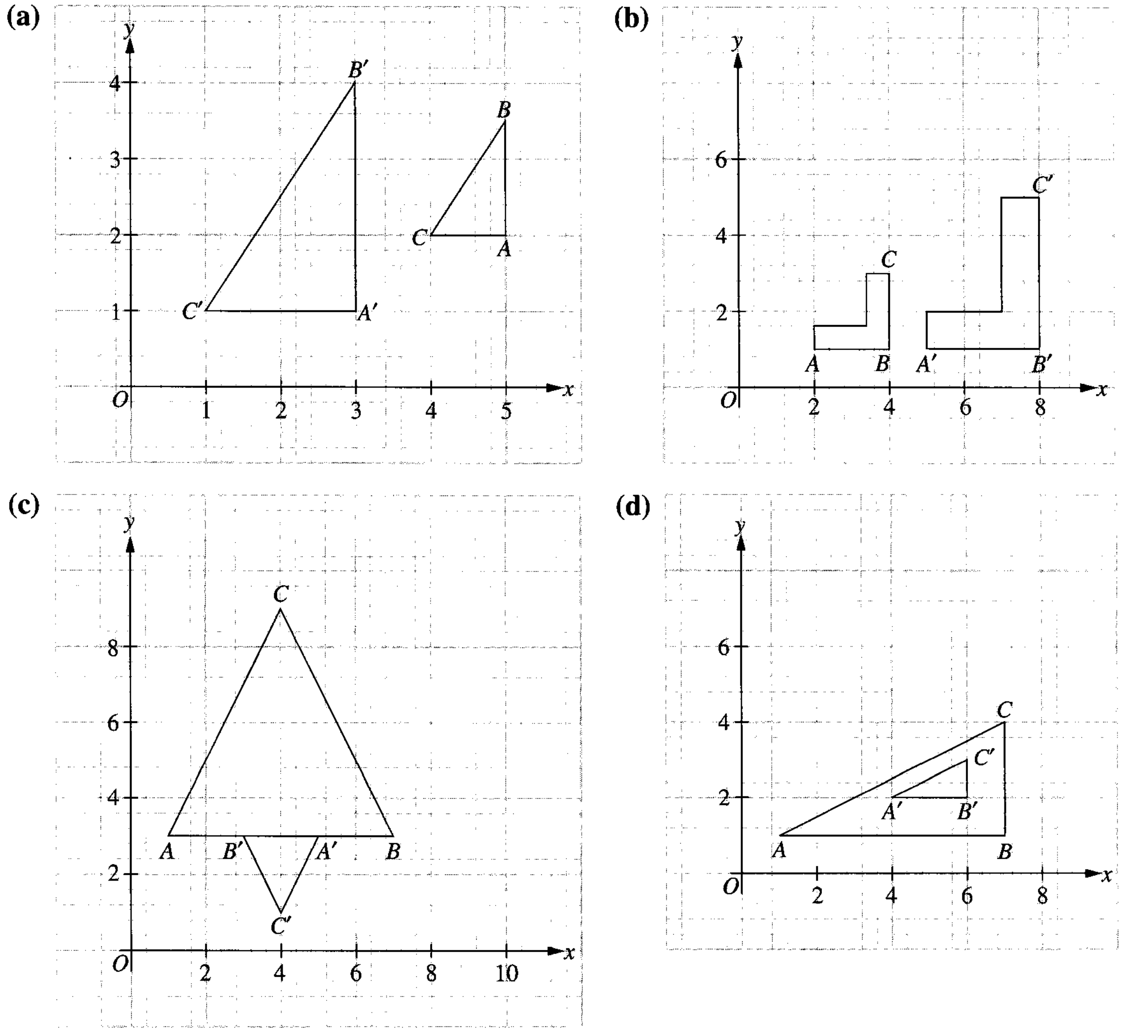


Fig. 5.7

3. $\triangle ABC$ with vertices $A(2, 1)$, $B(4, 3)$ and $C(3, 6)$ is transformed into $\triangle A'B'C'$ under an enlargement, with centre (1, 1) and scale factor 3. Illustrate these on a clearly-labelled diagram, marking the positions of $\triangle ABC$ and $\triangle A'B'C'$.

4. Enlarge the following triangles with the centre of enlargement E and scale factor k as given.
   (a) $A(1, 3)$, $B(2, 5)$ and $C(6, 1)$; $E(0, 0)$; $k = 2$
   (b) $P(1, 4)$, $Q(4, 1)$ and $R(5, 6)$; $E(1, 2)$; $k = -2$
   (c) $X(1, 1)$, $Y(2, 3)$ and $Z(4, 2)$; $E(1, 1)$; $k = 3$
   (d) $J(4, 4)$, $H(6, 7)$ and $K(3, 9)$; $E(8, 4)$; $k = \frac{1}{2}$
   (e) $L(4, 1)$, $M(4, 3)$ and $N(1, 3)$; $E(1, 0)$; $k = -3$

5. In each of the following parts, $\triangle ABC$ is mapped onto $\triangle PQR$ by an enlargement. Plot the object and image and find the centre of enlargement and the scale factor in each case:

	Object	*Image*
(a)	$A(4, 4)$, $B(7, 4)$, $C(4, 6)$	$P(1, 2)$, $Q(7, 2)$, $R(1, 6)$
(b)	$A(3, 3)$, $B(5, 1)$, $C(2, 1)$	$P(5, 7)$, $Q(11, 1)$, $R(2, 1)$
(c)	$A(10, 4)$, $B(10, 8)$, $C(6, 8)$	$P(13, 3)$, $Q(13, 9)$, $R(7, 9)$
(d)	$A(7, 3)$, $B(13, 3)$, $C(13, 0)$	$P(3, 7)$, $Q(1, 7)$, $R(1, 8)$
(e)	$A(2, 4)$, $B(5, 3)$, $C(4, 6)$	$P(1, 6)$, $Q(7, 4)$, $R(5, 10)$
(f)	$A(3, 1)$, $B(6, 3)$, $C(1, 4)$	$P(6, 13)$, $Q(0, 9)$, $R(10, 7)$

6. The vertices of $\triangle ABC$ are $A(1, 1)$, $B(1, -1)$ and $C(2, 2)$. Under an enlargement, $\triangle ABC$ is mapped onto $\triangle PQR$ whose vertices have coordinates $P(3, 2)$, $Q(3, -2)$ and $R(5, 4)$. Plot these two triangles on a sheet of graph paper and find
   (a) the coordinates of the centre of enlargement,
   (b) the scale factor.

7. Under an enlargement with centre (4, 3) and scale factor –2, the line AB with coordinates $A(9, 2)$ and $B(6, 6)$ is mapped onto the line PQ. Find
   (a) the coordinates of P and Q,
   (b) the length of PQ.

8. $\triangle ABC$ is enlarged onto $\triangle A'B'C'$ with the origin as centre of enlargement and scale factor –2. If the coordinates of $\triangle A'B'C'$ are $A'(-2, -2)$, $B'(-10, -4)$ and $C'(-4, -6)$, find the coordinates of $\triangle ABC$.

9. $\triangle ABC$ is enlarged onto $\triangle A'B'C'$ with B as the centre of enlargement and scale factor –2. If the coordinates of $\triangle A'B'C'$ are $A'(7, 7)$, $B'(3, 3)$ and $C'(9, 3)$, and B is the point (3, 3), find the coordinates of A and C.

10. The coordinates of $\triangle ABC$ are $A(1, 1)$, $B(7, 1)$ and $C(5, 6)$. $\triangle ABC$ is mapped onto $\triangle APR$ by an enlargement of scale factor 3. Find
    (a) the coordinates of the centre of enlargement,
    (b) the area of $\triangle APR$.

11. In Fig. 5.8, $ABPQ$ is a straight line. The square $PQRS$ is the image of $ABCD$ under an enlargement. Given that $AB = 10$ cm, $BP = 10$ cm and $RQ = 20$ cm, find the distance of the centre of enlargement from the point A.

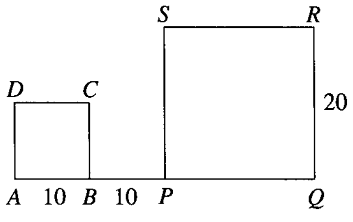


Fig. 5.8

12. Fig. 5.9 shows the front view of a hut. Copy this diagram and using K as the centre of enlargement, construct a similar view of the hut four times the original area.

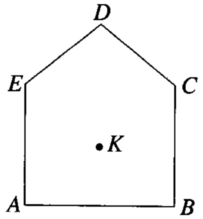


Fig. 5.9

13. In Fig. 5.10, $\triangle ABC$ is enlarged to $\triangle PQR$ with scale factor 3 and centre of enlargement E. Given that $BP = 4$ cm, $AB = 8$ cm, $BC = 6$ cm, $PQ = 24$ cm and $QR = 18$ cm, copy the diagram (not drawn to scale), using a scale of 1 cm to represent 4 cm, and locate E. Measure the length of EA.

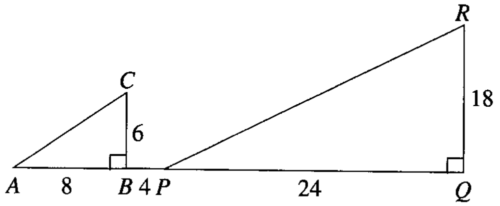
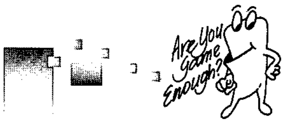


Fig. 5.10

★14. Draw accurately the parallelogram $ABCD$ in which $AD = 5$ cm, $DC = 9$ cm, and $A\hat{D}C = 60°$. Mark the point L on AB such that $AL = 3$ cm and draw the lines LD and AC to meet at O.

(a) State the scale factor and the centre of the enlargement which maps $\triangle ALO$ onto $\triangle CDO$.

(b) Draw a square which has one corner on OD, one corner on OC and two of its sides, both on the same side of O, parallel to DC. Hence draw the square which has one corner on OD, one corner on OC and two corners on DC. *(C)*



(1) How many pupils must you gather together so as to ensure that at least two pupils have birthdays falling in the same month?

(2) A secondary school has an enrolment of 1 100. Is it possible that there will be four pupils in the school whose birthdays fall on the same day of the year? Explain your answers.

Stretch

We shall now look at another non-isometric transformation, **stretch.** We shall only consider stretching along the x- and y-axes.

Hold up a rubber band with AB stationary and use another hand to stretch the points of CD to $C'D'$.

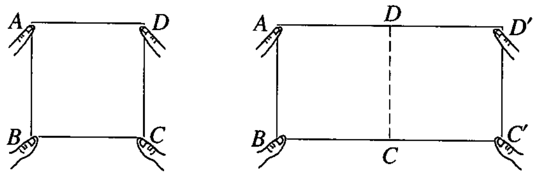


Fig. 5.11

This is an example of a stretch along the line BC or a stretch with AB as the invariant line.

Hold up another rubber band with PQ stationary and stretch the rubber band along PS to obtain $PQR'S'$.

This is an example of a stretch along PS or a stretch with PQ as the invariant line.

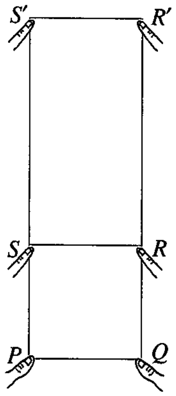


Fig. 5.12

Fig. 5.13 shows that the rectangle $OABC$ is mapped onto rectangle $OA'B'C$. Notice that every point in the rectangle is moved in the direction parallel to the x-axis except the points on the y-axis. The y-axis is the invariant line under this transformation.

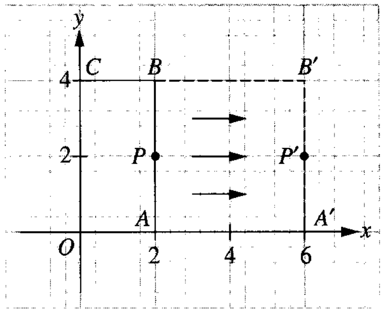


Fig. 5.13

The ratio $\left(\dfrac{\textbf{distance of } P' \textbf{ from the invariant line}}{\textbf{distance of } P \textbf{ from the invariant line}}\right)$ is a constant, where P is any point on AB and P' is the image of P.

i.e. $$\frac{OA'}{OA} = \frac{CB'}{CB} = \frac{6}{2} = 3.$$

The constant 3 is called the *stretch factor.*

We call this transformation **a stretch parallel to the x-axis** with stretch factor 3. Alternatively, it is also called a stretch with the y-axis as the invariant line and with stretch factor 3.

Notice that stretch will change both the shape and size of a figure. What is the value of $\left(\dfrac{\text{area } OA'B'C}{\text{area } OABC}\right)$?

If the stretch factor is –3, where will the image lie?

Fig. 5.14 shows that the rectangle $OABC$ is mapped onto rectangle $OAB'C'$. Notice that every point in the rectangle $OABC$ is moved in the direction parallel to the y-axis except the points on the x-axis. The x-axis is the invariant line under this transformation. Now

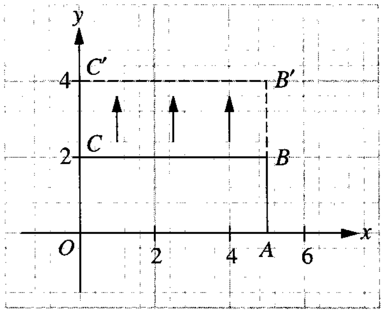


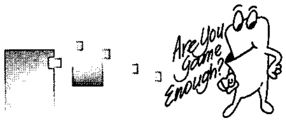
Fig. 5.14

$$\frac{OC'}{OC} = \frac{AB'}{AB} = \frac{4}{2} = 2 = \text{stretch factor.}$$

We call this transformation **a stretch parallel to the y-axis** with stretch factor 2. Alternatively, it is also called a stretch with the x-axis as the invariant line and stretch factor 2.

What is the value of $\left(\dfrac{\text{area } OAB'C'}{\text{area } OABC}\right)$?

If the stretch factor is –2, where will the image lie?



Adrian, Ben and Christopher decided to do some homework for the day. One did Additional Maths homework, one did Biology homework and one did Chemistry homework. Only one of the following four statements is true.

(1) Christopher did Additional Maths homework.
(2) Christopher did not do Biology homework.
(3) Adrian did not do Biology homework.
(4) Adrian did not do Chemistry homework.

Use the above information to deduce who did which homework for the day.

Example 3

For each of the following stretches,

(a) state the equation of the invariant line,
(b) find the stretch factor.

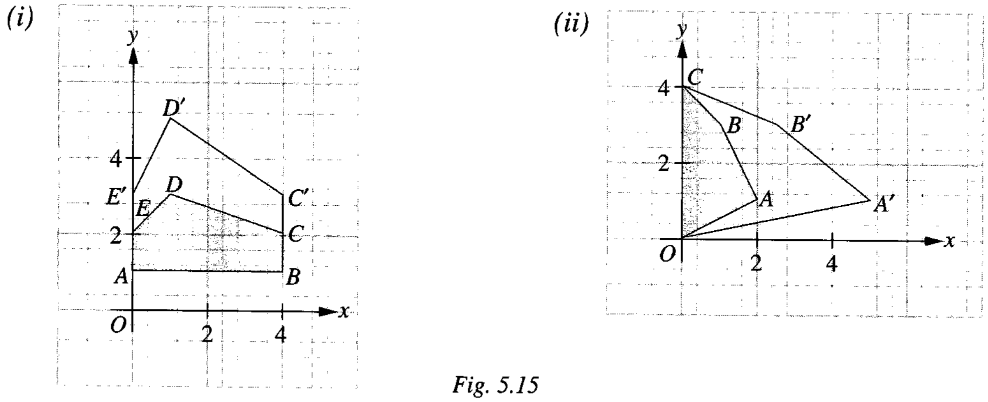


Fig. 5.15

Solution

(i) (a) A and B remain invariant and they are on the line $y = 1$.

$\therefore$ the invariant line is $y = 1$.

(b) $\frac{AE'}{AE} = \frac{2}{1} = 2.$

$\therefore$ the stretch factor is 2.

(ii) (a) The points on the y-axis are invariant.

$\therefore$ the invariant line is $x = 0$.

(b) $\frac{\text{distance of } A' \text{ from } x = 0}{\text{distance of } A \text{ from } x = 0} = \frac{5}{2} = 2\frac{1}{2}.$

$\therefore$ the stretch factor is $2\frac{1}{2}$.

Example 4

Draw on a sheet of graph paper the triangle formed by the points A(2, 1), B(2, 5) and C(4, 3). $\triangle ABC$ is transformed into $\triangle A'B'C'$ by a stretch of stretch factor 3 with invariant line x = 1. Draw $\triangle A'B'C'$ on the same diagram. What is the area of $\triangle A'B'C'$? If M is the midpoint of AC, find the image of M under this stretch.

Solution

From the graph, $HA' = 3HA$, $LC' = 3LC$ and $KB' = 3KB$.

Hence, we have $A'(4, 1)$, $B'(4, 5)$ and $C'(10, 3)$.

Area of $\triangle A'B'C'$ is $\frac{1}{2} \times 4 \times 6 = 12$ units^2.

The image of M is $\left(\frac{4 + 10}{2}, \frac{1 + 3}{2}\right) = (7, 2)$.

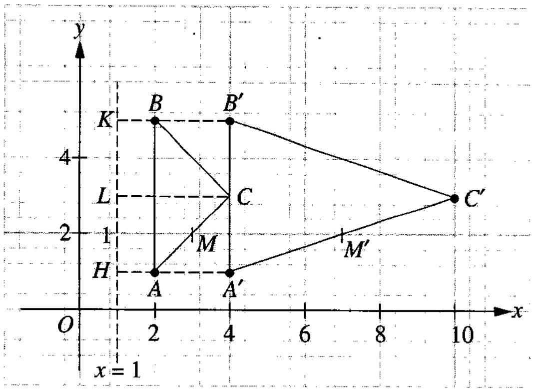


Fig. 5.16

Example 5

Draw on a sheet of graph paper, the triangle ABC with A(1, 4), B(3, 2) and C(–2, 0). △ABC is mapped onto △PQR under a stretch of factor 2 with the y-axis as the invariant line. Draw and label △PQR.

Solution

Since the stretch factor is 2 with the y-axis as the invariant line,

$LP = 2LA$, $MQ = 2MB$, and $OR = 2OC$.

$\therefore$ the coordinates of P is (2, 4), those of Q is (6, 2) and of R is (–4, 0).

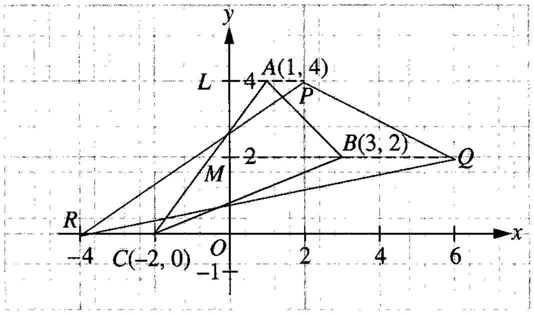


Fig. 5.17

From the example, we see that when the stretch factor is positive, a point and its image under a stretch lie on the same side of the invariant line. Similarly, when the stretch factor is negative, a point and its image under a stretch will lie on the opposite side of the invariant line.

Example 6

Draw on a sheet of graph paper, the triangle ABC with A(2, 2), B(8, 3) and C(5, –1). △ABC is mapped onto △PQR under a stretch with the x-axis as the invariant line and stretch factor –2. Draw and label △PQR.

Solution

Since the stretch factor is –2 with the x-axis as the invariant line, we have,

$LP = -2LA$, $MR = -2MC$ and $NQ = -2NB$,

where L, M and N are points on the x-axis.

The coordinates of P, Q and R are $P(2, -4)$, $Q(8, -6)$ and $R(5, 2)$.

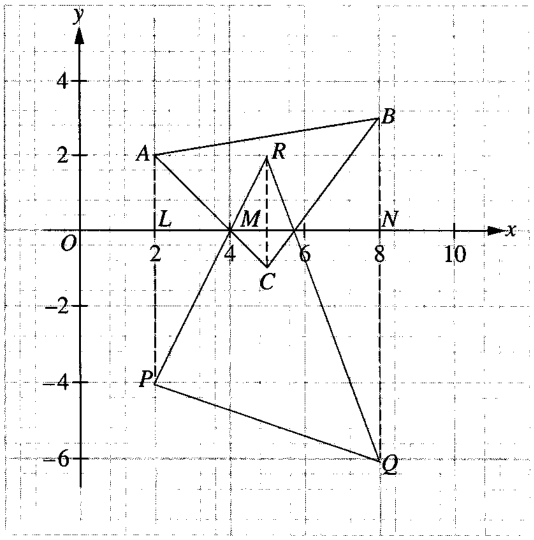


Fig. 5.18

Two-way Stretch or Double Stretch

Consider holding up a rubber band $ABCD$ in the form of a square. The rubber band undergoes a stretch with AD as the invariant line and stretch factor 3 to obtain $APSD$.

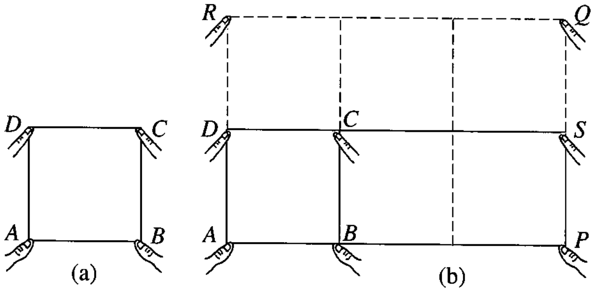


Fig. 5.19

$APSD$ undergoes another stretch with AP as the invariant line and stretch factor 2 to obtain $APQR$.

We say that $ABCD$ is transformed onto $APQR$ by a **two-way stretch** with stretch factor 3 along AB and stretch factor 2 along AD. This is the result of two one-way stretches where the invariant lines are perpendicular to one another. Hence, there are no invariant lines, leaving only one invariant point, the point of intersection of the two invariant lines.

If the stretch factors of a two-way stretch are equal, what would you call such a transformation? What will be the area of the image of a figure undergoing a two-way stretch with stretch factors k_1 and k_2?

Example 7

The diagram shows the pentagon ABCDE being transformed onto APQRS. Describe the transformation clearly.

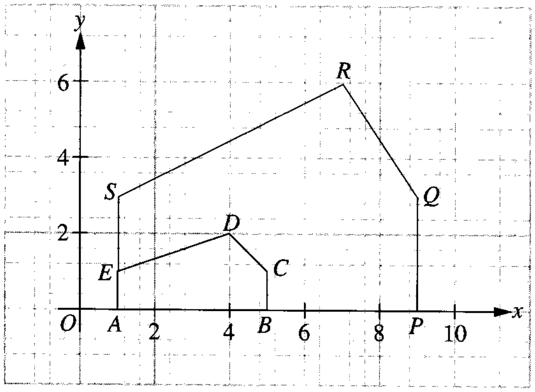


Fig. 5.20

Solution

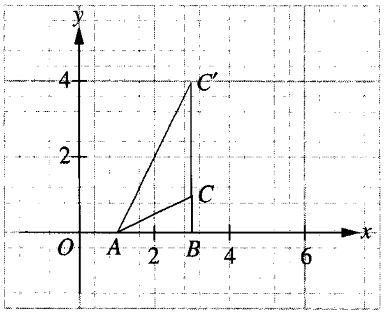
Notice that $AP = 2AB$, and $AS = 3AE$. The transformation that maps $ABCDE$ onto $APQRS$ is a two-way stretch with stretch factor 2 and invariant line $x = 1$ followed by a stretch with stretch factor 3 and the x-axis as the invariant line.

The invariant point is $A(1, 0)$, the point of intersection of the invariant lines $x = 1$ and $y = 0$.

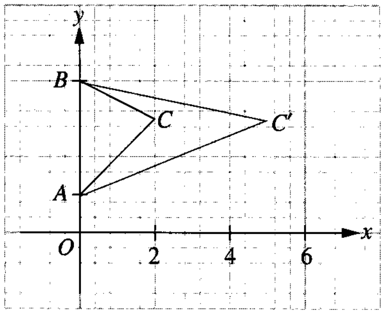
Exercise 5b

1. Describe each of the following stretches that maps $\triangle ABC$ onto $\triangle ABC'$.

(a)

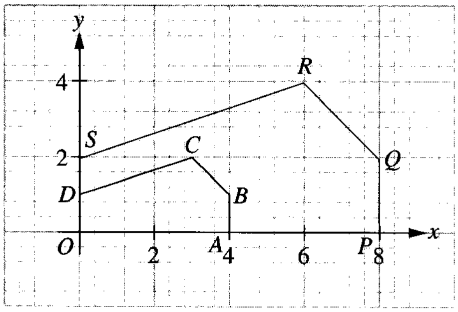


(b)

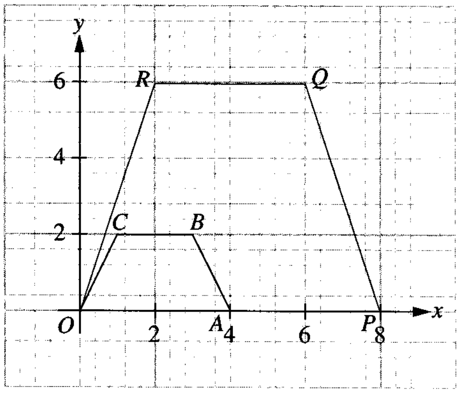


2. Describe each of the following stretches completely.

(a)

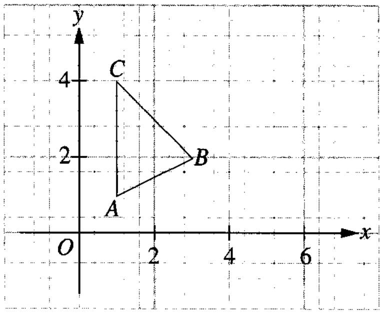


(b)

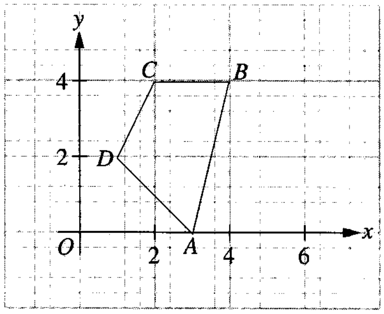


3. For each of the following, draw the image of the figure under a stretch with the y-axis as the invariant line and stretch factor 2.

(a)

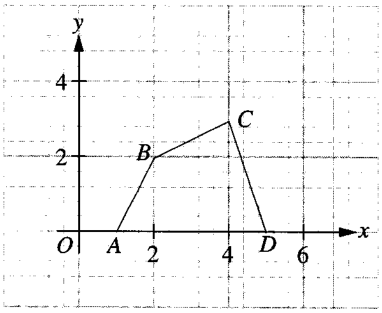


(b)

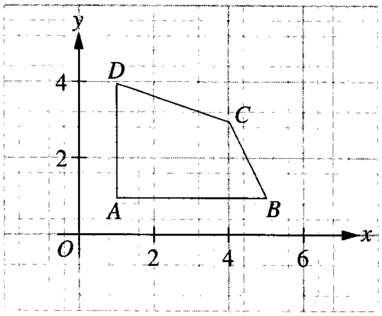


4. For each of the following, draw the image of the figure under a stretch with the x-axis as the invariant line and stretch factor $2\frac{1}{2}$.

(a)

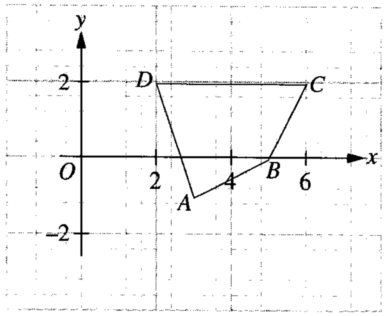


(b)



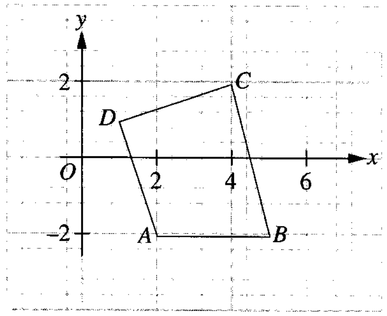
5. Draw the image figure under each of the given stretches.

(a)



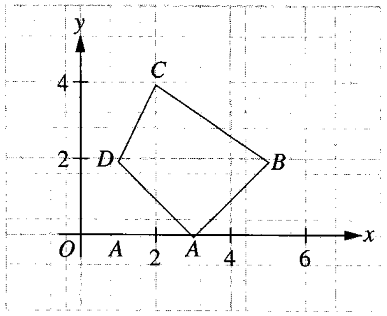
A stretch with y = 1 as the invariant line and stretch factor 2.

(b)



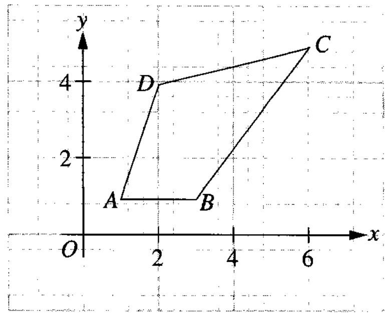
A stretch with y = −1 as the invariant line and stretch factor −2.

(c)



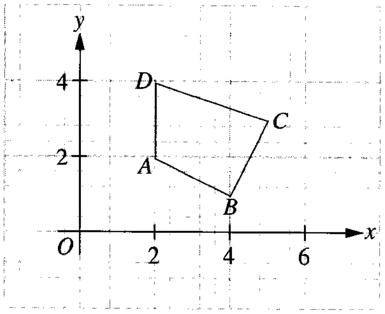
A stretch with x = 3 as the invariant line and stretch factor 2.

(d)



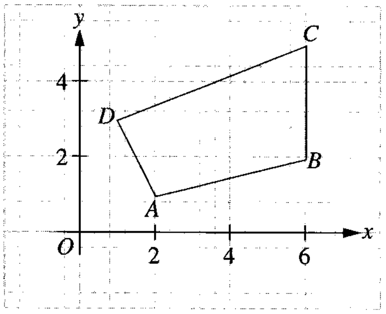
A stretch with x = 2 as the invariant line and stretch factor $-2\frac{1}{2}$.

(e)



A two-way stretch with x = 1 as the invariant line, stretch factor 2 and with y = 1 as the invariant line and stretch factor $1\frac{1}{2}$.

(f)



A two-way stretch with x = 4 as the invariant line, stretch factor 2 and with y = 2 as the invariant line, stretch factor −2.

6. Draw on a sheet of graph paper the triangle formed by the points (1, 0), (4, 0) and (3, 2) and also its images under the following one-way stretches.

   (a) invariant line the x-axis; scale factor 2,

   (b) invariant line the y-axis; scale factor 3.

   State the coordinates of the image points in each case.

7. Plot the points $A(0, 0)$, $B(2, 0)$ and $C(2, 2)$ on a sheet of graph paper. $\triangle ABC$ is mapped onto $\triangle PQR$ by a stretch with the x-axis as the invariant line and stretch factor 3. Plot the points P, Q and R on the same graph paper.

8. Plot the points $A(1, 1)$, $B(3, 1)$, $C(3, 6)$ and $D(1, 6)$ on a sheet of graph paper. $ABCD$ is mapped onto $PQRS$ by a two-way stretch with the x-axis as the invariant line, stretch factor 2 and with the y-axis as the invariant line and stretch factor 3. Plot the points P, Q, R and S on the same graph paper.

9. The triangle with vertices (2, 4), (2, –2) and (6, 0) is subjected to a one-way stretch whose invariant line is the y-axis. If the stretch maps the point (2, 0) onto (3, 0), find the area of the image triangle.

10. Draw on a sheet of graph paper $\triangle PQR$ with vertices $P(3, 1)$, $Q(3, 6)$ and $R(5, 3)$. $\triangle PQR$ is mapped onto $\triangle P'Q'R'$ by a stretch of factor 3 with $x = 2$ as the invariant line. Find the images of $\triangle PQR$ by construction.

11. Draw on a sheet of graph paper $\triangle ABC$ with vertices $A(2, 3)$, $B(5, 3)$ and $C(4, 4)$. $\triangle ABC$ is mapped onto $\triangle A'B'C'$ by a stretch of factor 2 with $y = 1$ as the invariant line. Draw $\triangle A'B'C'$ and find the area of $\triangle A'B'C'$.

12. S is a two-way stretch with y-axis as the invariant line, stretch factor 2 and x-axis as the invariant line and stretch factor x.
    - **(a)** Find the value of x when S maps (2, 1) onto (4, 3).
    - **(b)** With the value of x found in (a), find the coordinates of the point which will be mapped onto (2, 18) by S.
    - **(c)** Describe completely the transformation S would represent if $x = 2$.

Shear

A shear is another non-isometric transformation. Consider the side view of a pile of identical books placed in order as shown in Figure 5.21.

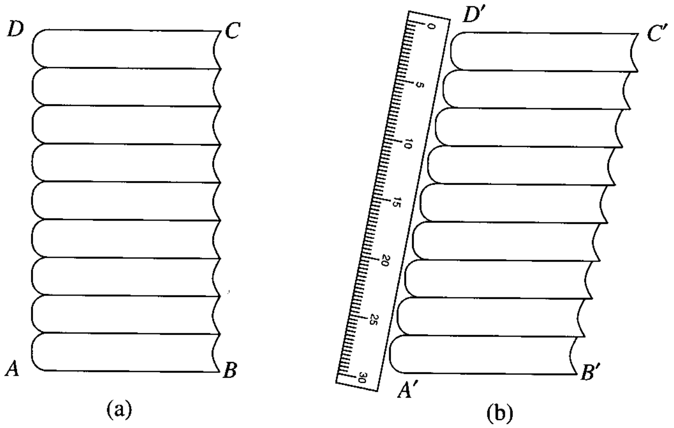


Fig. 5.21

With the help of a ruler, we can push the pile of books so that the book at the bottom remains stationary, while those above it are pushed progressively to the right.

The side view of the displaced books now looks like a parallelogram.

Consider the side view of a stack of playing cards when it is pushed sideways as shown in Fig. 5.22.

$ABCD$ represents the side view of a stack of playing cards. When it is pushed sideways parallel to AB, the side-view becomes $ABC'D'$ and the figure changes from a rectangle to a parallelogram. This transformation is called a **shear** parallel to AB. The invariant line is AB.

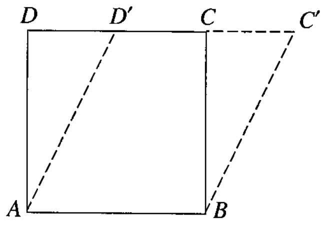


Fig. 5.22

Notice that the area of $ABCD$ is the same as the area of $ABC'D'$. Thus shearing is a non-isometric transformation that preserves the area of the figure.

The diagram below shows figure $OABC$ being transformed by a shear to $OAB'C'$.

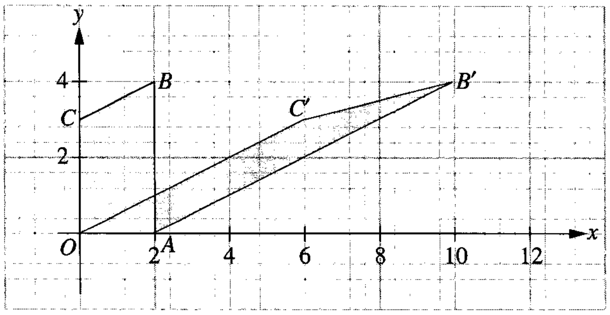


Fig. 5.23

Notice that the points on the x-axis remain unchanged under the shear. The x-axis is the invariant line.

If C' is the image of C and B' the image of B under the shear, the ratio $\left(\dfrac{\textbf{distance of } C \textbf{ from } C'}{\textbf{distance of } C \textbf{ from the invariant line}} \right)$ is a constant, i.e. $\dfrac{CC'}{OC} = \dfrac{BB'}{BA} = \dfrac{6}{3} = \dfrac{8}{4} = 2.$

The constant is called the **shear factor**.

We call this transformation a shear parallel to the x-axis with shear factor 2. Alternatively, it is also called a shear with the x-axis as the invariant line and shear factor 2.

Notice that all the other points in Fig. 5.24 are moved parallel to the invariant line in such a way that straight lines are mapped onto straight lines and parallel lines remain parallel. Thus, points on opposite sides of the invariant line move in opposite directions. Fig. 5.24 shows that P is mapped onto P' and Q onto Q'.

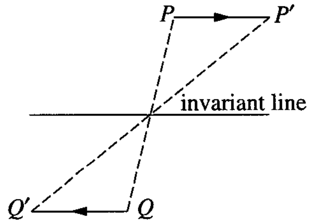
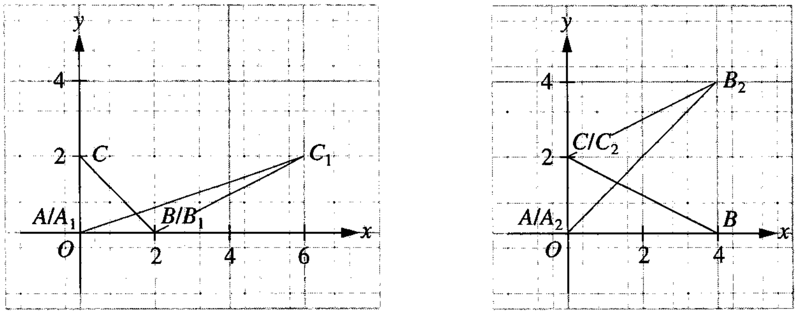


Fig. 5.24

Example 8

Describe each of the following shears completely.



(a)

(b)

Fig. 5.25

Solution

(a) The points on the x-axis are invariant.

The shear factor = $\frac{CC_1}{OC} = \frac{6}{2} = 3$.

$\therefore$ the transformation is a shear with the x-axis as the invariant line and shear factor 3.

(b) The points on the y-axis are invariant.

The shear factor = $\frac{BB_2}{OB} = \frac{4}{4} = 1$.

$\therefore$ the transformation is a shear with the y-axis as the invariant line and shear factor 1.

Example 9

The diagram shows a trapezium ABCD. S is a shear with the x-axis as the invariant line and shear factor 2. Draw the image of ABCD under S.

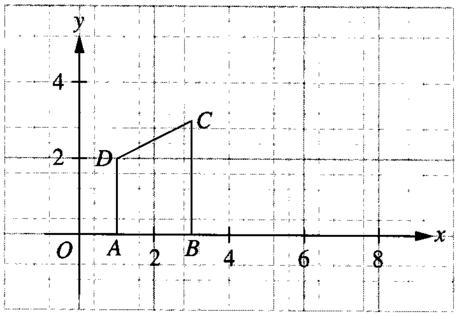


Fig. 5.26

Solution

Since the invariant line is the x-axis, A and B will remain unchanged under S. C and D will move parallel to the x-axis to C' and D'such that $\frac{CC'}{BC} = 2$ and $\frac{DD'}{AD} = 2$.

i.e. $\frac{CC'}{3} = 2$ and $\frac{DD'}{2} = 2$.

$\therefore$ $CC' = 6$ units and $DD' = 4$ units.

$\therefore$ C' is the point (9, 3) and $D'(5, 2)$.

Notice that a straight line and its image always meet on the invariant line. In Fig. 5.27, DA and DA' meet on the x-axis and so do CB and $C'B'$.

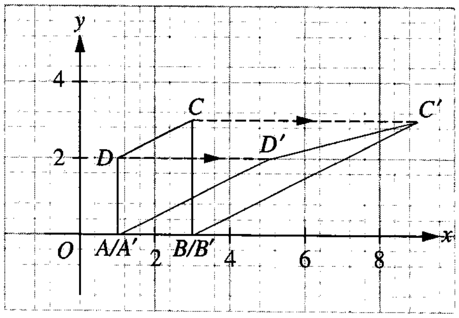


Fig. 5.27

Example 10

The coordinates of $\triangle ABC$ are $A(2, 3)$, $B(4, 3)$ and $C(4, 6)$. Under a shear, $\triangle ABC$ is mapped onto $\triangle PQR$ with coordinates $P(2, 5)$, $Q(4, 7)$ and $R(4, 10)$. Plot the two triangles on a sheet of graph paper and find the equation of the invariant line and the shear factor.

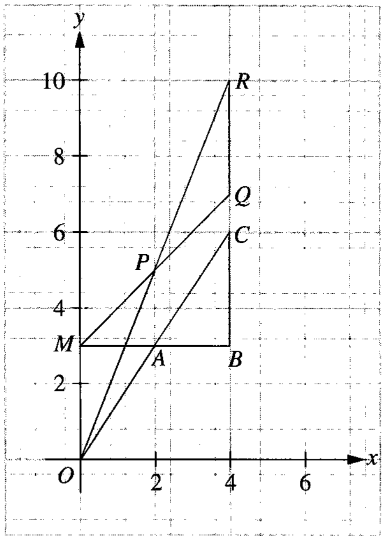


Fig. 5.28

Solution

Under a shear, a straight line and its image, which are not parallel, always meet at a point on the invariant line. BA produced and QP produced meet at M on the y-axis. CA produced and RP produced meet at O on the y-axis. Thus, the y-axis is the invariant line and its equation is $x = 0$.

The shear factor $= \frac{BQ}{BM} = \frac{4}{4} = 1$.

Example 11

Under a shear, $\triangle ABC$ whose coordinates are $A(2, 4)$, $B(5, 4)$ and $C(3, 6)$ is mapped onto $\triangle PQR$ with coordinates $P(6, 4)$, $Q(9, 4)$ and $R(11, 6)$. Plot the vertices of $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper. Find the equation of the invariant line and find the shear factor.

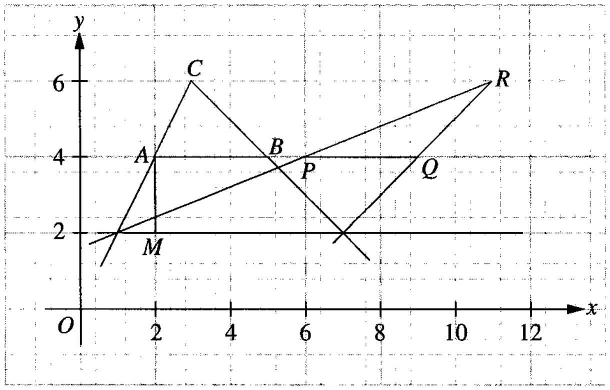


Fig. 5.29

Solution

CA produced and RP produced meet at a point on the line $y = 2$.

CB produced and RQ produced also meet at a point on the line $y = 2$.
$\therefore$ the equation of the invariant line under the shear is $y = 2$.

The shear factor $= \frac{AP}{AM} = \frac{4}{2} = 2$.

Example 12

In Fig. 5.30, $\triangle OAB$ is mapped onto $\triangle OAB'$ by a shear with the x-axis as the invariant line.

(a) Calculate the area of $\triangle OAB'$.

(b) Given that $\frac{\text{area of } \triangle OBB'}{\text{area of } \triangle OAB} = 3$, find the coordinates of B'.

(c) Calculate the shear factor.

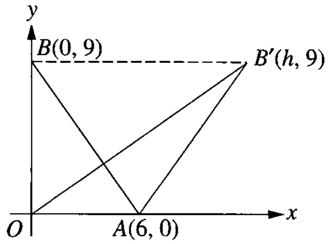


Fig. 5.30

Solution

(a) Area of $\triangle OAB' = \frac{1}{2}(6)(9) = 27$ units^2.

(b) Let the coordinates of B' be $(h, 9)$.

$$\frac{\text{area of } \triangle OBB'}{\text{area of } \triangle OAB} = \frac{\frac{1}{2}(9)(h)}{\frac{1}{2}(9)(6)} = 3.$$

$\therefore \quad h = 18$, i.e. the image of B is $B'(18, 9)$.

(c) Shear factor $= \frac{BB'}{OB} = \frac{18}{9} = 2.$

You can create interesting animations with the open tool, Geometer's Sketch Pad (GSP). Explore shearing with GSP. (Refer to Appendix A on page 346 for a sample worksheet on how to do animation on shearing.)

Example 13

In Fig. 5.31, the coordinates of $\triangle ABC$ are $A(2, 1)$, $B(4, 1)$ and $C(4, 5)$. Plot these points on a sheet of graph paper. $\triangle ABC$ is sheared with $y = 3$ as the invariant line such that C' is $(7, 5)$. Find the coordinates of A' and B' by drawing the image triangle.

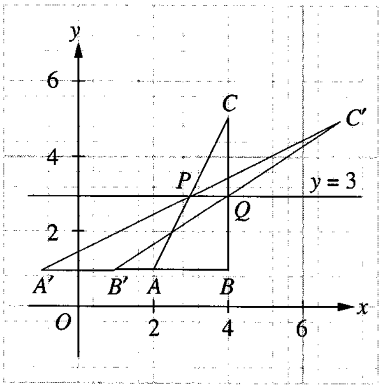


Fig. 5.31

Solution

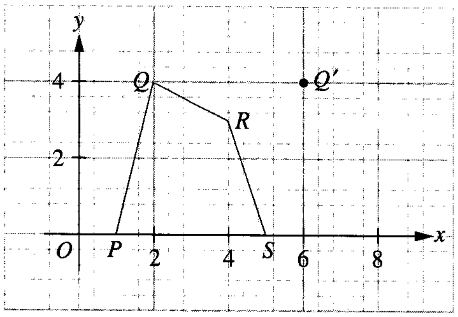
Let the invariant line $y = 3$ meets BC at Q and AC at P. Join $C'P$, $C'Q$ and produce the two lines to meet BA produced at A' and B' respectively.

From Fig. 5.31, the coordinates of A' and B' are $(-1, 1)$ and $(1, 1)$ respectively.

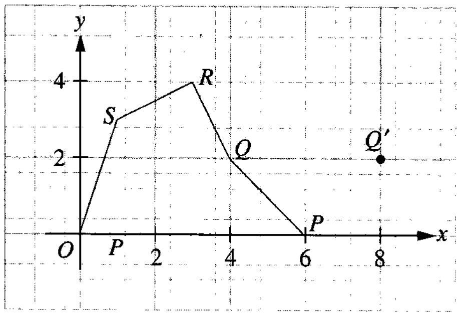
Exercise 5c

1. For each of the following, draw the image of the given figure under a shear with the x-axis as the invariant line which maps Q onto Q'. In each case, find the shear factor.

(a)

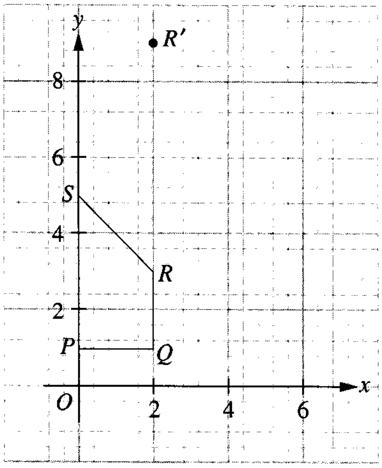


(b)

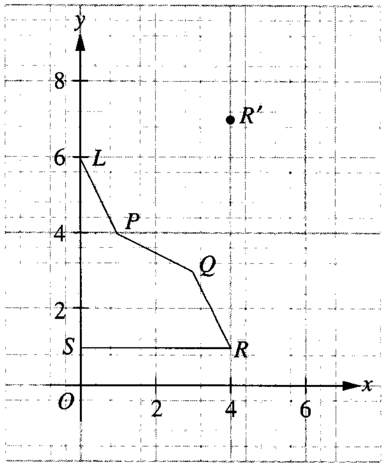


2. For each of the following, draw the image of the given figure under a shear with the y-axis as the invariant line which maps R onto R'. In each case, find the shear factor.

(a)

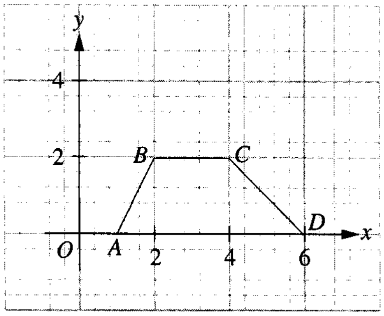


(b)



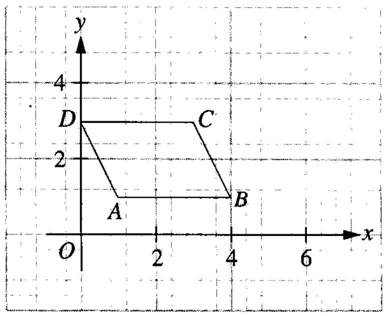
3. For each of the following, draw the image of the quadrilateral under a shear with the indicated invariant line and shear factor.

(a)



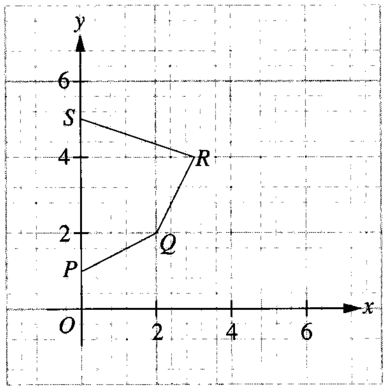
Invariant line is the x-axis.
Shear factor = 2

(b)



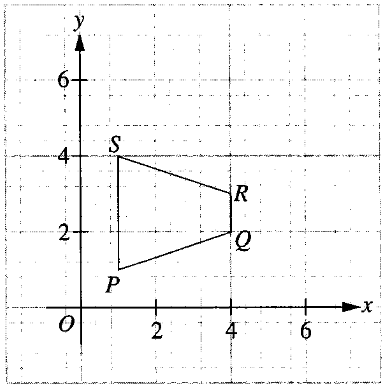
Invariant line is the x-axis.
Shear factor = $1\frac{1}{2}$

(c)



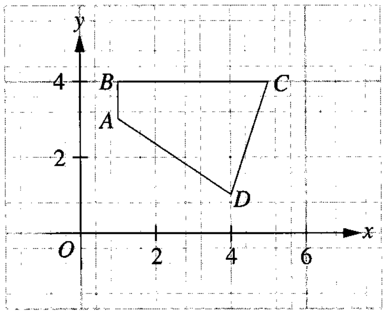
Invariant line is the y-axis.
Shear factor = 1

(d)



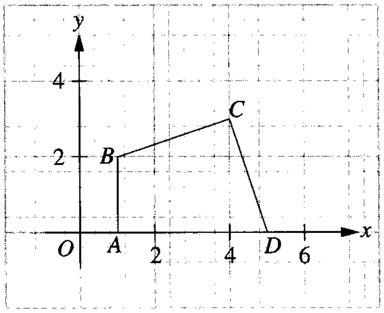
Invariant line is the y-axis.
Shear factor = 2

(e)



Invariant line is $x = 1$.
Shear factor = $\frac{1}{2}$

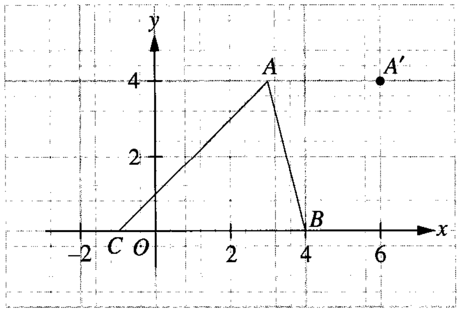
(f)



Invariant line is $y = -1$.
Shear factor = $1\frac{1}{2}$

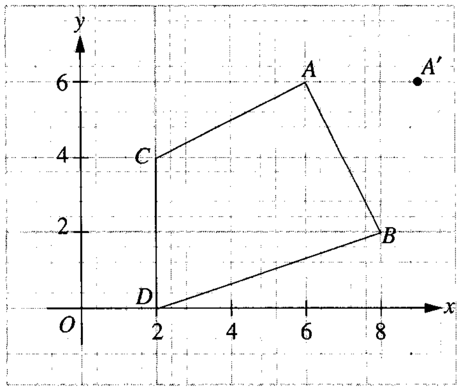
4. For each of the following, draw the image of the given figure under a shear with the indicated line as the invariant line which maps A onto A'. In each case, find the shear factor.

(a)



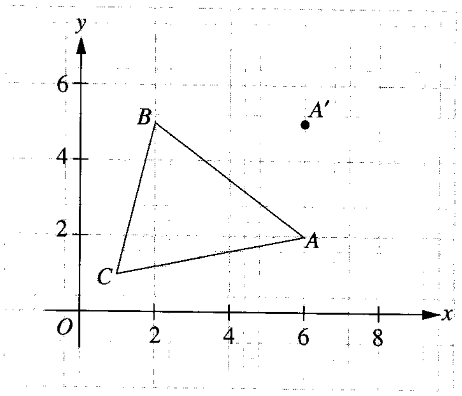
Invariant line is $y = 2$.

(b)



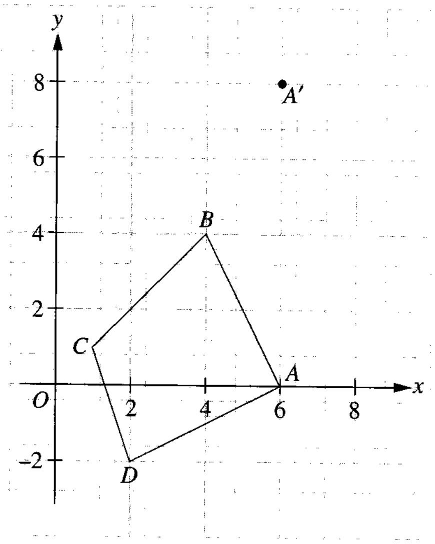
Invariant line is $y = 4$.

(c)



Invariant line is x = 4.

(d)

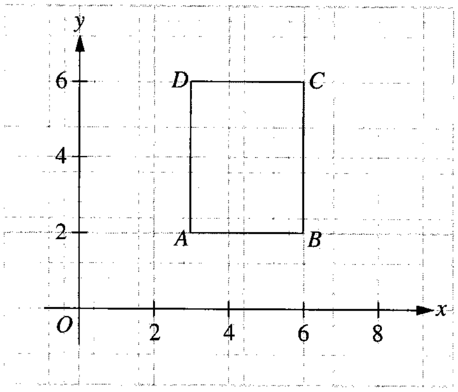


Invariant line is x = 2.

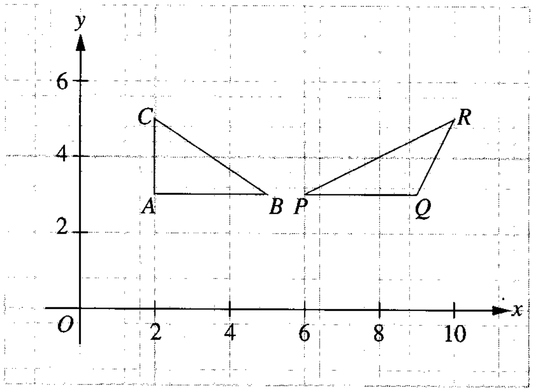
5. Draw on a sheet of graph paper the triangle with vertices (0, 2), (4, 2) and (0, 4). Draw also its images under the shear with the x-axis as the invariant line which maps the point (2, 1) onto (4, 1). State the coordinates of the image points.
6. Draw on a sheet of graph paper the figure $ABCD$ with vertices $A(2, 0)$, $B(2, -1)$, $C(-2, -1)$ and $D(-2, -3)$. Draw also the image of $ABCD$ under a shear with the line $y = -1$ as the invariant line and shear factor 2. State the coordinates of the image of $ABCD$ under the shear.
7. A shear with the line $y = 0$ as the invariant line and shear factor k maps the point (–3, 1) onto a point M on the y-axis. Find the value of k.
8. A shear H with the x-axis as the invariant line and shear factor $2x$ maps the point (1, 2) onto (5, 2). Find
   (a) the value of x, **(b)** the coordinates of the point P whose image under H is (12, 7).
9. A shear S with the y-axis as the invariant line and shear factor –1 maps the point (2, 3) onto P'. Find
   (a) the coordinates of P',
   (b) the coordinates of the point which will be mapped onto (3, –2) under S.
10. Draw on a sheet of graph paper $\triangle ABC$ with vertices $A(2, 6)$, $B(2, -1)$ and $C(6, -1)$. Draw also its images under the shear with the line $y = 2$ as the invariant line which maps $A(2, 6)$ onto $A'(6, 6)$. State the coordinates of B' and C'.
11. A shear H with $y = 0$ as the invariant line maps (2, 1) onto (5, 1). Calculate the image of the point (0, 4) under H.
12. H is a shear with the y-axis as the invariant line and shear factor –2. Given that H maps the point $P(2, 4)$ onto a point P' on the x-axis, state the coordinates of P'. H also maps the point $Q(-3, -5)$ onto Q'. Find the coordinates of Q'.
13. The square $OABC$ with vertices $O(0, 0)$, $A(1, 0)$, $B(1, 1)$ and $C(0, 1)$ is mapped onto the parallellogram $OPQR$ with vertices $O(0, 0)$, $P(1, 0)$, $Q(6, 1)$ and $R(5, 1)$ respectively under a transformation P.
    (a) Describe P completely. **(b)** Find the image of the point (4, 7) under P.
14. A shear H has the x-axis as the invariant line and shear factor 3.
    (a) Calculate the image of (2, 1) under H.
    (b) $\triangle ABC$ has vertices $A(1, 2)$, $B(1, -2)$ and $C(4, -2)$. Find the image of $\triangle ABC$ under H. Represent the coordinates of $\triangle ABC$ and its image on a sheet of graph paper.

15. P is the set of all points on the line $x + y = 3$. Under a shear with the y-axis invariant and shear factor 2, the set P is mapped onto the set Q. Describe the set Q geometrically and write down the coordinates of the invariant point of the transformation.

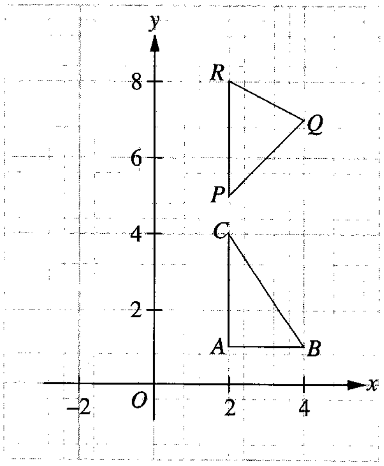
16. The diagram shows a rectangle with vertices $A(3, 2)$, $B(6, 2)$, $C(6, 6)$ and $D(3, 6)$.
    - **(a)** Construct the image of $ABCD$ under a shear with the x-axis as the invariant line and the image of A is $A'(5, 2)$.
    - **(b)** Construct the image of $ABCD$ under a shear with $y = 2$ as the invariant line and the image of C is $C'(10, 6)$.
    - **(c)** Construct the image of $ABCD$ under a shear with $y = 3$ as the invariant line and the image of D is $D'(9, 6)$.



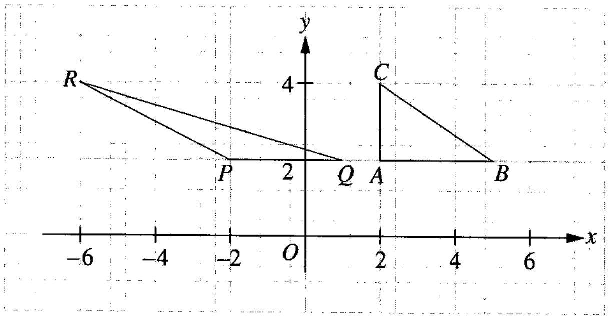
17. $\triangle ABC$ is mapped onto $\triangle PQR$ under a shear. Find
    - **(a)** the equation of the invariant line,
    - **(b)** the shear factor.



18. $\triangle ABC$ is mapped onto $\triangle PQR$ under a shear. Find
    - **(a)** the equation of the invariant line,
    - **(b)** the shear factor.



19. Under a shear, $\triangle ABC$ is mapped onto $\triangle PQR$. Find
    - **(a)** the equation of the invariant line,
    - **(b)** the shear factor.



Combining Transformations

In Chapter 4 as well as the earlier part of this chapter, we studied each of the various transformations on its own. We shall now examine the effects of combining some of these transformations.

First, let us introduce the concept of inverse transformation.

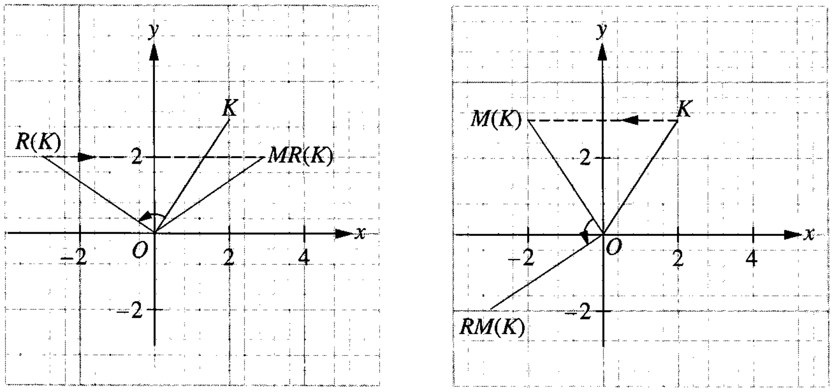
If P represents the transformation that maps the figure A onto the figure B, then the transformation Q that will map figure B onto A is called the inverse transformation of P, written as P^{-1}, (i.e. $Q = P^{-1}$).

Now, let us look at how transformations can be combined.

If M represents a reflection in the *y*-axis and R represents a 90° anticlockwise rotation about the origin, then MR represents a 90° anticlockwise rotation about the origin followed by a reflection in the *y*-axis while RM denotes a reflection in the *y*-axis followed by a 90° anticlockwise rotation about the origin.

Consider the point *K*(2, 3). Under MR, *K* will be mapped onto (–3, 2) under R and then onto (3, 2) under M, i.e. MR(2, 3) = (3, 2) [see Fig. 5.32(a)].

Under RM, *K*(2, 3) will be mapped onto (–2, 3) under M and then onto (–3, –2) under R, i.e. RM(2, 3) = (–3, –2) [see Fig. 5.32(b)].



(a) (b)

Fig. 5.32

i.e. We observe that MR ≠ RM.

Note: In general, the combination of two transformations is non-commutative.

In most instances, we use symbols to represent transformations in order to simplify statements. For example, if we represent the enlargement with centre at origin and scale factor 2 as E and a shear along the x-axis as H, then HE represents the transformation of an enlargement followed by a shear and EH represents a shear followed by an enlargement, HH (normally written as H^2) represents a shear followed by another shear, while $EE(E^2)$ is an enlargement followed by another enlargement.

Example 14

State the inverse of each of the following transformations.

(a) An anticlockwise rotation of 90° about the origin.

(b) A translation $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

(c) An enlargement with centre (1, 1) and scale factor 3.

Solution

The inverse transformation is

(a) a clockwise rotation of 90° about the origin,

(b) a translation $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$, since $\begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

(c) an enlargement with centre (1, 1) and scale factor $\frac{1}{3}$.

Can you think of reasons for the answers in (a) and (c)?

Example 15

Under the transformation P, the image of every point (x, 2) is (x, 2 + x). Find

(a) the coordinates of the point which is invariant under P,

(b) the image of the points A(1, 2) and B(2, 2). Plot the points of A, B and their images on a sheet of graph paper. Hence, describe the transformation P completely.

Solution

(a) If $(x, 2)$ is an invariant point, then $(x, 2) \rightarrow (x, 2)$ under P.

$\therefore \quad 2 = 2 + x$, i.e. $x = 0$

Hence, the invariant point is (0, 2).

(b) Since $(x, 2)$ is a point on the line $y = 2$, we shall consider $P(x, 2) \rightarrow (x, 2 + x)$.

When $x = 1$, $(1, 2) \rightarrow (1, 3)$ and when $x = 2$, $(2, 2) \rightarrow (2, 4)$.

From Fig. 5.33, P represents a shear parallel to the y-axis. *BA* produced and *B′A′* produced meet at *K* on the y-axis.

$$\text{Shear factor} = \frac{AA'}{AK} = \frac{BB'}{BK} = 1.$$

Thus P is a shear with the y-axis as the invariant line and shear factor 1.

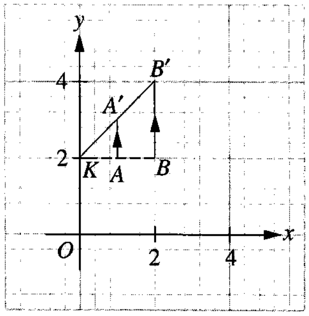


Fig. 5.33

Example 16

The transformation T is a translation of 2 units upwards, parallel to the y-axis, and the transformation M is a reflection in the line y = x.

Given that P is the point (1, 2), find the coordinates of the image of P under the following transformations.

(a) T^2 *(b)* M^2 *(c)* TM *(d)* MT

Solution

(a) The translation T is represented by $\begin{pmatrix}0\\2\end{pmatrix}$.

$$\therefore \quad T^2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}1\\6\end{pmatrix}$$

(b) Under M, $(a, b) \to (b, a)$.

$$M^2\begin{pmatrix}a\\b\end{pmatrix} = M\begin{pmatrix}b\\a\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$$

$$M^2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

Notice that M^2 maps any point onto itself.

(c) $$TM\begin{pmatrix}1\\2\end{pmatrix} = T\begin{pmatrix}2\\1\end{pmatrix}$$
$$= \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

(d) $$MT\begin{pmatrix}1\\2\end{pmatrix} = M\left[\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix}\right]$$
$$= M\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix}$$

Example 17

In Fig. 5.34, $OA = AC$*,* $DA = 2AB$ *and* $B\hat{A}O = 90°$*. Describe three successive transformations so that* $\triangle OAB$ *may be transformed into* $\triangle OCD$*.*

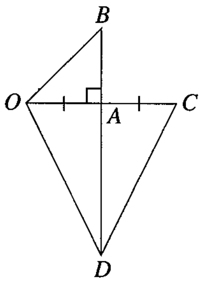


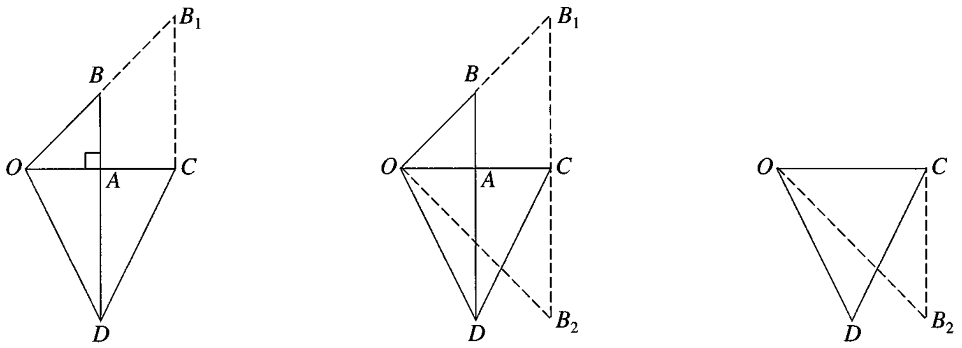
Fig. 5.34

Solution

Under an enlargement with centre O and scale factor 2, $\triangle OAB$ is transformed into $\triangle OCB_1$ as can be seen in Fig. 5.35(a).

Under reflection in the line OC, $\triangle OCB_1$ is transformed into $\triangle OCB_2$ as can be seen in Fig. 5.35(b).

Under a shear with CO as the invariant line and shear factor $\frac{B_2D}{B_2C}$, point B_2 is moved to point D as can be seen in Fig. 5.35(c).



(a) (b) (c)

Fig. 5.35

Can you do it in another way?

Example 18

$\triangle_1$, $\triangle_2$, $\triangle_3$ and $\triangle_4$ are 4 triangles. $\triangle_2$ is the reflection of $\triangle_1$ in the line $x + y = 0$. $\triangle_3$ is the anticlockwise rotation of $\triangle_2$ through 90° about the point (1, 1). $\triangle_4$ is the enlargement of $\triangle_3$ with centre of enlargement at the origin and scale factor 3. If the area of $\triangle_1$ is 8 cm², calculate the area of $\triangle_2$, $\triangle_3$ and $\triangle_4$.

Solution

Reflection is isometric,

$\therefore$ area of $\triangle_2$ = area of $\triangle_1$ = 8 cm²

Rotation is isometric,

$\therefore$ area of $\triangle_3$ = area of $\triangle_2$ = 8 cm²

Enlargement is non-isometric and area of $\triangle_4 = 3^2 \times$ (area of $\triangle_3$)

$$= 9 \times 8$$
$$= 72 \text{ cm}^2$$

Example 19

Using a scale of 1 cm to represent 1 unit on both axes, draw x- and y-axes for $0 \leqslant x \leqslant 9$ and $0 \leqslant y \leqslant 11$. Draw the rectangle PQRS whose coordinates are P(0, 0), Q(4, 0), R(4, 2) and S(0, 2) and the rectangle ABCD whose coordinates are A(1, 4), B(1, 0), C(3, 0) and D(3, 4).

(a) Draw the axis of symmetry of the figure formed by the two rectangles and write down its equation.

(b) Given that PQRS can be mapped onto ABCD by a rotation, find the centre of rotation by construction and state the angle of rotation.

(c) PQRS is mapped onto $P_1Q_1R_1S_1$ by a shear with the y-axis as the invariant line and shear factor 2. Find the coordinates of P_1, Q_1, R_1 and S_1. Plot $P_1Q_1R_1S_1$ on the graph and find the area of $P_1Q_1R_1S_1$.

(d) $P_1Q_1R_1S_1$ is mapped onto $P_2Q_2R_2S_2$ by a stretch with $x = 0$ as the invariant line and stretch factor 2. Find the coordinates of P_2, Q_2, R_2 and S_2. Plot $P_2Q_2R_2S_2$ on the graph. Find the area of the quadrilateral $P_2Q_2R_2S_2$.

Solution

(a) Refer to Fig. 5.36. The equation of the axis of reflection is $x = 2$.

(b) The centre of rotation is $\left(2\frac{1}{2}, 1\frac{1}{2}\right)$. The angle of rotation is 90° clockwise.

(c) Under the shear with the y-axis as the invariant line, P and S remain invariant.

Shear factor = 2

$$\Rightarrow \frac{QQ_1}{OQ} = \frac{RR_1}{SR} = 2$$

i.e. $\frac{QQ_1}{4} = \frac{RR_1}{4} = 2.$

$\therefore$ $QQ_1 = 8 = RR_1$ and Q_1 is the point (4, 8) and R_1 is the point (4, 10).

Since there is no change in the area under a shear, area of $P_1Q_1R_1S_1$ = area of $PQRS = 2 \times 4 = 8$ units^2.

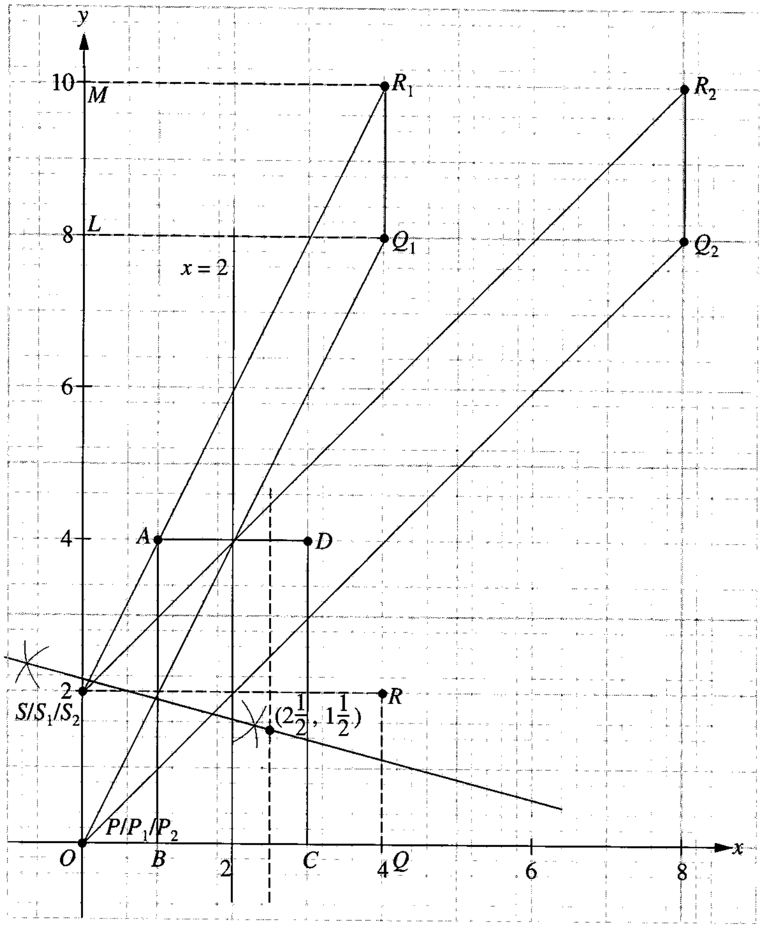


Fig. 5.36

(d) Under the stretch with $x = 0$ as the invariant line, P_1 and S_1 will remain invariant.

Stretch factor = 2 $\Rightarrow \frac{LQ_2}{LQ_1} = \frac{MR_2}{MR_1} = 2$ i.e. $\frac{LQ_2}{4} = \frac{MR_2}{4} = 2.$

Thus, Q_2 is the point (8, 8) and R_2(8, 10).

Area of $P_2Q_2R_2S_2$ = 2(area of $P_1Q_1R_1S_1$)
= $2 \times 8 = 16$ units^2.

Example 20

Sketch and describe three successive transformations under which OABC will be transformed into PQRS.

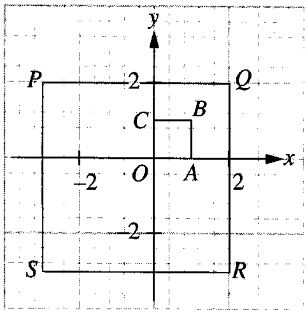


Fig. 5.37

Solution

(a) *OABC* is transformed into $P_1Q_1R_1S_1$ under an enlargement with centre *O*, scale factor 5 (Fig. 5.38(a)).

(b) $P_1Q_1R_1S_1$ is reflected in the line $y = 2\frac{1}{2}$ to $P_2Q_2R_2S_2$ (Fig. 5.38(b)).

(c) $P_2Q_2R_2S_2$ is then translated to *PQRS* by $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ (Fig. 5.38(c)).

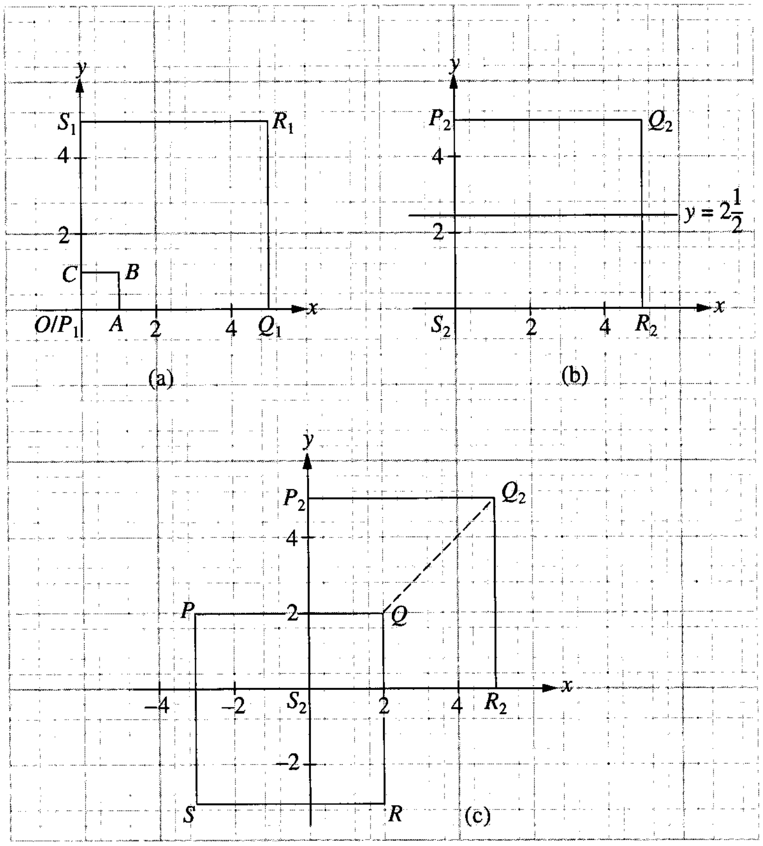


Fig. 5.38

Example 21

Sketch and describe three successive transformations under which PQRS is the image of OABC.

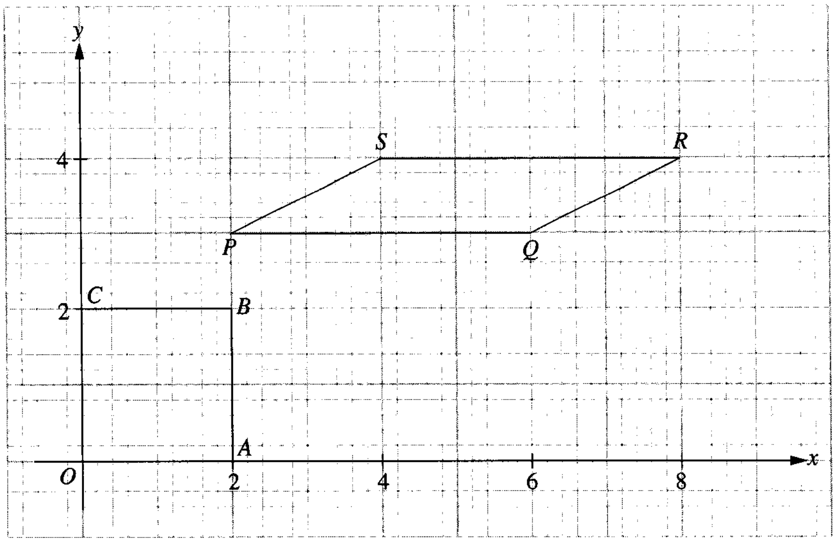


Fig. 5.39

Solution

(a) A two-way stretch with scale factor 2 along the x-axis and scale factor $\frac{1}{2}$ along the y-axis gives $OA_1B_1C_1$ (Fig. 5.40(a)).

(b) $OA_1B_1C_1$ is transformed into $OA_1B_2C_2$ under a shear along the x-axis with shear factor 2 (Fig. 5.40(b)).

(c) $OA_1B_2C_2$ is transformed into $PQRS$ by a translation $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (Fig. 5.40(c)).

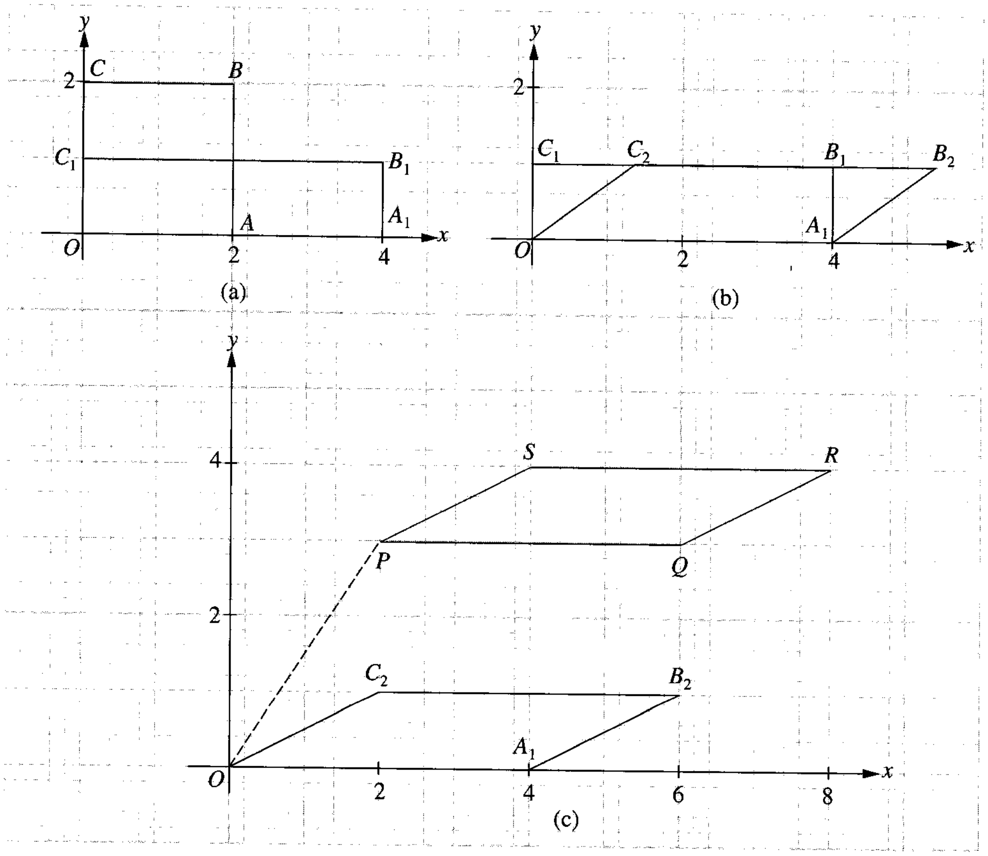


Fig. 5.40

Example 22

Using a scale of 1 cm to 1 unit on each axis, draw the x- and y-axes for $-3 \leqslant x \leqslant 6$ and $-5 \leqslant y \leqslant 5$.

(a) The vertices of $\triangle ABC$ are $A(0, 1)$, $B(3, 5)$ and $C(1, 5)$. Draw and label $\triangle ABC$.

(b) $\triangle ABC$ is mapped onto $\triangle A_1B_1C_1$ by a reflection in the y-axis. $\triangle A_1B_1C_1$ is then translated to $\triangle A_2B_2C_2$ by a translation T represented by $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$. Draw $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ on your graph. State the single transformation that will map $\triangle ABC$ directly onto $\triangle A_2B_2C_2$.

(c) $\triangle ABC$ is mapped onto $\triangle A_3B_3C_3$ by an enlargement scale factor –1 and centre at origin. Draw $\triangle A_3B_3C_3$ in your graph. Give another geometrical transformation description of this enlargement.

Solution

(a) See Fig. 5.41.

(b) The transformation that will map $\triangle ABC$ onto $\triangle A_2B_2C_2$ is a reflection in the line $x = 3$.

(c) The transformation represents a rotation of 180° about the origin.

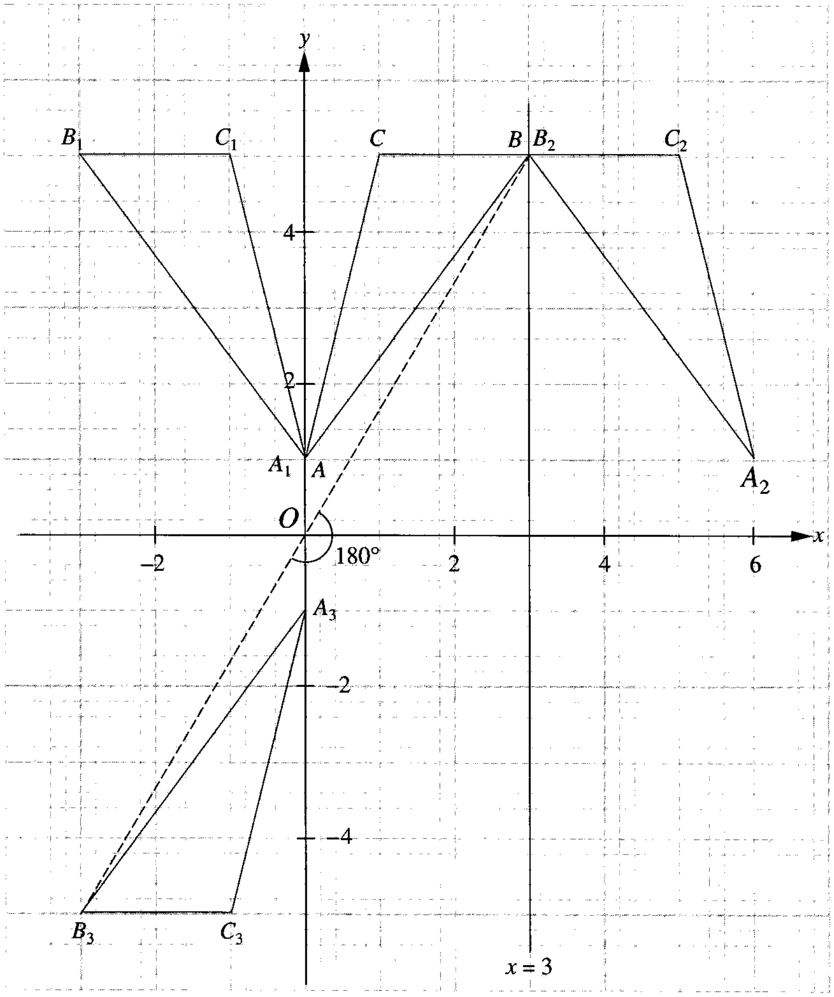


Fig. 5.41

Example 23

Using a scale of 1 cm to 1 unit on each axis, draw the x- and y-axes for $-5 \leqslant x \leqslant 5$ and $-5 \leqslant y \leqslant 5$.

(a) The vertices of $\triangle ABC$ are A(1, 2), B(4, 1) and C(3, 4). Draw and label $\triangle ABC$.

(b) $\triangle ABC$ undergoes a double transformation: a reflection in the y-axis (M) followed by an anticlockwise rotation of 90° about the origin (R). Plot the image of $\triangle ABC$ under (i) M, (ii) RM.

(c) Describe a single transformation that will map $\triangle ABC$ onto $\triangle A''B''C''$ where $\triangle A''B''C''$ is the image of $\triangle ABC$ under RM.

Solution

(c) In Fig. 5.42, by construction, line $y + x = 0$ is the mirror line of $\triangle ABC$ and $\triangle A''B''C''$. Hence, RM is the reflection in the line $y + x = 0$.

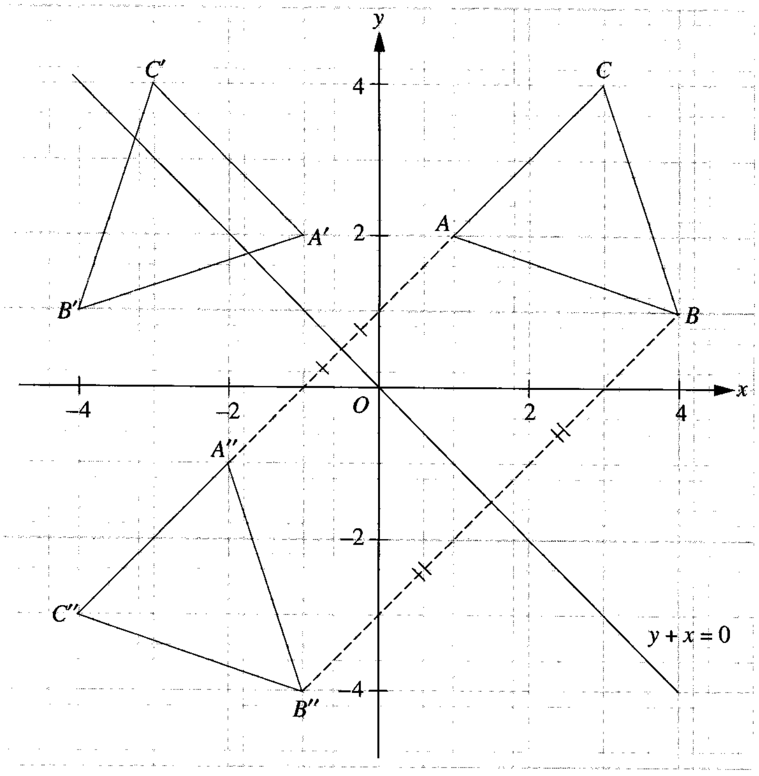


Fig. 5.42

Example 24

Answer the whole of this question on a sheet of graph paper.

$\triangle ABC$ has vertices $A(2, 0)$, $B(4, 1)$ and $C(1, 4)$.
$\triangle PQR$ has vertices $P(6, 4)$, $Q(5, 6)$ and $R(2, 3)$.
$\triangle LMN$ has vertices $L(-1, 4)$, $M(-3, 5)$ and $N(0, 8)$.

Draw these triangles on the sheet of graph paper and label the vertices.

(a) $\triangle ABC$ can be mapped onto $\triangle PQR$ by a rotation. Find the centre of rotation and the angle of rotation.

(b) $\triangle LMN$ is the image of $\triangle ABC$ after a reflection in the line l followed by a translation parallel to the y-axis. Draw the line l on the graph and write down its equation. Also write down the column vector representing the translation.

(c) $\triangle XYZ$ is the image of $\triangle ABC$ after an enlargement of scale factor 2 with centre of enlargement at O. Draw the vertices of $\triangle XYZ$ and write down the ratio area of $\triangle ABC$: area of $\triangle XYZ$.

Solution

(a) By constructing the perpendicular bisectors of AP and CR, the centre of rotation is found to be (2, 4). The angle of rotation is 90° anticlockwise.

(b) The equation of the line l is $x = \frac{1}{2}$.

The column vector representing the translation is $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

(c) Area of $\triangle ABC$: area of $\triangle XYZ = 1^2 : 2^2$

$= 1 : 4$.

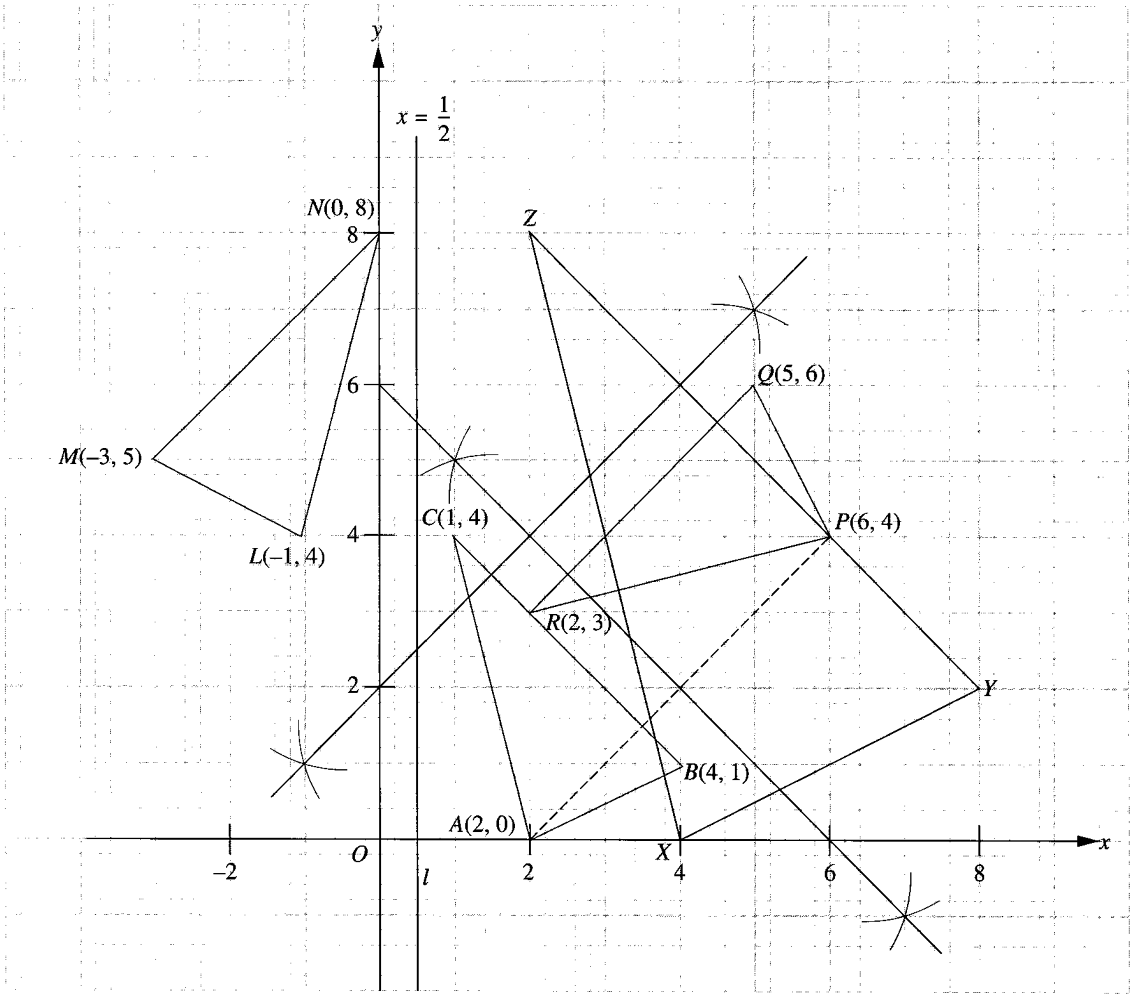


Fig. 5.43

Example 25

(a) *Using a scale of 1 cm to 1 unit on each axes, draw the x- and y-axes for $0 \leqslant x \leqslant 10$ and $-2 \leqslant y \leqslant 10$. Draw on the same axes the line $y = x - 1$ and label it as m. Also draw the line $x + y = 8$ and label it as l. The vertices of a triangle A have coordinates (1, 3), (3, 3) and (1, 6). Draw and label triangle A.*

(b) *The transformation M_l is a reflection in the line l and the transformation M_m is a reflection in the line m. Draw and label triangles $M_l(A)$ and $M_mM_l(A)$.*

(c) *The single transformation which maps A onto $M_mM_l(A)$ is a rotation. State the angle of rotation and find the coordinates of the centre of rotation.*

Solution

(a) and (b) See Fig. 5.44.

(c) The angle of rotation is 180° and the centre of rotation is the intersection of the lines h and k. Its coordinates are $\left(4\frac{1}{2}, 3\frac{1}{2}\right)$.

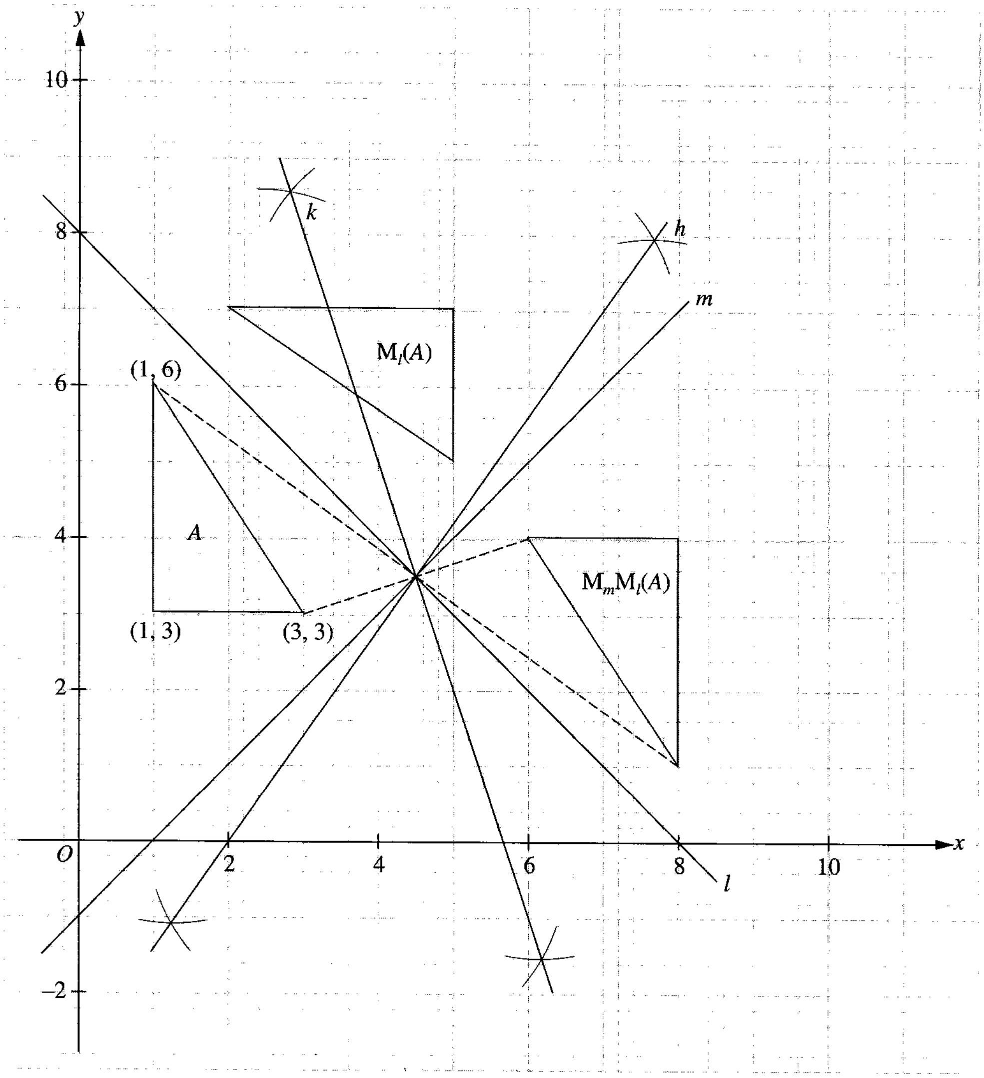


Fig. 5.44

Exercise 5d

1. If M is a reflection in the y-axis and T is a translation represented by $\begin{pmatrix}3\\0\end{pmatrix}$, find
   (a) MT(2, 3),
   (b) (x, y) if TM$(x, y) = (4, 3)$.

2. If M denotes reflection in the x-axis and R denotes rotation about the origin through 90° anticlockwise, find
   (a) MR(2, 3),
   (b) (x, y), if RM$(x, y) = (3, 1)$.

3. Under a reflection in the line $y = 3$, the point $A(5, 1)$ is mapped onto A_1. Find the coordinates of A_1. A reflection in the line $y = 8$ will map the point A_1 onto the point A_2. Find the coordinates of A_2. Given that A_2 is the reflection of A in the line $y = k$, find the value of k.

4. The coordinates of $\triangle ABC$ are $A(7, 4)$, $B(7, 0)$ and $C(5, 4)$. The coordinates of $\triangle PQR$ are $P(10, 4)$, $Q(10, 10)$ and $R(7, 4)$. Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and describe two successive transformations that will map $\triangle ABC$ onto $\triangle PQR$.

5. A reflection in the line $y = 0$ will map the point P onto P_1 and a reflection in the line $x = 0$ will map the point P_1 onto P_2. Describe a single transformation that will map P directly onto P_2.

6. The transformation T is the translation $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and the transformation M is a reflection in the line $y + x = 0$. Given that P is the point (2, 5), write down the coordinates of the image of P under the following transformations.

(a) T^2 **(b)** M^2 **(c)** M^8
(d) MT **(e)** TM

7. The coordinates of $\triangle ABC$ are $A(2, 2)$, $B(5, 2)$ and $C(3, 4)$. $\triangle ABC$ is mapped onto $\triangle PQR$ by means of the following successive transformations.

(a) A clockwise rotation of 90° about (0, 0).
(b) A reflection in the line $x = 0$.

Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and describe a single transformation that will map $\triangle ABC$ onto $\triangle PQR$.

8. The coordinates of $\triangle ABC$ are $A(4, 0)$, $B(5, 0)$ and $C(5, 2)$. $\triangle ABC$ is mapped onto $\triangle PQR$ by means of the following successive transformations.

(a) An enlargement scale factor 2, centre at (4, 0).
(b) A reflection in the line $x = 4$.
(c) A 90° clockwise rotation about the point (1, 1).

Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and label the vertices clearly.

9. The transformation R is a 90° clockwise rotation about (0, 2) and the translation T is given by the column vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. If A is the point (4, 5), find the coordinates of R(A) and TR(A).

10. A is the point (5, 1) and R is a transformation which gives a 90° anticlockwise rotation about the origin. T is a translation represented by the column vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. If $\mathrm{RT}(A) = B$, $\mathrm{TR}(A) = C$ and $\mathrm{R}^2(A) = D$, find the coordinates of B, C and D.

⋆11. The rectangle $ABCD$ is divided into 16 equal rectangles. The point P is such that the area of $\triangle APB$ is equal to one quarter of the area of rectangle $ABCD$, and the point Q, lying on AB, is such that $\triangle AEB$ is an enlargement of $\triangle APQ$.

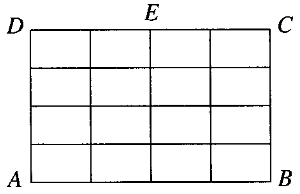


Fig. 5.45

Mark P and Q clearly on a copy of the diagram. (*C*)

12. The coordinates of $\triangle ABC$ are $A(1, -1)$, $B(1, 0)$ and $C(3, -1)$. $\triangle ABC$ is transformed into $\triangle A_1B_1C_1$ by the following successive transformations.

(a) A reflection in the x-axis.
(b) An enlargement, centre origin, scale factor 3.

Draw $\triangle ABC$ on a sheet of graph paper and construct $\triangle A_1B_1C_1$ on the same graph. State the coordinates of A_1.

13. A square $PQRS$ has the line $y = x$ as an axis of symmetry, and three vertices of the square are $P(0, 6)$, $Q(6, 0)$ and $R(12, 6)$. Draw the square on a sheet of graph paper and state the coordinates of S. Find the equation of the axis of symmetry perpendicular to the given axis of symmetry. If this new axis of symmetry cuts the x-axis at the point A and the y-axis at the point B, compare the area of the triangle OAB with that of the square $PQRS$.

14. R represents a 90° anticlockwise rotation about O and M represents a reflection in the x-axis.

(a) Sketch on a copy of the diagram, the image of the square $ABCD$, correctly lettered, under the transformation MR (i.e. R followed by M).

(b) State a single transformation which is equivalent to MR.

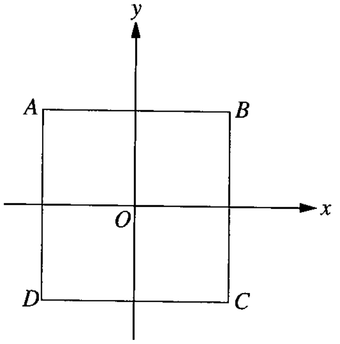


Fig. 5.46

15. $\triangle ABC$ with vertices $A(1, 1)$, $B(2, 3)$ and $C(-1, 4)$ is first reflected in the x-axis and then rotated through 90° anticlockwise about the origin. Calculate the new coordinates of A, B and C. A square $PQRS$ is transformed by the above transformations into a square whose vertices are at the points with coordinates (0, 5), (3, 5), (3, 8) and (0, 8) respectively. Calculate the coordinates of P, Q, R and S.

16. A square $ABCD$ with vertices $A(2, 4)$, $B(4, 4)$, $C(4, 6)$ and $D(2, 6)$ is enlarged with centre $P(0, 5)$ and scale factor 3, to $A_1B_1C_1D_1$. Construct the images formed and state the coordinates of C_1. Taking A as the centre of enlargement and scale factor -3, construct the images of $A_2B_2C_2D_2$ formed under this transformation and state the coordinates of C_2.

17. Triangle ABC has vertices $A(6, 12)$, $B(8, 4)$ and $C(2, 2)$.

(a) Triangle FGH has vertices $F(-12, 6)$, $G(-4, 8)$ and $H(-2, 2)$. Describe fully the single transformation that maps $\triangle ABC$ onto $\triangle FGH$.

(b) Write down the column vector which represents the translation that maps C onto B.

(c) $ABCP$ is a trapezium in which $\overrightarrow{PA}$ is parallel to $\overrightarrow{CB}$ and $PA = \frac{1}{2}CB$. Find the coordinates of P.

(d) An enlargement maps $\triangle ABC$ onto $\triangle ADE$.

- **(i)** Which point is the centre of the enlargement?
- **(ii)** Given that D is the point $(k, 0)$, find
  - **(a)** the value of k,
  - **(b)** the scale factor of the enlargement,
  - **(c)** the coordinates of E,
  - **(d)** the ratio of the area of $\triangle ABC$ to the area of $\triangle ADE$. (*C*)

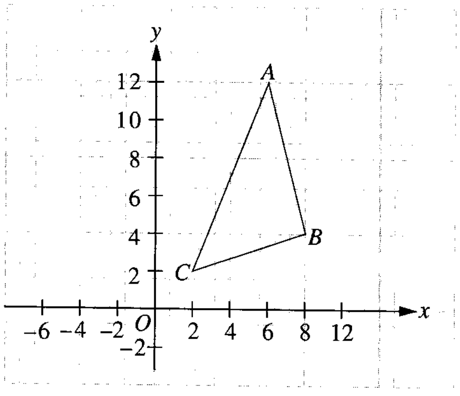


Fig. 5.47

18. In Fig. 5.48, $ABCD$, $ALMN$ and $LPQR$ are rectangles. M is the mid-point of BD and Q is the mid-point of BM, the rectangles being consequently similar.

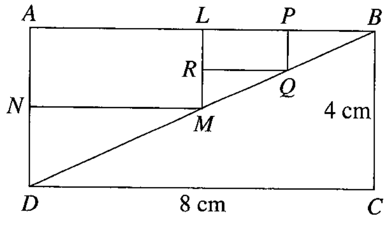


Fig. 5.48

(a) State the centre of enlargement under which $ALMN$ is the image of $LPQR$. Also state the scale factor of the enlargement.

(b) Copy the figure and construct, in full scale drawing, the centre of enlargement under which *ABCD* is the image of *LPQR*. State the scale factor of this enlargement.

(c) State two successive transformations under which $\triangle MND$ will be mapped onto $\triangle DCB$.

19. Fig. 5.49 shows triangles *A*, *B*, *C*, *D*, *E* and *F*.

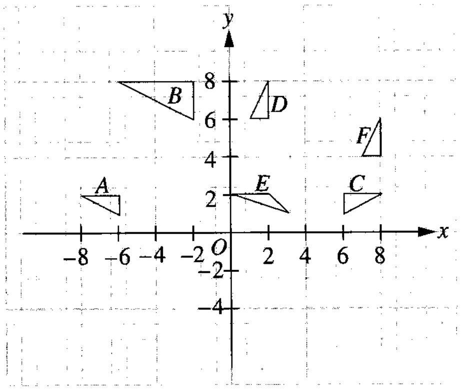


Fig. 5.49

(a) Triangle *A* is mapped onto triangle *B* by an enlargement. Find
 (i) the scale factor,
 (ii) the coordinates of the centre of the enlargement.

(b) Triangle *A* is mapped onto triangle *C* by a single transformation. Describe this transformation completely.

(c) Triangle *C* is mapped onto triangle *D* by a single transformation. Describe fully this transformation.

(d) Triangle *C* is mapped onto triangle *E* by a shear. Find the shear factor of this transformation.

(e) Triangle *A* is mapped onto triangle *F* by a transformation V which is made up of two separate transformations. The first is a clockwise rotation of 90° about *O*; it is followed by a translation represented by the column vector $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$. Describe fully the single transformation which is equivalent to V. (*C*)

20. Fig. 5.50 shows a parallelogram *ABCD*. *A* is (1, 6), *B* is (9, 2), *C* is (9, 0) and *D* is (1, 4).

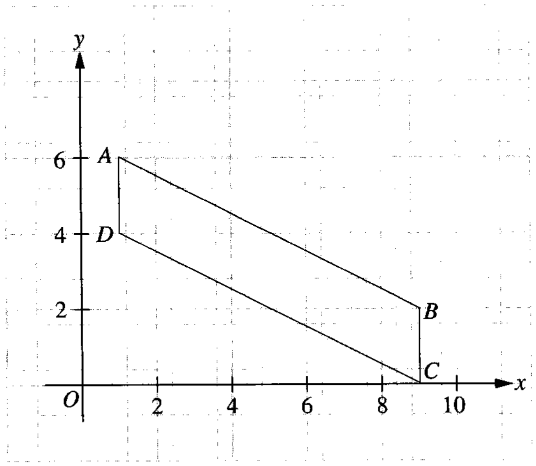


Fig. 5.50

(a) Find the gradient of *DC*.

(b) Calculate the area of the parallelogram *ABCD*.

(c) The parallelogram is reflected in the line $y = 2$. Find the coordinates of the image of the point *D*.

(d) Find the coordinates of the point about which *ABCD* has rotational symmetry of order 2. (*C*)

21. Answer the whole of this question on a sheet of graph paper.

Triangle *A* has vertices (4, 1), (4, –1) and (5, 1).

Triangle *B* has vertices (1, 4), (–1, 4) and (1, 5).

(a) Using a scale of 1 cm to represent 1 unit on each axis, draw axes for values of *x* and *y* in the ranges $-6 \leqslant x \leqslant 6$ and $-6 \leqslant y \leqslant 6$. Draw and label the triangles *A* and *B*.

(b) Describe fully the single transformation which maps triangle *A* onto triangle *B*.

(c) A 90° clockwise rotation about the origin maps triangle *A* onto triangle *C*. Draw and label triangle *C*.

(d) Describe the transformation which maps triangle *B* onto triangle *C* completely. (*C*)

Summary

In this chapter, we have studied the following non-isometric transformations.

1. An **enlargement** is defined by its centre and scale factor. The scale factor affects the size and position of the image. The centre of enlargement is the only possible invariant point. Under an enlargement with the scale factor k,

$$\textbf{image area} = k^2 \times \textbf{(area of original figure).}$$

2. Under a **stretch** with the line l as the invariant line and stretch factor k, if A' is the image of A under the stretch, then

$$k = \left(\frac{\textbf{distance of } A' \textbf{ from } l}{\textbf{distance of } A \textbf{ from } l}\right).$$

If $\triangle A'B'C'$ is the image of $\triangle ABC$ under the stretch, then

$$\textbf{area of } \triangle A'B'C' = k \times \textbf{(area of } \triangle ABC\textbf{).}$$

For a double stretch parallel to the x- and y-axes with factors k_1 and k_2 respectively

$$\textbf{image area} = k_1k_2 \times \textbf{(original area).}$$

3. Under a **shear** with the line l as the invariant line and shear factor k, if A' is the image of A under the shear, then

$$k = \left(\frac{\textbf{distance of } A \textbf{ from } A'}{\textbf{distance of } A \textbf{ from } l}\right).$$

A shear is a non-isometric transformation that preserves the area of the figure.

Review Questions 5

1. Under an enlargement, centre (0, 3), the line PQ with coordinates $P(2, 5)$ and $Q(3, 1)$ are mapped onto the points $(4, p)$ and (m, n). Find the values of p, m and n.

2. For each of the following, draw the image of the figure under a stretch with the y-axis as the invariant line and stretch factor $1\frac{1}{2}$.

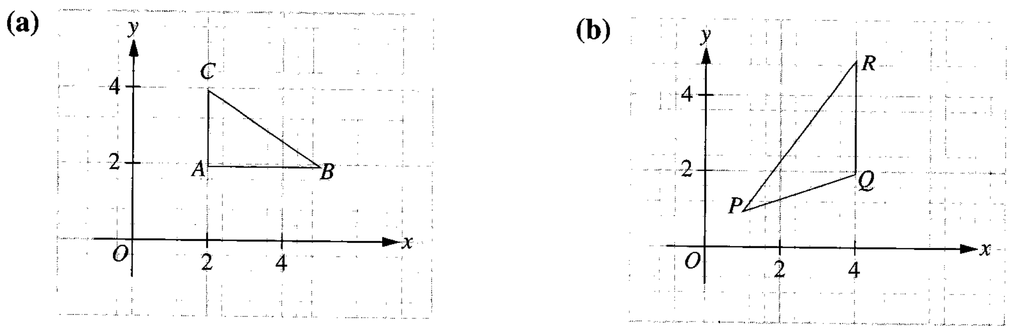
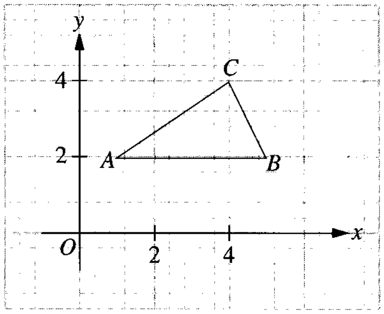


Fig. 5.51

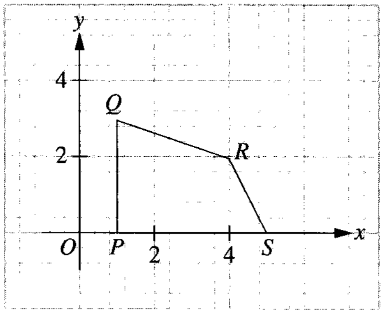
3. For each of the following, copy the figure and draw its image under each of the given stretches.

(a)



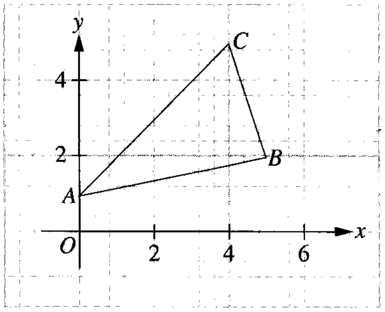
A stretch with the x-axis as the invariant line and stretch factor 1.5.

(b)



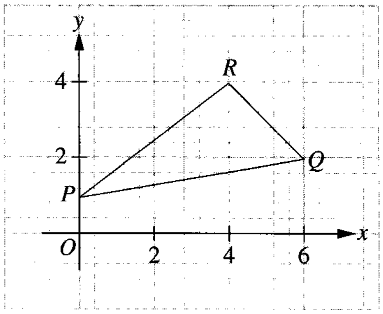
A stretch with $y = 0$ as the invariant line and stretch factor 2.

(c)



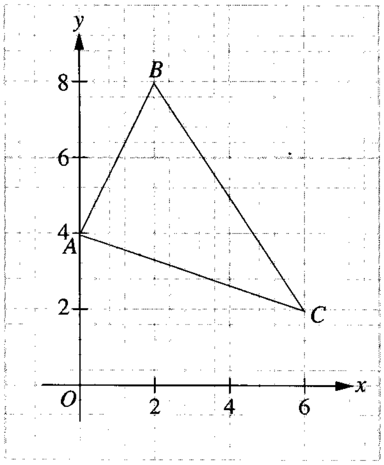
A stretch with $x = 2$ as the invariant line and stretch factor 2.

(d)



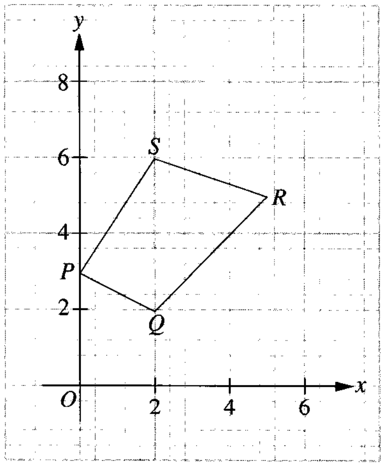
A stretch with $y = 2$ as the invariant line and stretch factor –2.

(e)



A stretch with $y = 4$ as the invariant line and stretch factor $1\frac{1}{2}$.

(f)



A stretch with $x = 2$ as the invariant line and stretch factor $-1\frac{1}{2}$.

Fig. 5.52

4. The vertices of $\triangle ABC$ are $A(4, 2)$, $B(2, 1)$ and $C(2, 5)$. $\triangle ABC$ is mapped onto $\triangle LMN$ by a stretch with $x = 1$ as the invariant line and stretch factor 2. Draw the vertices of $\triangle LMN$ on a sheet of graph paper.

$\triangle LMN$ is then mapped onto $\triangle PQR$ under a shear with $y = 0$ as the invariant line and shear factor 2. Draw the vertices of $\triangle PQR$ on the same sheet of graph paper.

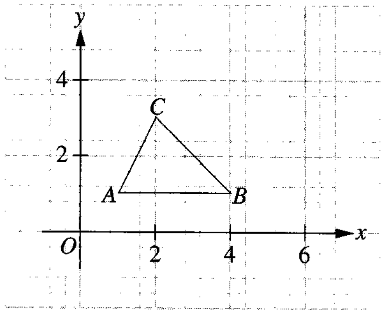
Write down the area of

(a) $\triangle ABC$, **(b)** $\triangle LMN$, **(c)** $\triangle PQR$.

5. Draw on a sheet of graph paper $\triangle ABC$ with vertices $A(1, 2)$, $B(5, 2)$ and $C(4, 4)$. $\triangle ABC$ is mapped onto $\triangle PQR$ with vertices $P(-1, 2)$, $Q(11, 2)$ and $R(8, 4)$. Draw $\triangle PQR$ on the same sheet of graph paper. Describe fully the transformation which maps $\triangle ABC$ onto $\triangle PQR$.

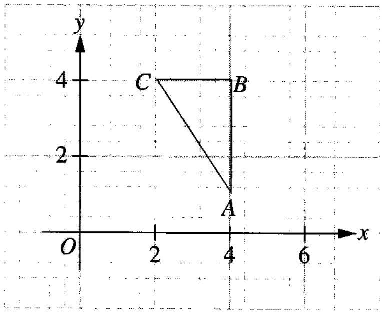
6. For each of the following, draw the image of $\triangle ABC$ under a shear with the given invariant line and shear factor.

(a)



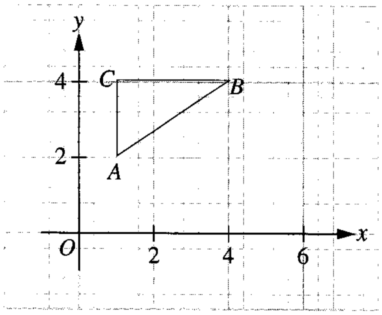
Invariant line is the x-axis and shear factor = 2.

(b)



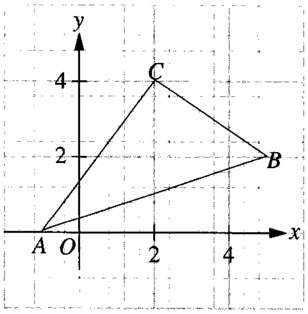
Invariant line is the y-axis and shear factor = $1\frac{1}{2}$.

(c)



Invariant line is $y = 1$ *and shear factor =* $1\frac{1}{2}$.

(d)



Invariant line is $y = 1$ *and shear factor = 2.*

Fig. 5.53

7. A transformation H followed by another transformation K will map $\triangle ABC$ onto $\triangle PQR$. Describe H and K completely.

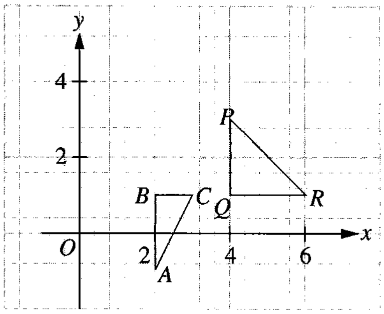


Fig. 5.54

8. A transformation H followed by another transformation K will map $\triangle ABC$ onto $\triangle PQR$. Describe H and K completely.

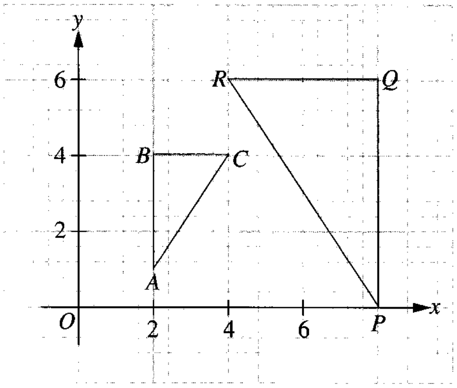


Fig. 5.55

9. The diagram shows five triangles A, B, C, D and E. Describe a single transformation that will map

(a) $\triangle A$ onto $\triangle B$,
(b) $\triangle B$ onto $\triangle C$,
(c) $\triangle B$ onto $\triangle E$,
(d) $\triangle D$ onto $\triangle C$.

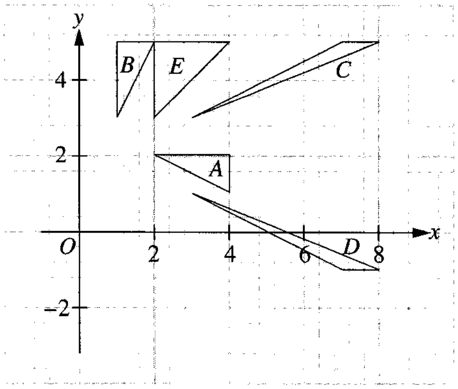


Fig. 5.56

10. The diagram shows five triangles A, B, C, D and E. Describe a single transformation that will map

(a) $\triangle A$ onto $\triangle B$,
(b) $\triangle A$ onto $\triangle C$,
(c) $\triangle B$ onto $\triangle D$,
(d) $\triangle A$ onto $\triangle E$.

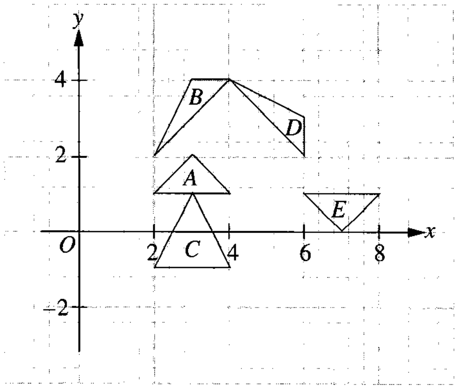


Fig. 5.57

11. $\triangle ABC$ is mapped onto $\triangle A'B'C'$ by means of an enlargement centre A and scale factor 4. Write down the values of the ratios of

(a) the side $B'C'$ to the side BC,
(b) the size of $A\hat{B}'C'$ to the size of $A\hat{B}C$,
(c) the area of $\triangle AB'C'$ to the area of $\triangle ABC$.

12. In Fig. 5.58, $\triangle A_1B_1C_1$ is an enlargement of $\triangle ABC$. If O is the centre of enlargement, copy the diagram and find O by construction. State the value of the ratio

(a) $AB : A_1B_1$,

(b) area of $\triangle A_1B_1C_1$: area of $\triangle ABC$.

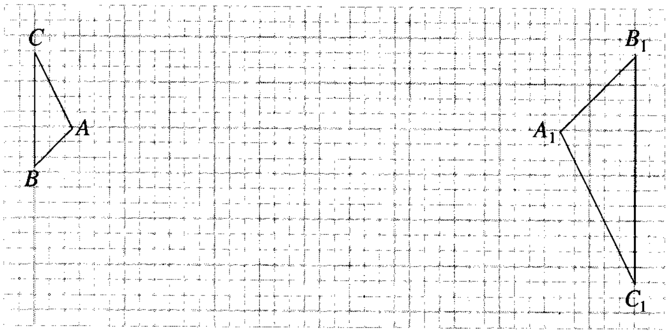


Fig. 5.58

13. Under an enlargement E, the point $A(1, 3)$, is mapped onto $A'(1, 7)$ and $B(3, 2)$ is mapped onto $B'(7, 4)$. Plot the points A, B, A' and B' on a sheet of graph paper and find

(a) the coordinates of the centre of enlargement,

(b) the scale factor of the enlargement,

(c) the coordinates of the image of the point (2, 2) under E,

(d) the coordinates of the point whose image is (4, 7).

14. The coordinates of the vertices of $\triangle ABC$ are $A(2, 0)$, $B(2, 2)$ and $C(6, 2)$. Under an enlargement centre at origin and scale factor –2, $\triangle ABC$ is mapped onto $\triangle A_1B_1C_1$. Find

(a) the coordinates of A_1, B_1 and C_1,

(b) the ratio of A_1B_1 to AB,

(c) the ratio of area of $\triangle ABC$ to area of $\triangle A_1B_1C_1$.

15. $\triangle ABC$ is mapped onto $\triangle APQ$ by an enlargement with centre A and scale factor k. Illustrate, with freehand sketches, the cases when

(a) $k = 1\frac{1}{2}$, **(b)** $k = -2\frac{1}{2}$.

Given that the length of $CQ = 35$ cm, find the length of AQ in each case.

16. The transformation U is a shear with $y = 0$ as the invariant line and shear factor –2.

(a) Find the image of (10, 2) under U.

(b) Given that U maps the point $(m, 4)$ onto a point on the y-axis, find the value of m.

(c) Given that the image of the point $P(h, k)$ under U is the point (–8, 2), find the value of h and of k.

17. A transformation X maps $\triangle ABC$ with vertices $A(0, 0)$, $B(1, 0)$ and $C(4, 3)$ onto $\triangle PBC$ where the coordinates of P is $(1, -1)$. Describe this transformation completely and determine the equation of the invariant line. What is the area of $\triangle APC$?

18. The transformation G is a reflection in the y-axis.

(a) The point $A(3, 4)$ is mapped onto the point B under G. Find the coordinates of B and write down the equation of the invariant line.

(b) The point B is now mapped onto the point C by a transformation P which is a 90° anticlockwise rotation about the origin. Find the coordinates of C.

(c) Describe the single transformation which is equivalent to G followed by P.

19. In Fig. 5.59, $ABCD$ is a parallelogram. P is the mid-point of AC and R is the mid-point of AP. The parallelogram $PQBK$ and $RSKH$ are drawn. Describe fully the transformations which will map the following.

(a) $\triangle ARH$ onto $\triangle ACB$ **(b)** $\triangle APK$ onto $\triangle PCQ$

(c) $\triangle ABC$ onto $\triangle CDA$ **(d)** $RSKH$ onto $PQBK$

(e) $HKSR$ onto $ABCD$ **(f)** $PQBK$ onto $DCBA$

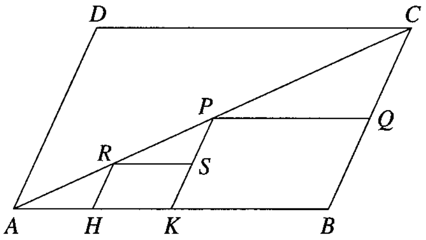


Fig. 5.59

20. **(a)** A translation maps $\triangle A$ onto $\triangle B$.
Write down the column vector representing this translation.

(b) A rotation maps $\triangle A$ onto $\triangle C$.
Write down the coordinates of the centre of this rotation.

(c) An enlargement maps $\triangle A$ onto $\triangle D$.
Write down the coordinates of the centre of this enlargement and its scale factor.

(d) A shear maps $\triangle A$ onto $\triangle E$.
Write down the shear factor. (*C*)

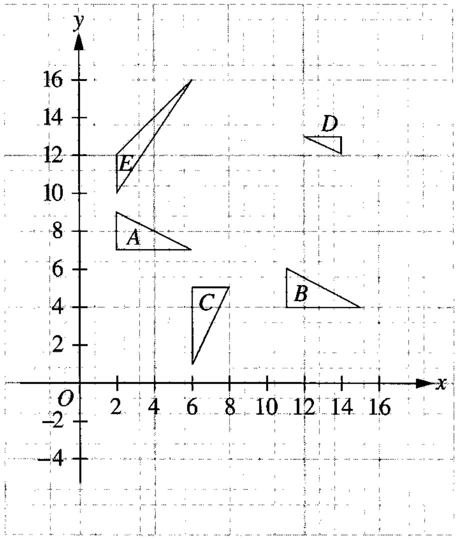


Fig. 5.60

21. The line l and $\triangle ABC$ are shown on the graph.

(a) **(i)** Write down the gradient of the line AB.

(ii) Find the equation of the line AB.

(iii) The line $x = 2$ is the axis of symmetry of the quadrilateral $ABCD$.

Write down the coordinates of the point D.

(b) The triangle ABC is reflected in the line l.
Find the coordinates of the image of the point C.

(c) An enlargement, scale factor 2, maps triangle ABC onto triangle LMN.
The point A maps onto the point $L(2, 2)$.

(i) Find the coordinates of the centre of the enlargement.

(ii) Find the coordinates of N, the image of C.

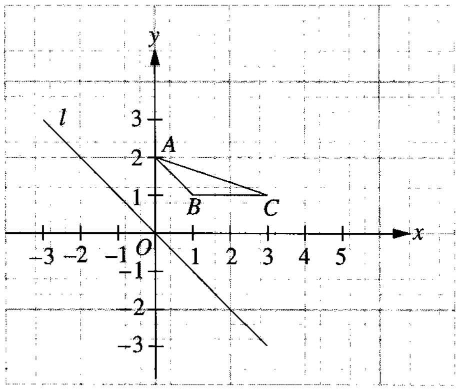


Fig. 5.61

(d) A shear, with the x-axis invariant, maps triangle ABC onto triangle PQR.
The point A maps onto the point P (6, 2).

(i) Find the coordinates of Q, the image of B.

(ii) State the length of QR. (*C*)

22. Answer the whole of this question on a sheet of graph paper.

(a) Using a scale of 1 cm to represent 1 unit on each axis, draw x- and y-axes for $-8 \leqslant x \leqslant 10$ and $-4 \leqslant y \leqslant 18$. Draw and label the triangle whose vertices are $A(1, 4)$, $B(2, 4)$ and $C(2, 1)$.

(b) The enlargement E has centre at the origin and maps $\triangle ABC$ onto $\triangle A_1B_1C_1$. Given that A_1 is the point (4, 16),

(i) draw and label $\triangle A_1B_1C_1$,

(ii) write down the scale factor of enlargement E.

(c) The point $B_2(-4, -2)$ is the image of B under a reflection in the line l. Draw and label the line l and find its equation.

(d) The transformation R, a clockwise rotation of 90° about the origin, maps $\triangle ABC$ onto $\triangle A_3B_3C_3$. Draw and label $\triangle A_3B_3C_3$.

(e) The transformation X is a stretch with $x = 0$ as the invariant line and stretch factor -3 and it maps $\triangle ABC$ onto $\triangle A_4B_4C_4$. Find the coordinates of A_4, B_4 and C_4, then draw and label this triangle. *(C)*

23. Answer the whole of this question on a sheet of graph paper.

Using a scale of 2 cm to 1 unit on each axis, draw the x- and y-axes for $-4 \leqslant x \leqslant 5$ and $-3 \leqslant y \leqslant 9$.

(a) The vertices of $\triangle ABC$ are $A(1, 1)$, $B(2, 2)$ and $C(0, 3)$. Draw and label $\triangle ABC$. The vertices of $\triangle A_1B_1C_1$ are $A_1(-2, 2)$, $B_1(-3, 3)$ and $C_1(-4, 1)$. Draw and label $\triangle A_1B_1C_1$.

Describe fully the single transformation which maps $\triangle ABC$ onto $\triangle A_1B_1C_1$.

(b) $\triangle ABC$ can also be mapped onto $\triangle A_1B_1C_1$ by a rotation of 90° anticlockwise about the origin followed by a translation. Write down the column vector which represents this translation.

(c) The vertices of $\triangle A_2B_2C_2$ are $A_2(2, 0)$, $B_2(3, 1)$ and $C_2(4, -1)$. Draw and label $\triangle A_2B_2C_2$.

$\triangle ABC$ is mapped onto $\triangle A_2B_2C_2$ by a reflection in the line l.

Draw and label the line l. Write down the equation of l.

(d) $\triangle ABC$ is mapped onto $\triangle A_3B_3C_3$ by a shear with the y-axis invariant and shear factor k. Given that A_3 is the point (1, 4), find the value of k and the coordinates of B_3. *(C)*

★24. Answer the whole of this question on a sheet of graph paper.

(a) Using a scale of 1 cm to represent 1 unit on each axis, draw the x- and y-axes for $-4 \leqslant x \leqslant 14$ and $-6 \leqslant y \leqslant 12$. Draw and label triangle P with vertices (2, 6), (4, 3) and (4, 6).

(b) The single transformation X maps the triangle P onto the triangle X(P) with vertices (6, 12), (12, 3) and (12, 12). Draw and label triangle X(P) and describe the transformation X fully.

(c) The single transformation Y maps the triangle P onto the triangle Y(P) which has vertices (3, 4), (6, 2) and (6, 4). Draw the triangle Y(P) and describe the transformation Y fully.

(d) The transformation Z is a shear with $y = 0$ as the invariant line and shear factor k and it maps the point (4, 3) onto the point (13, 3). Find the value of k and hence, find the image of the point (2, 6) under Z.

(e) The transformation R is a 90° clockwise rotation about the origin and T is the translation $\begin{pmatrix} -4 \\ 4 \end{pmatrix}$. Draw and label the triangle TR(P).

★**25.** Fig. 5.62 shows triangles *A*, *B*, *C*, *D* and *E*.

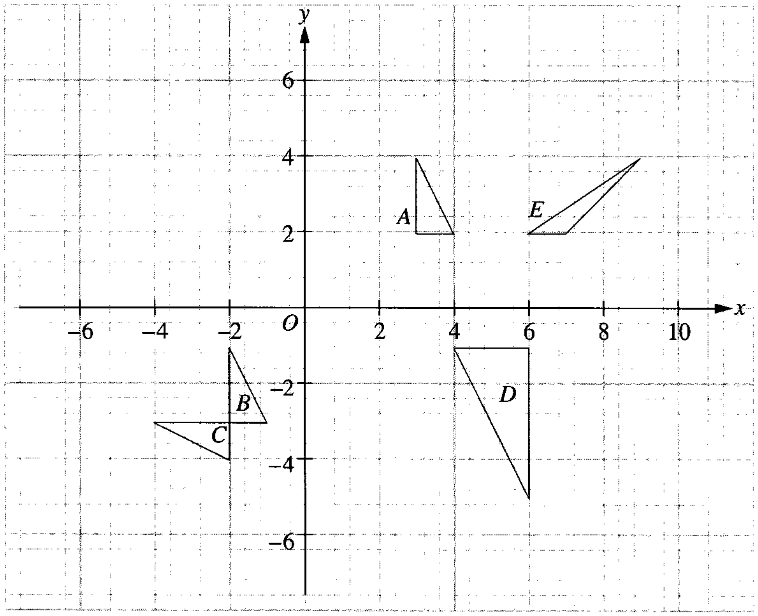


Fig. 5.62

(a) Triangle *A* is mapped onto triangle *B* by a translation. Write down the column vector of the translation.

(b) **(i)** Triangle *A* is mapped onto triangle *C* by a reflection in the line *l*. Write down the equation of the line *l*.

(ii) Triangle *B* is mapped onto triangle *C* by a reflection in the line *m*. Find the equation of the line *m*.

(c) Triangle *A* is mapped onto triangle *D* by an enlargement.

(i) Write down the coordinates of the centre of the enlargement.

(ii) Write down the scale factor of the enlargement.

(iii) Find the value of the ratio

$$\frac{\text{area of triangle } A}{\text{area of triangle } D},$$

expressing your answer in its lowest terms.

(d) Triangle *A* is mapped onto triangle *E* by a single transformation. Describe fully this single transformation. *(C)*

Problem Solving

Example 26

Five pupils A, B, C, D and E took part in a writing competition. Before announcing the outcome the teacher asked each of them to predict the results of two of their friends. The following were their guesses:

A's guesses: B gets 3rd, C gets 5th
B's guesses: A gets 1st, E gets 4th
C's guesses: D gets 2nd, E gets 4th
D's guesses: C gets 1st, B gets 2nd
E's guesses: D gets 2nd, A gets 3rd.

After checking their guesses against the actual results the teacher said, "Each of you have exactly one guess correct." Can you rearrange the positions based on what they have predicted?

Solution

The predictions are as shown in Table 5.1(a).

	1st	2nd	3rd	4th	5th
A			B		C
B	A			E	
C		D		E	
D	C	B			
E		D	A		

(a)

	1st	2nd	3rd	4th	5th
A			B✓		C✗
B	A✓			E✗	
C		D✓		E✗	
D	C✗	B✓			
E		D✓	A✗		

(b)

Table 5.1

Consider the results shown on Table 5.1(b) in which our suppositions are marked with either a tick or a cross.

Suppose A's first guess is right then B gets 3rd place. Hence, E's guess that A is 3rd is wrong and D must be 2nd.

This means that B's and C's guesses that E is 4th must be wrong.

Then B's guess that A is 1st must be right.

If A is 1st then D's guess that C is 1st must be wrong. His other guess must be right that is B is 2nd.

But we earlier concluded that D must be 2nd.

So A's guess that B is 3rd has led to a contradiction.

Since B cannot be 3rd, C must be 5th. This will give us the following result as show in Table 5.1(c).

	1st	*2nd*	*3rd*	*4th*	*5th*
A			B✗		C✓
B	A✗			E✓	
C		D✗		E✓	
D	C✗	B✓			
E		D✗	A✓		

Table 5.1(c)

1. *A, B, C and D met at a function. All the four are bilingual in Chinese, Malay, Tamil or English. There is no one common language among the four of them. However, one of the languages is common to three of them. Based on the following facts, establish the languages that they each speak.*

   ① *B speaks no English but when A and C want to have a conversation he will be invited to be the interpreter.*

   ② *B, C and D wish to have a conversation but could not find a common language.*

   ③ *A knows Tamil but not D. However they are able to converse without difficulty.*

   ④ *None of the four is bilingual in both Tamil and Malay.*

2. *Five cards are coloured red, blue, yellow, green and purple on one face but are identical on the back. They are placed face down on a table. Five boys, A, B, C, D and E are allowed to pick 2 cards and guess their colours.*

   Their guesses are:
   A – 2nd card purple,
   3rd card yellow
   B – 2nd card blue,
   4th card red
   C – 1st card red,
   5th card green

From Table 5.1(c), we have C on 5th, B on 2nd, E on 4th, A on 3rd. Thus D must be 1st.

To check, run through each row to ensure only one entry is correct and each column should, at the most, contain one correct entry.

The above problem is solved by *using tabulations, making a supposition* and *eliminating* the unlikely guesses. If the supposition leads to a contradiction, make a new supposition and start the elimination all over again until the problem is solved.

D – 3rd card blue,
4th card green
E – 2nd card yellow,
5th card purple

When the cards are turned over, each boy is found to have guessed exactly one card correctly. Also each card was guessed correctly by exactly one boy. Can you identify which of the cards each of them has guessed correctly?

☆ ☆ ☆ ☆ ☆

Example 27

There are 4 persons A, B, C and D living in a neighbourhood. One of them is an accountant, the others are a banker, a consultant and a dentist. Match them according to their occupation based on the following information.

① *A and B are neighbours and they car-pool to work.*
② *A is older than C.*
③ *A teaches D how to play chess.*
④ *The accountant walks to work everyday.*
⑤ *The banker and the dentist are not neighbours.*
⑥ *The consultant and the dentist do not know each other.*
⑦ *The dentist is older than the banker and the consultant.*

Solution

We set up, first of all, a table and eliminate each impossible condition by putting a '✗' and each possible one by a '✓'.

① and ④ lead to

	Accountant	*Banker*	*Consultant*	*Dentist*
A	✗			
B	✗			
C				
D				

Table 5.2(a)

Suppose A is the dentist,

⑤ leads to

	Accountant	*Banker*	*Consultant*	*Dentist*
A	✗	✗	✗	✓
B	✗	✗		✗
C				✗
D				✗

Table 5.2(b)

and ① and ⑥ lead to

	Accountant	*Banker*	*Consultant*	*Dentist*
A	✗	✗	✗	✓
B	✗	✗	✗	✗
C				✗
D				✗

Table 5.2(c)

From Table 5.2(c), we conclude that the supposition that A is the dentist is incorrect.

Suppose A is the banker,

① and ⑤ lead to

	Accountant	*Banker*	*Consultant*	*Dentist*
A	✗	✓	✗	✗
B	✗	✗	✓	✗
C		✗	✗	
D		✗	✗	

Table 5.2(d)

We continue with Table 5.2(d) and make the supposition that C is the accountant.

This leads to

	Accountant	*Banker*	*Consultant*	*Dentist*
A	✗	✓	✗	✗
B	✗	✗	✓	✗
C	✓	✗	✗	✗
D	✗	✗	✗	✓

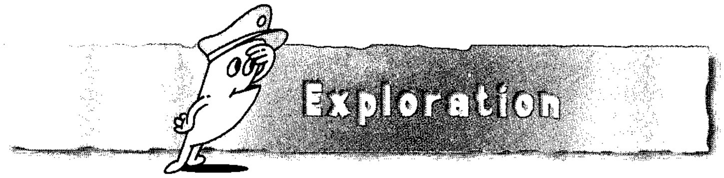
Table 5.2(e)

and is consistent with all the clues given.

If we assume C to be the dentist, then clue 2 and clue 7 would contradict.

Hence, we conclude that A is the banker, B the consultant, C the accountant and D the dentist.

In this example, we solve the problem by making *suppositions, tabulations* and *eliminating possibilities.*



Exploration

1. The diagram shows $\triangle ABC$ in which $A\hat{C}B = 90°$. $ACDE$ and $ABFG$ are squares. Prove that the area of $\triangle ABC$ = area of $\triangle AEG$ by considering three geometrical transformations. Describe these transformations clearly.

   Can you also prove that the area of $\triangle ABC$ = area of $\triangle AEG$ by using another method?

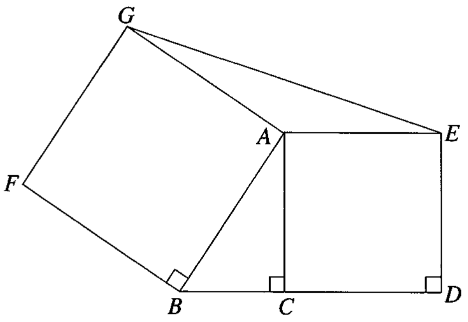


Fig. 5.63

2. In Fig. 5.64, $ABCD$ is a rectangle. Given that $QD = AB$, $D\hat{X}C = A\hat{Y}B = 90°$, describe two successive transformations that will map $\triangle ABY$ onto $\triangle QDX$.

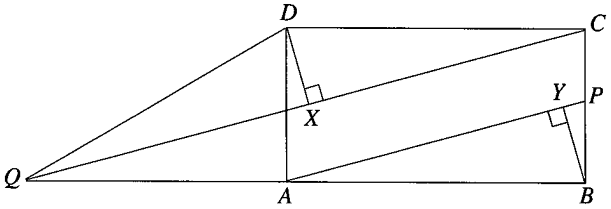


Fig. 5.64

3. In Fig. 5.65, $PQRS$ is a square, $LMNP$ is a rectangle and MR is parallel to LS. Copy the diagram and, by describing three successive shears which transforms the square into the rectangle $LMNP$, or otherwise, show that they have the same area. Given in addition that SR = 15 cm and MN = 20 cm, calculate
   (a) LM,
   (b) LS,
   (c) the distance between the lines MR and LS.

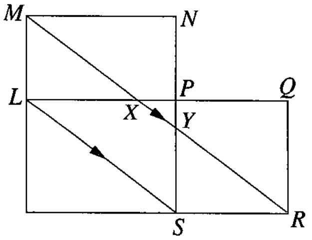


Fig. 5.65

CHAPTER 6

Probability

In this chapter, you will learn how to

- solve problems involving probabilities of simple and combined events;
- use possibility diagrams and tree diagrams to solve probability problems.

Preliminary Problem

The theory of probability originated as a method for studying games of chance.

The picture shows a queue waiting to bet on Toto, which is a favourite game of chance among Singaporeans. One of these hopefuls may be a big winner. What do you think is the probability of winning the jackpot prize in a Toto draw?

Introduction

Probability Theory was first used primarily to solve problems in gambling. An Italian, Girolamo Cardano (1501–1576), wrote a gambler's manual which made use of the theory. In response to a gambling problem posed to Blaise Pascal (1623–1662), a French mathematician, by Chevalier de Méré in 1654, Pascal and another French mathematician, Pierre Fermat, laid the foundations for probability.

This theory has become a powerful and widely applicable branch of mathematics. It has widespread use in business, science and industry. Its uses range from the determination of life insurance premiums to the description of the behaviour of molecules in a gas and also the prediction of outcomes in an election.

Directly or indirectly, probability or chance plays a role in all activities. We often make statements each of which involves the *likelihood* or the *chance of occurrence* of an *event*.

(a) 'It will *probably* rain today.'

The event is 'It will rain today.'

(b) 'Though we are sending the national team, we cannot confidently *predict* that we shall be in the finals again.'

The event is 'We shall be in the finals.'

(c) 'There is a 50 : 50 *chance* of our school winning the National School Basketball Championship.'

The event is 'Our school wins the National School Basketball Championship.'

Can you give other statements which involve probability?

Each statement suggests an event whose occurrence or non-occurrence involves an element of uncertainty. However because of past information or the present statistics of the event, we can say, with some degree of confidence, that the statement is valid. For example, we may make the statement (a) because most of the days we have observed were rainy days.

In conversation we might say that it was 'very warm' yesterday. However, the expert from the Meteorological Office would state the maximum temperature in °C. Similarly, a person might describe himself as 'big-footed', but when it comes to buying shoes, a more exact description (shoe size) is needed. Therefore, terms like 'most likely' and 'probably' used in the above statements are *too vague* for many purposes and so ways of measuring probability have been devised.

Back in Time

Pierre de Fermat (1601–1665) was a magistrate by profession. His extraordinary mathematical ingenuity enabled him to contribute profoundly to higher mathematics and analytical geometry. When he claimed that he had proved some mathematical theorems, he was always proved right. His famous 300-year-old 'Last Theorem' was finally proved in 1994 by Andrew Wiles. Fermat and Blaise Pascal were responsible for laying the foundations for the ***theory of probability****.*

Experiments

In the study of probability, an **experiment** is an operation or a process with a result or an *outcome* whose occurrence depends on **chance**. The following are some examples of such experiments.

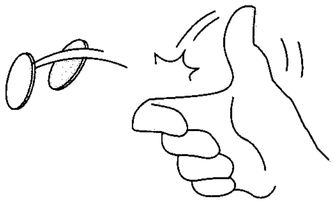
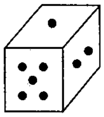
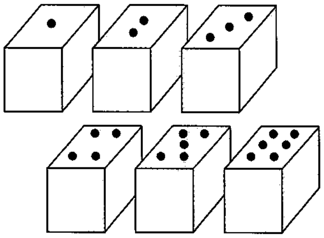
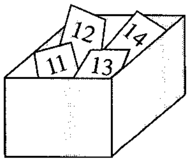
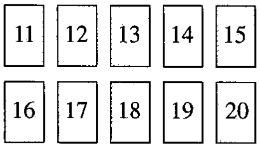
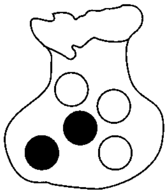
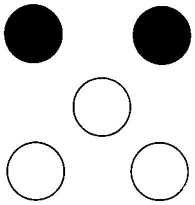
Experiment	Possible Outcomes
1.  Tossing a coin	 Head Tail
2.  Tossing a die	
3.  Ten cards numbered 11, 12, ... , 20 are placed in a box. One card is drawn.	
4.  Three white balls and two black balls are placed in a bag. One ball is picked from the bag.	

Fig. 6.1

Classical Definition of Probability

In Experiment 1, if the coin is fair, i.e. balanced or unbiased, then each outcome is equally likely to occur. Hence, in one toss, the chances that a head will appear is 1 in 2. We say that the probability of tossing a head is $\frac{1}{2}$.

In Experiment 2, if the die is fair, then each of the six outcomes is equally likely to show up on the top face. Therefore, the chances that a six will appear on the top face are 1 in 6. We say that the probability that a six will appear on the top face is $\frac{1}{6}$. What is the probability that a six will not appear on the top face?

In Experiment 3, if the card is drawn at random, i.e. the cards are thoroughly mixed in the box and we draw the card without looking, then it is equally likely that any one of the ten cards will be selected. There are 4 prime numbers from 11 to 20. These are 11, 13, 17 and 19. So, the chances of picking a card bearing a prime number are 4 in 10.

We say that the probability of selecting a card bearing a prime number is $\frac{4}{10} = \frac{2}{5}$.

What is the probability of selecting a card bearing (a) an even number, (b) an odd number, (c) a number which is neither prime nor even?

In Experiment 4, if a ball is drawn at random from the bag, then each ball will have an equal chance of being selected. The chances that a black ball is picked are 2 in 5. We say that the **probability** of drawing a black ball is $\frac{2}{5}$. What is the probability of selecting a white ball?

In an experiment, we often call the result we are interested in **an event**. The results of tossing a head in Experiment 1, getting six on the top face in Experiment 2, selecting a card bearing a prime number in Experiment 3 and selecting a white ball in Experiment 4, are all examples of events.

In general, in an experiment resulting in n equally likely outcomes, if m of these outcomes favour the occurrence of an event E, then the probability of event E happening, written $P(E)$, is defined as

$$P(E) = \frac{\textbf{No. of outcomes favourable to the occurrence of } E}{\textbf{Total number of equally likely outcomes}}$$
$$= \frac{m}{n}.$$

Casinos, lottery and other operators in the gambling business make use of the theory of probability to fix rules that will ensure that they are always on the winning side in the long run so that they will not go out of business.

Example 1

A card is drawn from a pack of 52 playing cards (well-shuffled so that the drawing is random).

(a) What is the total number of possible outcomes of this experiment?

(b) How many of these outcomes have the occurrence of

(i) a black card, *(ii) a red ace,*

(iii) a diamond, *(iv) a card which is not a diamond?*

Write down the probability of each event.

Solution

(a) There are 52 well-shuffled cards. Hence, there are 52 equally likely possible outcomes of this experiment.

(b) (i) There are 26 black cards in the pack. Hence, 26 outcomes have the occurrence of a black card.

$\therefore$ the probability of drawing a black card is $\frac{26}{52} = \frac{1}{2}$

or simply P(drawing a black card) = $\frac{1}{2}$.

(ii) There are 2 red aces, i.e. the ace of hearts and the ace of diamonds. Thus, the number of outcomes having the occurrence of a red ace = 2.

$\therefore$ P(drawing a red ace) = $\frac{2}{52} = \frac{1}{26}$

(iii) There are 13 diamonds in the pack. Therefore, 13 outcomes favour the occurrence of a diamond.

$\therefore$ P(drawing a diamond) = $\frac{13}{52} = \frac{1}{4}$

(iv) Since there are 13 diamonds in the pack, the remaining 39 cards in the pack are not diamonds. Hence, 39 outcomes favour the non-occurrence of a diamond.

$\therefore$ P(drawing a card which is not a diamond) = $\frac{39}{52} = \frac{3}{4}$

The CD, Tangible Math: Probability Constructor, by Logal Software provides simulations on the random behaviours when coins are tossed, dice rolled, colour wheels spun, marbles taken out of a jar, or dots placed in areas. Many Applets on probability are also available in the internet, for example, colour wheels which can simulate tossing coins, tetrahedral dice, six-sided dice and so on by choosing to have each wheel divided into 2, 4, 6 equal sectors and so on. You can search for the sites using the phrase, 'Applets on Probability'.

Example 2

A spinner in the form of a regular hexagon is constructed as shown in Fig. 6.2. When the pointer is spun, what is the probability that the pointer will stop on

(a) 6, *(b) an odd-numbered space,*
(c) a space whose number is a multiple of 3,
(d) a space whose number is less than 6,
(e) a space whose number is greater than 6,
(f) a space whose number is less than 7?

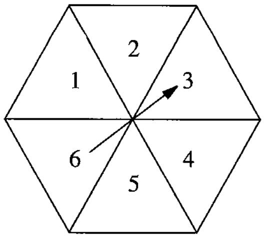


Fig. 6.2

Solution

When the pointer is spun, it is equally likely to stop on any one of the six spaces, assuming that the pointer does not stop on a line.

(a) P(the pointer will stop on 6) = $\frac{1}{6}$

(b) Since there are 3 odd-numbered spaces,

$\therefore$ P(the pointer will stop on an odd-numbered space) = $\frac{3}{6} = \frac{1}{2}$

(c) Since there are 2 spaces whose numbers are multiples of 3,

$\therefore$ P(the pointer will stop on a space whose number is a multiple of 3) = $\frac{2}{6} = \frac{1}{3}$

(d) There are 5 spaces whose numbers are less than 6,

$\therefore$ P(the pointer will stop on a space whose number is less than 6) = $\frac{5}{6}$

(e) There is no space whose number is greater than 6,

$\therefore$ P(the pointer will stop on a space whose number is greater than 6) = $\frac{0}{6} = 0$

(f) There are 6 spaces whose numbers are less than 7,

$\therefore$ P(the pointer will stop on a space whose number is less than 7) = $\frac{6}{6} = 1$

Example 3

A circle is divided into four sectors coloured yellow, red, blue and green as shown in Fig. 6.3.

A point is selected at random in the circle. Find the probability that it lies in the

(a) yellow sector,
(b) blue sector,
(c) black sector?

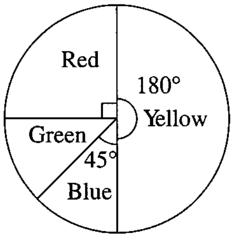
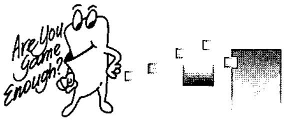


Fig. 6.3

Which one of the two probabilities is greater: P(the point lies in the red sector) or P(the point lies in the green sector)?



Myrna, Alison and Shelly repeatedly take turns tossing a die. Myrna begins; Alison always follows Myrna; Shelly always follows Alison; and Myrna always follows Shelly. What is the probability that Shelly will be the first one to toss a six?

Solution

Since a point is selected at random, any point in the circle will have the same chance of being selected.

(a) Number of points in the yellow sector is proportional to the area of the sector which is also proportional to the angle of the sector.

$\therefore$ P(selecting a point in the yellow sector)

$$= \frac{\text{Area of the yellow sector}}{\text{Area of the circle}}$$

$$= \frac{180°}{360°} = \frac{1}{2}$$

(b) Similarly, P(selecting a point in the blue sector)

$$= \frac{\text{Area of the blue sector}}{\text{Area of the circle}}$$

$$= \frac{45°}{360°} = \frac{1}{8}$$

(c) Since there is no black sector in the circle,

$\therefore$ P(selecting a point in the black sector) = $\frac{\text{Area of the black sector}}{\text{Area of the circle}}$

$$= \frac{0}{\text{Area of the circle}} = 0$$

As the area of the red sector is greater than the area of the green sector, P(the point lies in the red sector) > P(the point lies in the green sector).

Have you observed that for each of the events considered so far, the probability of it occurring has taken the values ranging from 0 to 1? Are there events for which the probabilities have values outside this range?

Recall that the probability of event E happening is given by

$$P(E) = \frac{m}{n},$$

when n represents the total number of equally likely outcomes and m, the outcomes favouring the occurrence of E.

Obviously, $0 \leqslant m \leqslant n$.

Dividing by n, we have $0 \leqslant \frac{m}{n} \leqslant 1$.

What kind of events give rise to the values of probability 0 and 1?

For any event E, $0 \leqslant P(E) \leqslant 1$.

If $P(E) = 0$, then the event cannot possibly occur.

If $P(E) = 1$, then the event will certainly occur.

Exercise 6a

1. A bag contains 40 marbles, 25 green ones and 15 red ones. A marble is picked at random from the bag. What is the probability of picking a red marble?

2. In a class of 30 pupils, 6 are short-sighted. If a pupil is selected at random, what is the probability that the selected pupil will be short-sighted?

3. In a class of 30 pupils, 12 are girls and two of them are short-sighted. Among the 18 boys, 6 are short-sighted. If a pupil is selected at random, what is the probability that the pupil chosen will be

(a) a girl, **(b)** short-sighted,
(c) a short-sighted boy?

4. Each of the letters of the word MATHEMATICS is written on a card. All the eleven cards are well-shuffled and placed down on a table. If a card is turned over, what is the probability that the card bears

(a) the letter '*M*', **(b)** a vowel,
(c) the letter '*P*'?

5. If we take a standard pack of 52 well-shuffled playing cards, what is the probability of drawing

(a) the ace of clubs,
(b) a king, queen or jack,
(c) a joker?

6. A two-digit number is written down at random. Find the probability that the number will be

(a) smaller than 20, **(b)** even,
(c) a multiple of 5.

7. If a number is chosen at random from the numbers 1 to 30 inclusive, what is the chance that a prime number will be picked?

8. In a car park, there are 100 vehicles, 75 of which are cars, 15 are lorries and 10 are buses. If they are all equally likely to leave, what is the probability of

(a) a lorry leaving first,
(b) a car leaving second if a lorry had left?

9. A box of 2 dozen pencils contains 8 pencils with broken points. What is the probability of picking one pencil without a broken point?

10. Peter has 10 books, 5 Chinese books and 5 English books, in his bag. Three of his ten books are science fiction which are in English. If a book is chosen at random, find the probability of choosing

(a) an English book,
(b) a Malay book,
(c) either a Chinese book or an English book,
(d) a science fiction book,
(e) an English book which is not a science fiction book,
(f) a book which is not a science fiction book.

11. A bag contains 15 balls of which x are red. Write an expression for the probability that a ball drawn at random from the bag is red. When 5 more red balls are added to the bag, the probability becomes $\frac{3}{4}$. Find the value of x.

12. Fig. 6.4 shows a circle divided into sectors of different colours. If a point is selected at random in the circle, calculate the probability that it lies in the

(a) yellow sector,
(b) red sector,
(c) blue sector,
(d) green or white sector.

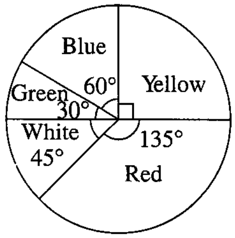


Fig. 6.4

Sample Space and Events

We mentioned earlier that an experiment results in several possible outcomes. The collection of all the possible outcomes is called the **sample space** or the **probability space** and is usually denoted by S. In general, if an experiment results in n outcomes, e_1, e_2, e_3, …, e_n, the collection of these n outcomes can be put in braces { } and thus, the sample space S of this experiment can be written as

$$S = \{e_1, e_2, e_3, \ldots, e_n\}.$$

The sample space of Experiment 1 which we discussed on page 160 (Fig. 6.1) is

$$S = \{H, T\},$$

where H represents the outcome of tossing a head and T, the outcome of tossing a tail.

In Experiment 2, the sample space is

$$S = \{1, 2, 3, 4, 5, 6\},$$

where each number represents the number of dots on the top face.

In Experiment 3, the sample space is

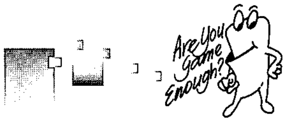
$$S = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}.$$

In Experiment 4, the sample space is

$$S = \{B_1, B_2, W_1, W_2, W_3\}$$

where B_1, B_2 represent the two black balls and W_1, W_2, W_3 represent the three white balls.

In Experiment 2, let A denote the event 'getting an even number on the top face of the die'. The definition of A collects the outcomes, namely



1. A couple gave birth to 10 girls in a row. What is the probability that the 11th child will be a boy?

2. A girl types 3 letters and 3 different addresses on 3 envelopes. She puts the letters into the envelopes randomly and sent them to 3 of her friends, A, B and C. What is the probability that
(a) only one of them will receive the correct letter?
(b) only two of them will receive the correct letters?

3. Dylan and David were eating some grapes. After some time, they decided to play a game. Each boy had to throw two dice each time. A grape would be given to John if the sum was 4 or 5 and to David if the sum was 10 or 11. Do you think the game is fair?

2, 4 and 6 from the sample space S and we write $A = \{2, 4, 6\}$. Thus, A is part of S and there are 3 possible ways out of 6 in which A can occur.

If **n(*A*)** denotes the number of outcomes in the event A and **n(*S*)** the number of all possible outcomes in the sample space, then

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}.$$

Similarly, in Experiment 3, if B denotes the event of selecting a card bearing a prime number, then

$$B = \{11, 13, 17, 19\}.$$
$$S = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}.$$
$$n(B) = 4,\ n(S) = 10 \text{ and } P(B) = \frac{n(B)}{n(S)} = \frac{4}{10} = \frac{2}{5}.$$

In general, an event E contains some or all of the outcomes of the sample space S that favour the occurrence of E and the probability of an event E occurring is given by

$$P(E) = \frac{n(E)}{n(S)},$$

where $n(E)$ is the number of outcomes in E and $n(S)$ is the total number of possible outcomes in the sample space S.

Possible Outcomes in the Sample Space

When an experiment is more complex, listing all possible outcomes in the sample space may be a problem. Systematic and effective methods are needed for the purpose. In the following sections, we will learn the methods of using probability or possibility diagrams and tree diagrams.

Possibility Diagrams

Let us look at some examples of finding probability using possibility diagrams.

Example 4

Fig. 6.5 shows two spinners each of which is divided into four equal sectors. Each spinner has a pointer which, when spun, is equally likely to come to rest in any of the four equal sectors.

In a game, each pointer is spun once.

(a) Find the probability that the points will stop

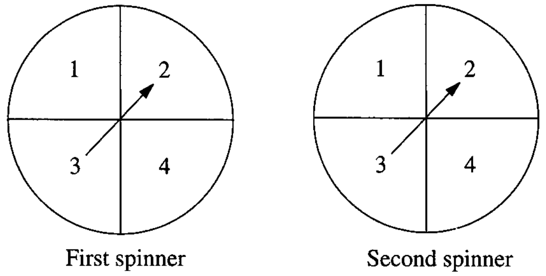


Fig. 6.5

(i) at the same number,
(ii) at different numbers.

(b) What is the probability that the first spinner shows the larger number?

Solution

We can use the ordered pair (a, b), where a refers to the number in the first spinner and b the number in the second spinner, to denote an outcome in the sample space. Thus, (1, 3) means a '1' from the first spinner and a '3' from the second spinner.

Hence,

the sample space, $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

For a more systematic and effective way of listing the outcomes in S, we can plot the ordered pairs as coordinates. Thus, we show all the possible outcomes on a diagram. (See Fig. 6.6.)

Fig. 6.6 is called a **possibility diagram**. How many dots are there in the diagram? What do they represent?

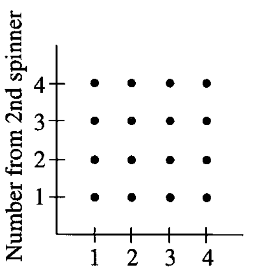


Fig. 6.6

From Fig. 6.6, n(S) = 16 (4 × 4 = 16).

We define the following events as follows:

A: getting the same numbers on the two spinners,
B: getting different numbers on the two spinners,
C: the first spinner shows the larger numbers.

The dots enclosed in the loop in Fig. 6.7 represent the possible outcomes in event A.

$\therefore \quad n(A) = 4$

and $\quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{4}{16} = \dfrac{1}{4}.$

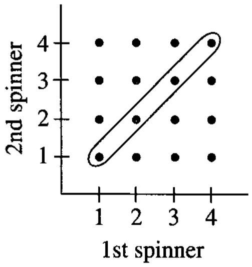


Fig. 6.7

In Fig. 6.8, the dots enclosed in the two triangles together represent the possible outcomes in event B.

$\therefore \quad n(B) = 12$

and $\quad P(B) = \dfrac{12}{16} = \dfrac{3}{4}.$

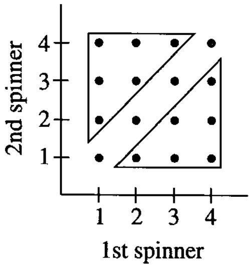


Fig. 6.8

The lower triangle in Fig. 6.8 contains the dots representing the outcomes where the first spinner shows the larger number.

$\therefore \quad n(C) = 6 \text{ and } P(C) = \dfrac{6}{16} = \dfrac{3}{8}.$

The numbers on the two spinners are either the same, or different. Do you notice that event B consists of all the outcomes in the sample space S that are not in A? If A' denotes the event 'not getting the same number on the two spinners', are A' and B the same event?

A and B are called **complementary events.** B is said to be complementary to A, i.e. $B = A'$ and vice versa, i.e. $A = B'$.

We note that

$$P(A) + P(B) = \frac{1}{4} + \frac{3}{4} = 1$$

i.e. $P(A) + P(A') = 1$ or $P(A) = 1 - P(A')$

and $P(B') + P(B) = 1$ or $P(B) = 1 - P(B')$.

In general, the complementary event to the event E, denoted by E' contains all the outcomes of the sample space S that are not in E. P(E) and P(E') satisfy the equation

$$P(E) + P(E') = 1 \text{ or } P(E) = 1 - P(E').$$

Note that E' denotes the event 'E does not occur.'

Example 5

Two dice are thrown together. Find the probability that the sum of the resulting numbers is

(a) odd, *(b) even,* *(c) a prime number,*

(d) a multiple of 4, *(e) at least 7.*

Solution

First die

Second die

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Fig. 6.9

Since we have to find several probabilities concerning the sum of two numbers on the die, it is more efficient to construct a possibility diagram as shown in Fig. 6.9.

All the possible sums are displayed in the diagram.

Do you notice a pattern in which the sums appear? If S denotes the sample space, from the diagram, n(S) = 36.

We define the following events:

A: the sum is odd. B: the sum is even. C: the sum is a prime number.

D: the sum is a multiple of 4. E: the sum is at least 7.

(a) From the possibility diagram, $n(A) = 2 + 4 + 6 + 4 + 2 = 18$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

(b) The sum is either even or odd.

$\therefore$ A and B are complementary events.

$$\therefore \quad P(B) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

Alternatively, from the possibility diagram, $n(B) = 1 + 3 + 5 + 5 + 3 + 1 = 18$

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

(c) The sums 2, 3, 5, 7 and 11 are prime numbers appearing in the possibility diagram. There are one '2', two '3's, four '5's, six '7's and two '11's.

$\therefore$ $n(C) = 1 + 2 + 4 + 6 + 2 = 15$

$$P(C) = \frac{n(C)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

What is the event C'?

(d) The sums 4, 8 and 12 are multiples of 4.

$n(D) = 3 + 5 + 1 = 9$ (three '4's, five '8's and one '12')

$$\therefore \quad P(D) = \frac{n(D)}{n(S)} = \frac{9}{36} = \frac{1}{4}$$

(e) From the possibility diagram, $n(E) = 6 + 5 + 4 + 3 + 2 + 1 = 21$

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{21}{36} = \frac{7}{12}$$

What is the event E'?

Can you write down the probability that the sum is less than 7?

Example 6

A circular card is divided into 3 equal sectors with scores of 1, 2 and 3. The card has a pointer pivoted at its centre. The pointer is spun twice.

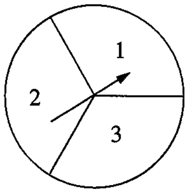


Fig. 6.10

(a) Find the probability that
- *(i) each score is 1,*
- *(ii) at least one of the scores is 3.*

(b) In a game, a player spins the pointer twice. His final score is the larger of the two individual scores if they are different and their common value if they are the same. The possibility diagram in Fig. 6.11 shows his final score.
- *(i) Copy and complete the possibility diagram.*
- *(ii) Using the diagram, find the probability that his final score is even.*
- *(iii) Using the same diagram, find the probability that his final score is a prime number.*

	1	2	3
1	1		
2			
3		3	

Fig. 6.11

Solution

(a) Let S denote the sample space.

From the possibility diagram,

$n(S) = 9$

(i) If A denotes the event 'each score is 1',

then $n(A) = 1$

and $P(A) = \frac{n(A)}{n(S)} = \frac{1}{9}$ (See Fig. 6.12.)

(ii) If B denotes the event 'at least one of the scores is 3',

then $n(B) = 5$

and $P(B) = \frac{n(B)}{n(S)} = \frac{5}{9}$ (See Fig. 6.12.)

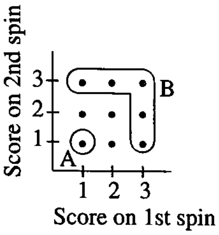


Fig. 6.12

(b) (i)

	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

Fig. 6.13

From the possibility diagram, n(S) = 9

(ii) Let C denote the event 'his final score is even'.

$$n(C) = 3 \quad \text{and} \quad P(C) = \frac{n(C)}{n(S)} = \frac{3}{9} = \frac{1}{3}$$

(iii) Let D denote the event 'his final score is a prime number'.

$$n(D) = 8 \quad \text{and} \quad P(D) = \frac{n(D)}{n(S)} = \frac{8}{9}.$$

✶Exercise 6b

1. A large fish tank contains 36 goldfishes of which 8 are red and 28 are black. One goldfish is picked at random.

(a) What is the probability that it will be red?

(b) Assuming that a red goldfish is picked the first time and put aside, what is the probability that the second goldfish picked will be red?

2. A box contains 5 red, 3 yellow and 2 blue discs. Two discs are drawn at random from the box one after another.

(a) What is the probability that the first disc drawn will be red?

(b) If the first disc drawn is blue and it is not replaced, what is the probability of drawing a yellow disc on the second draw?

3. All the clubs are removed from a pack of ordinary playing cards.

(a) A card is drawn at random from the remaining cards in the pack. Find the probability of drawing

(i) a red card, **(ii)** a heart,

(iii) a picture card, i.e. jack, queen or king,

(iv) a card that is not an ace.

(b) Assuming that a jack of spade is drawn the first time and put aside, find the probability that the second card drawn is

(i) a black card, **(ii)** an ace,

(iii) not a picture card.

4. A class has 16 boys and 24 girls. Of the 16 boys, 3 are left-handed and of the 24 girls, 2 are left-handed.

(a) Mrs Tee, the form teacher, selects a pupil to run an errand. Assuming that she is equally likely to select a pupil, what is the probability that she selects

(i) a boy,

(ii) a left-handed pupil?

(b) While waiting for the pupil to return, she needs another pupil to clean the blackboard. She selects one of the remaining pupils at random. Assuming that the first pupil she sent away is a girl who is not left-handed, find the probability that she selects

(i) a left-handed girl,

(ii) a left-handed boy.

5. A die is first thrown and a coin is then tossed. List all the possible outcomes of the sample space of the experiment using a possibility diagram. With the help of the possibility diagram, find the probability of the events A, B, C and D as defined below.

- A: getting a prime number on the die and the coin showing head.
- B: getting an even number on the die and the coin showing tail.
- C: getting a number greater than 4 on the die.
- D: getting a number less than 5 on the die.

6. A box contains three cards bearing the numbers, 1, 2 and 3. A second box contains four cards bearing the numbers 2, 3, 4 and 5. A card is chosen at random from each box.

(a) Display all the possible outcomes of the experiment using a possibility diagram.

(b) With the help of the possibility diagram, calculate the probability that

- **(i)** the cards bear the same number,
- **(ii)** the numbers on the cards are different,
- **(iii)** the larger of the two numbers on the cards is 3,
- **(iv)** the sum of the two numbers on the cards is less than 7,
- **(v)** the product of the two numbers on the cards is at least 8.

7. $X = \{4, 5, 6\}$ and $Y = \{7, 8, 9\}$. An element x is selected randomly from X and an element y is selected randomly from Y. The possibility diagrams in Fig. 6.14 display separately some of the values of $x + y$ and xy.

		x		
	+	4	5	6
	7	11		
y	8			
	9		14	

		x		
	×	4	5	6
	7			42
y	8		40	
	9			

Fig. 6.14

(a) Copy and complete the possibility diagrams.

(b) Find the probability that the product xy is

(i) odd, **(ii)** even,
(iii) at most 40.

(c) Find the probability that the sum $x + y$ is

(i) prime, **(ii)** greater than 12,
(iii) at most 14.

8. Six cards numbered 0, 1, 2, 3, 4, and 5 are placed in a box and well-mixed. A card is drawn at random from the box and the number on the card is noted before it is replaced in the box. The cards in the box are thoroughly mixed again, a second card is drawn at random from the box and the number on it is noted. The sum of the two numbers is then obtained.

(a) Copy and complete Fig. 6.15, giving all the possible sums of the two numbers. Some of the possible sums have been found.

		1st number					
	+	0	1	2	3	4	5
2nd number	0						
	1	1			4		
	2						
	3						
	4		5				
	5						

Fig. 6.15

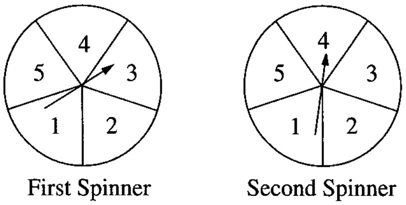
(b) How many possible outcomes are there in the sample space of this experiment?

(c) What is the probability that the sum of the two numbers

- **(i)** will be 7,
- **(ii)** will be a prime number,
- **(iii)** will not be a prime number,
- **(iv)** will be even,
- **(v)** will not be even?

(d) Which sum is more likely to occur, the sum of 7 or the sum of 8?

9. In an experiment, two spinners are constructed with spinning pointers as shown in Fig. 6.16. Both pointers are spun. Construct the sample space for this experiment.



First Spinner Second Spinner

Fig. 6.16

(a) How many possible outcomes are there in the sample space? Use a possibility diagram to show this.

(b) Find the probability that the pointers will stop
  - **(i)** at numbers on the spinners whose sum is 6,
  - **(ii)** at the same numbers on both spinners,
  - **(iii)** at different numbers on the spinners,
  - **(iv)** at two different prime numbers.

(c) What is the probability that the number on the first spinner will be less than the number on the second spinner?

10. In a game, the player throws a coin and a six-faced die simultaneously. If the coin shows a head, the player's score is the score on the die. If the coin shows a tail, then the player's score is twice the score on the die. Some of the player's possible scores are shown in the possibility diagram given in Fig. 6.17.

	Die						
		1	2	3	4	5	6
Coin	H	1					
	T			6			

Fig. 6.17

(a) Copy and complete the possibility diagram.

(b) Using the diagram, find the probability that the player's score is
  - **(i)** odd, **(ii)** even,
  - **(iii)** a prime number,
  - **(iv)** less than or equal to 8,
  - **(v)** a multiple of 3.

11. Two six-sided dice were thrown together and the difference of the resulting numbers on their faces was calculated. Some of the differences are shown in the possibility diagram given in Fig. 6.18.

(a) Copy and complete the possibility diagram.

(b) Using the diagram, find the probability that the difference of the two numbers is

		1st die					
		1	2	3	4	5	6
2nd die	1	0					
	2			1			4
	3						
	4						
	5						
	6		4				0

Fig. 6.18

  - **(i)** 1,
  - **(ii)** non-zero,
  - **(iii)** odd,
  - **(iv)** a prime number,
  - **(v)** more than 2.

12. Charles is sticking stamps to the value of 30 cents on each of a large number of envelopes. He has many 5-cent and 10-cent stamps with which to do this.

(a) Draw a small circle around each point in the diagram given in Fig. 6.19 which represents a possible choice of stamps he could make. (One small circle, representing the choice of one 10-cent stamp and four 5-cent stamps, has already been drawn for you.)

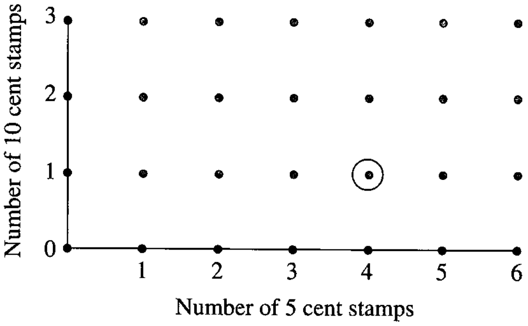


Fig. 6.19

(b) When he has finished, Charles notices that there is an equal number of envelopes bearing each possible choice of stamps. He chooses one envelope at random. Find the probability that it has
  - **(i)** exactly two 5-cent stamps,
  - **(ii)** more 5-cent stamps than 10-cent stamps. *(C)*

Tree Diagrams

Consider the experiment of tossing two fair coins.

The possibility diagram of the experiment is shown in Fig. 6.20.

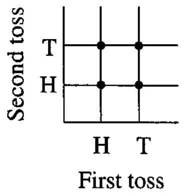


Fig. 6.20

The diagram drawn in the form as shown in Fig. 6.21 illustrates another way of listing all the possible outcomes of the experiment. It is called a **tree diagram**.

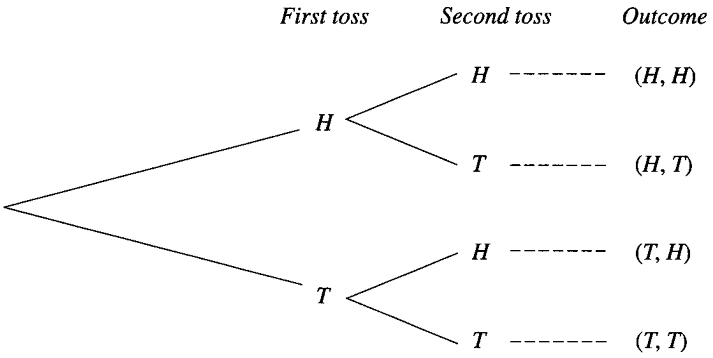


Fig. 6.21

Two line segments are drawn from a point to represent the two possible outcomes of tossing the first coin. From each of these outcomes, two line segments are drawn to represent the two possible outcomes of tossing the second coin.

Each 'branch' of the tree leads to an outcome of the experiment. For example, the first 'branch' from the top leads to the outcome of obtaining two heads represented by (H, H).

From the possibility diagram or the tree diagram,

$$S = \{(\text{H, H}), (\text{H, T}), (\text{T, H}), (\text{T, T})\} \text{ and thus } n(S) = 4.$$

We define the following events:

E: at least one head is obtained from tossing two coins
A: exactly one head is obtained from tossing two coins
B: exactly two heads are obtained from tossing two coins

$E = \{(\text{H, T}), (\text{T, H}), (\text{H, H})\}$, giving $n(E) = 3$ and thus, $\text{P}(E) = \frac{3}{4}$.

$A = \{(\text{H, T}), (\text{T, H})\}$, giving $n(A) = 2$ and thus, $\text{P}(A) = \frac{2}{4} = \frac{1}{2}$.

$B = \{(\text{H, H})\}$, giving $n(B) = 1$ and thus, $\text{P}(B) = \frac{1}{4}$.

Addition of Probabilities

In the above experiment, can the two events A and B occur simultaneously, i.e. when we toss two coins can we obtain exactly one head and exactly two heads at the same time?

The answer is **no**. If A and B can happen together, they must have some outcomes in common. Surely, $A = \{(H, T), (T, H)\}$ and $B = \{(H, H)\}$ do not have any outcomes in common.

A and B are known as **mutually exclusive** events, i.e. the occurrence of event A excludes the occurrence of event B and vice versa. Do you notice that event $E = \{(H, T), (T, H), (H, H)\}$ combines the events A and B such that E 'obtaining at least one head' occurs if **either** event A 'obtaining exactly one head' occurs **or** event B 'obtaining two heads' occurs?

Back in Time

Pascal's Triangle, named after the French mathematician, Blaise Pascal (1623–1662), is a triangular array of numbers. Blaise Pascal did extensive work with it and applied it to the field of probability.

Thus,

$$\text{P}(E \text{ occurs}) = \text{P(either } A \text{ occurs } \textbf{or } B \text{ occurs)}.$$

In short, we write

$$\text{P}(E) = \text{P}(A \textbf{ or } B).$$

We observe that

$$\text{P}(E) = \frac{3}{4} \quad \text{and} \quad \text{P}(A) + \text{P}(B) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Therefore,

$$\text{P}(A \textbf{ or } B) = \text{P}(A) + \text{P}(B).$$

In general, for two mutually exclusive events A and B,
P(A occurs or B occurs) = P(A or B) = P(A) + P(B).

This result can be extended to cover more than 2 mutually exclusive events. For three mutually exclusive events A, B and C,

$$\text{P}(A \text{ occurs or } B \text{ occurs or } C \text{ occurs}) = \text{P}(A \text{ or } B \text{ or } C) = \text{P}(A) + \text{P}(B) + \text{P}(C).$$

Example 7

Box A contains 4 pieces of paper numbered 1, 2, 3, 4. Box B contains 2 pieces of paper numbered 1, 2. One piece of paper is removed at random from each box. Draw a tree diagram to display all the possible outcomes of the experiment.

(a) Using the diagram, find the probability of obtaining at least one '1'.

(b) *By adding the possible sums and products of the two numbers obtained for the diagram, find the probability that*
   - (i) *the sum of the two numbers is 3,*
   - (ii) *the sum of the two numbers is 5,*
   - (iii) *the sum of the two numbers is odd,*
   - (iv) *the product of the two numbers is exactly divisible by 4,*
   - (v) *the product of the two numbers is* ***not*** *a prime,*
   - (vi) *the product of the two numbers is at least 4,*
   - (vii) *the sum is equal to the product.*

Solution

Do you think it is easy to represent all the possible outcomes of this experiment using dots in a possibility diagram? Can you suggest a way to do it?

The tree diagram is drawn and the possible sums and products of the two numbers are also displayed as shown in Fig. 6.22.

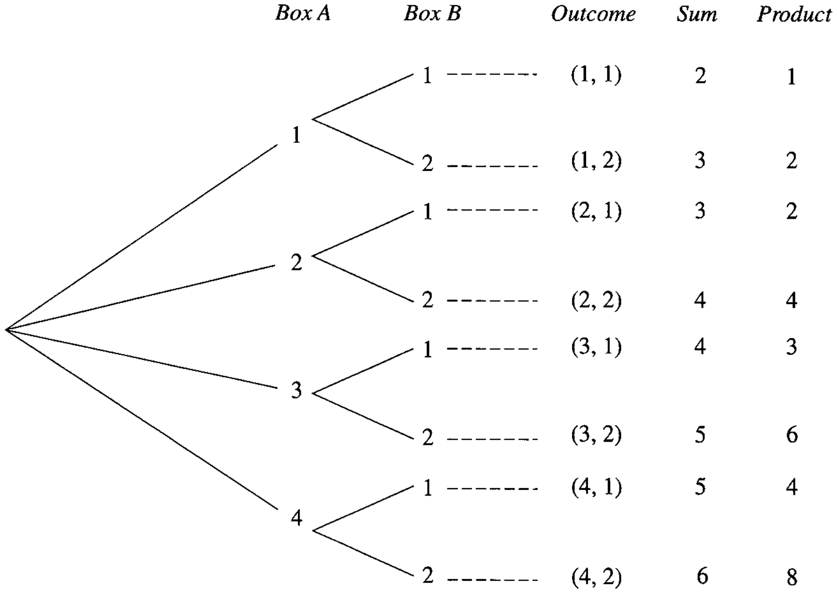


Fig. 6.22

From the diagram, n(S) = 8.

(a) Let A denote the event 'at least one "1"'.

A = {(1, 1), (1, 2), (2, 1), (3, 1), (4, 1)} and $n(A) = 5$.

$\therefore \quad P(A) = \frac{5}{8}$.

(b) We define the following events:

B: the sum of the two numbers is 3
C: the sum of the two numbers is 5
D: the sum of the two numbers is odd
E: the product of the two numbers is exactly divisible by 4
F: the product of the two numbers is not a prime
G: the product of the two numbers is at least 4
H: the sum is equal to the product

From the 'sum' column in Fig. 6.22, by counting, $n(B) = 2$; $n(C) = 2$ and thus,

(i) $P(B) = \frac{2}{8} = \frac{1}{4}$;

(ii) $P(C) = \frac{2}{8} = \frac{1}{4}$.

Sum of the two numbers is odd ⇒ either sum of the two numbers is 3 or 5.

(iii)
$$\begin{aligned} P(D) &= P(B \text{ or } C) \\ &= P(B) + P(C) \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

From the 'product' column in Fig. 6.22, by counting, $n(E) = 3$; $n(F) = 5$; $n(G) = 4$ and thus,

(iv) $P(E) = \frac{3}{8}$;

(v) $P(F) = \frac{5}{8}$;

(vi) $P(G) = \frac{4}{8} = \frac{1}{2}$.

From the 'sum' and 'product' columns, by comparing and counting $n(H) = 1$ and thus,

(vii) $P(H) = \frac{1}{8}$.

1. In (v), the event 'product is not a prime' and the event 'product is a prime' are complementary.

∴ $P(F) = 1 - P($product of the two numbers is a prime$)$

$$= 1 - \frac{3}{8} = \frac{5}{8}.$$

2. In (vi), 'product is at least 4' is equivalent to 'product is 4 or 6 or 8'.

∴ $P(G) = P($product is 4$) + P($product is 6$) + P($product is 8$)$

$$= \frac{2}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}.$$

Example 8

The probabilities of three teams, L, M and N, winning a football competition are $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{10}$ respectively. Calculate the probability that

(a) either L or M wins,

(b) neither L nor N wins.

Solution

We assume that only one team can win, so the events are mutually exclusive.

(a)
$$\begin{aligned} P(L \text{ or } M \text{ wins}) = P(L \text{ wins}) + P(M \text{ wins}) &= \frac{1}{4} + \frac{1}{8} \\ &= \frac{3}{8} \end{aligned}$$

(b)
$$\begin{aligned} P(L \text{ or } N \text{ wins}) = P(L \text{ wins}) + P(N \text{ wins}) &= \frac{1}{4} + \frac{1}{10} \\ &= \frac{7}{20} \end{aligned}$$

$$\begin{aligned} P(\text{neither } L \text{ nor } N \text{ wins}) = 1 - P(L \text{ or } N \text{ wins}) &= 1 - \frac{7}{20} \\ &= \frac{13}{20} \end{aligned}$$

Example 9

A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card is

(a) an ace or king, *(b) a heart or a diamond,*
(c) neither a king nor a queen.

Solution

We define the following events:

A: the card drawn is an ace. H: the card drawn is a heart. D: the card drawn is a diamond.
K: the card drawn is a king. Q: the card drawn is a queen.

(a) $\text{P}(A) = \frac{4}{52} = \frac{1}{13}$ and $\text{P}(K) = \frac{4}{52} = \frac{1}{13}$

Now, the events A and K are mutually exclusive since they cannot happen at the same time, i.e. a card cannot be both an ace and a king.

$$\therefore \quad \text{P}(A \text{ or } K) = \text{P}(A) + \text{P}(K) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

(b) $\text{P}(H) = \frac{13}{52} = \frac{1}{4}$ and $\text{P}(D) = \frac{13}{52} = \frac{1}{4}$

The events H and D are mutually exclusive.

$$\therefore \quad \text{P}(H \text{ or } D) = \text{P}(H) + \text{P}(D) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

(c) $\text{P}(K \text{ or } Q) = \text{P}(K) + \text{P}(Q) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$

$$\therefore \quad \text{P}(\text{neither } K \text{ nor } Q) = 1 - \text{P}(K \text{ or } Q) = 1 - \frac{2}{13} = \frac{11}{13}$$

★Exercise 6c

1. A die in the form of a tetrahedron (solid regular triangular pyramid) has the numbers 1, 2, 3 and 4 printed on its four faces.

- **(a)** When the die is thrown what is the probability that
  - **(i)** it will land with the face printed 4 down,
  - **(ii)** it will land so that the sum of the three upper faces is an odd number.
- **(b)** If the die and a coin are thrown together, list all possible outcomes of the experiment using a tree diagram.

2. A spinner as shown in Fig. 6.23 and a coin are used in a game. The spinner is spun once and the coin is tossed once. Draw a tree diagram to list all possible outcomes. With the help of the tree diagram, calculate the probability of getting

- **(a)** red on the spinner and tail on the coin,
- **(b)** blue or yellow on the spinner and head on the coin.

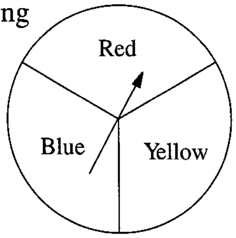


Fig. 6.23

3. Two spinners similar to the spinner in Question 2 are spun simultaneously. Display all possible outcomes in a tree diagram. Hence, find the probability that
   - **(a)** the pointers will stop on sectors of different colours,
   - **(b)** one of the pointers stops on red and the other pointer stops on yellow,
   - **(c)** the first pointer stops on blue and the second pointer stops on red or yellow.

4. A bag contains 4 cards numbered 1, 3, 5, 7. A second bag contains 3 cards numbered 1, 2, 7. One card is drawn at random from each bag.
   - **(a)** Draw a tree diagram for the experiment.
   - **(b)** With the help of your tree diagram, calculate the probability that the two numbers obtained
     - **(i)** have the same value,
     - **(ii)** are both odd,
     - **(iii)** are both prime,
     - **(iv)** have a sum greater than 4,
     - **(v)** have a sum that is even,
     - **(vi)** have a product that is prime,
     - **(vii)** have a product that is greater than 20,
     - **(viii)** have a product that is divisible by 7.

5. Fig. 6.24 shows two cards, each with a pointer pivoted at its centre. The first card is divided into 4 equal sectors scoring 1, 2, 4 and 5. The second card is divided into 4 equal sectors scoring 0, 1, 3 and 5. In a game, the pointers are spun.

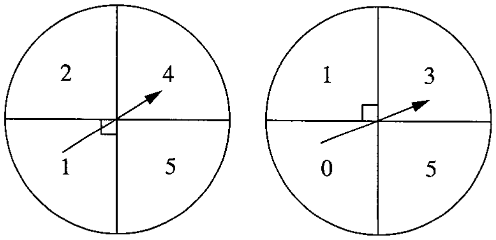


Fig. 6.24

   Using a tree diagram, find the probability that
   - **(a)** the score on each card is the same,
   - **(b)** the score on each card is prime,
   - **(c)** the sum of the scores is odd,
   - **(d)** the sum of the scores is divisible by 5,
   - **(e)** the sum of the scores is 6 or less,
   - **(f)** the product of the scores is not zero,
   - **(g)** the product of the scores is greater than 11.

6. Eleven cards numbered 11, 12, 13, 14, ... 21 are placed in a box. A card is removed at random from the box. Find the probability that the number on the card is
   - **(a)** even,
   - **(b)** prime,
   - **(c)** either even or prime,
   - **(d)** divisible by 3,
   - **(e)** either even or divisible by 3,
   - **(f)** odd,
   - **(g)** divisible by 4,
   - **(h)** either odd or divisible by 4.

7. A bag contains 7 red, 5 green and 3 blue counters. A counter is selected at random from the bag. Find the probability of selecting
   - **(a)** a red counter,
   - **(b)** a green counter,
   - **(c)** either a red or a green counter,
   - **(d)** neither a red nor a green counter.

8. The letters of the word '*MUTUALLY*' and the word '*EXCLUSIVE*' are written on individual cards and the cards are put into a box. A card is picked at random. What is the probability of picking
   - **(a)** the letter '*U*',
   - **(b)** the letter '*E*',
   - **(c)** the letter '*U*' or '*E*',
   - **(d)** a consonant,
   - **(e)** the letter '*U*' or a consonant,
   - **(f)** the letter '*U*' or '*E*' or '*L*'?

9. A coin is tossed three times. Display all the possible outcomes of the experiment using a tree diagram.

   From your tree diagram, find the probability of obtaining
   - **(a)** three heads,
   - **(b)** exactly two heads,
   - **(c)** at least two heads.

10. The probability of a football team winning any match is $\frac{7}{10}$ and the probability of losing any match is $\frac{2}{15}$.

(a) What is the probability that the team wins or loses a particular match?

(b) What is the probability that the team neither wins nor loses a match?

∗11. When a golfer plays any hole, the probabilities that he will take 4, 5 or 6 strokes are $\frac{1}{14}$, $\frac{2}{7}$ and $\frac{3}{7}$ respectively. He never takes less than 4 strokes. Calculate the probability that in playing a hole, he will take

(a) 4 or 5 strokes,

(b) 4, 5 or 6 strokes,

(c) more than 6 strokes.

∗12. In a basketball tournament, three of the participating teams, Panda, Spaceship and Rocket have the probabilities $\frac{4}{15}$, $\frac{1}{10}$ and $\frac{1}{5}$ respectively, of winning the tournament. Find the probability that

(a) Panda or Rocket will win the tournament,

(b) Panda, Spaceship or Rocket will win the tournament,

(c) neither Panda nor Rocket will win the tournament,

(d) none of these three teams will win the tournament.

Multiplication of Probabilities – Probability Tree

A bag contains 2 red balls and 4 blue balls. Mary takes a ball at random from the bag and replaces it back in the bag. She mixes the balls in the bag well before repeating the process of picking a ball at random from the bag. Fig. 6.25 displays the possibility diagram for the above experiment.

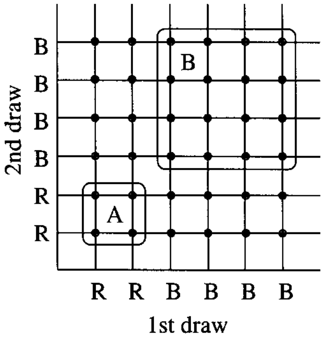


Fig. 6.25

In the possibility diagram, R and B represent the outcomes of drawing a red ball and a blue ball in each draw respectively.

Consider the following events:

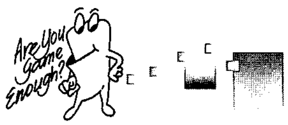
A: Mary draws a red ball in both occasions

B: Mary draws a blue ball in both occasions

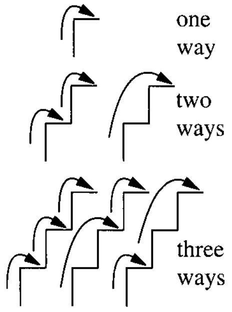
From Fig. 6.25, we have,

$$P(A) = \frac{4}{36} = \frac{1}{9} \quad \text{and} \quad P(B) = \frac{16}{36} = \frac{4}{9}.$$

Do you think it is practical to use the above method if, in each draw, Mary takes a ball from a bag with say 30 red and blue balls?



1. Consider a boy who can climb up the stairs in one step, or at most, two steps in one stride. For example, there is only one way to climb a one-step stair, two ways to climb a two-step stair and three ways to climb a three-step stairs.



Find the number of ways to climb four-, five-, six- and seven-step stairs. Copy and complete the following table.

Let us imagine constructing a tree diagram for the above problem. The possibility diagram in Fig. 6.25 and the tree diagram in Fig. 6.22 may suggest a rather complicated diagram with 36 'branches'. If we were to construct a tree diagram this way, there will be 30 × 30 = 900 'branches' if there were 30 balls in the bag. A way of simplifying the problem is to construct a **probability tree** as shown in the following.

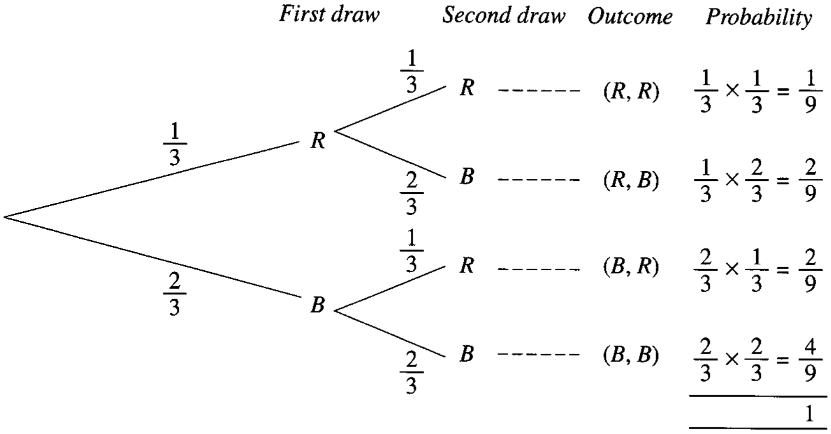


Fig. 6.26

In the probability tree, instead of repeating the 'R's and the 'B's, appropriate probabilities are added as shown. The probabilities are obtained as follows.

In each draw, P(Mary takes a red ball) = $\frac{2}{6} = \frac{1}{3}$

and P(Mary takes a blue ball) = $\frac{4}{6} = \frac{2}{3}$

or P(Mary takes a blue ball) = 1 – P(Mary takes a red ball)

$$= 1 - \frac{1}{3} = \frac{2}{3}.$$

If Mary performs the experiment by drawing balls from a bag containing 30 red and blue balls, $\frac{1}{3}$ of which are red, will the probability tree for the experiment be different from that in Fig. 6.26?

To obtain P(*A*) from the probability tree, we simply **multiply** the probabilities along the 'branch' leading to the outcome (R, R), i.e.

$$\begin{aligned} P(A) &= \frac{1}{3} \times \frac{1}{3} \\ &= \frac{1}{9}. \end{aligned}$$

Similarly,

$$\begin{aligned} P(B) &= \frac{2}{3} \times \frac{2}{3} \\ &= \frac{4}{9}. \end{aligned}$$

Now, consider the events

A_1: Mary takes a red ball in her first draw; A_2: Mary takes a red ball in her second draw;

$P(A_1) = \frac{2}{6} = \frac{1}{3}$ $P(A_2) = \frac{2}{6} = \frac{1}{3}$

No. of steps	*1*	*2*	*3*	*4*	*5*	*6*	*7*
No. of ways	*1*	*2*	*3*				

Formulate a rule to help you find the number of ways to climb eight-, nine- and ten-step stairs.

2. *Consider a man who can climb up the stairs in one step, two steps or, at most, three steps in one stride. Find the number of ways he can climb four-, five-, six- and seven-step stairs. Copy and complete the following table.*

No. of steps	*1*	*2*	*3*	*4*	*5*	*6*	*7*
No. of ways	*1*	*2*	*4*				

Form a general rule to help the man determine the number of ways he can climb eight-, nine- and ten-step stairs.

Clearly, event A occurs only if event A_1 **and** event A_2 occur.

Thus, P(A occurs) = P(A_1 occurs **and** A_2 occurs).

We write simply as $P(A) = P(A_1 \textbf{ and } A_2)$.

From above, $P(A) = \frac{1}{3} \times \frac{1}{3} = P(A_1) \times P(A_2)$.

Similarly, $P(B) = P(B_1 \textbf{ and } B_2) = P(B_1) \times P(B_2) = \frac{2}{3} \times \frac{2}{3}$, where

B_1: Mary takes a blue ball in her first draw; B_2: Mary takes a blue ball in her second draw;

$P(B_1) = \frac{4}{6} = \frac{2}{3}$ $P(B_2) = \frac{4}{6} = \frac{2}{3}$.

If C denotes the event 'Mary draws two balls of different colours', then C' is the event 'Mary draws two balls of the same colour',

we have $P(C') = P(A \textbf{ or } B) = P(A) + P(B) = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$.

Thus, $P(C) = 1 - P(C') = 1 - \frac{5}{9} = \frac{4}{9}$.

Alternatively,

$$
\begin{aligned}
P(C) &= P[(A_1 \textbf{ and } B_2) \textbf{ or } (B_1 \textbf{ and } A_2)] \\
&= P(A_1 \textbf{ and } B_2) + P(B_1 \textbf{ and } A_2) \\
&= \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} \\
&= \frac{4}{9}.
\end{aligned}
$$

Now, suppose Mary makes a second draw **without** replacing the first ball drawn in the bag. The possibility diagram and probability tree are as shown in Fig. 6.27 and Fig. 6.28 respectively.

Notice that this diagram is the same as that in Fig. 6.25 except for the missing dots along the diagonal from left to right. Why?

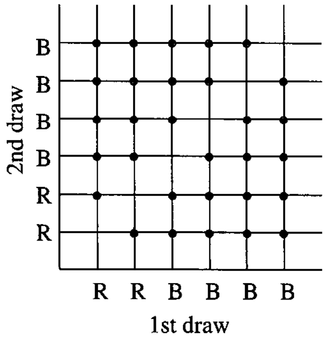


Fig. 6.27

Notice that in Fig. 6.28, the probabilities corresponding to outcomes of the second draw are adjusted to $\frac{1}{5}$, $\frac{4}{5}$ and $\frac{2}{5}$, $\frac{3}{5}$ respectively depending on whether Mary first draws a red ball or a blue ball. Why?

Notice also that $\frac{2}{6} + \frac{4}{6} = \frac{1}{5} + \frac{4}{5} = \frac{2}{5} + \frac{3}{5} = 1$.

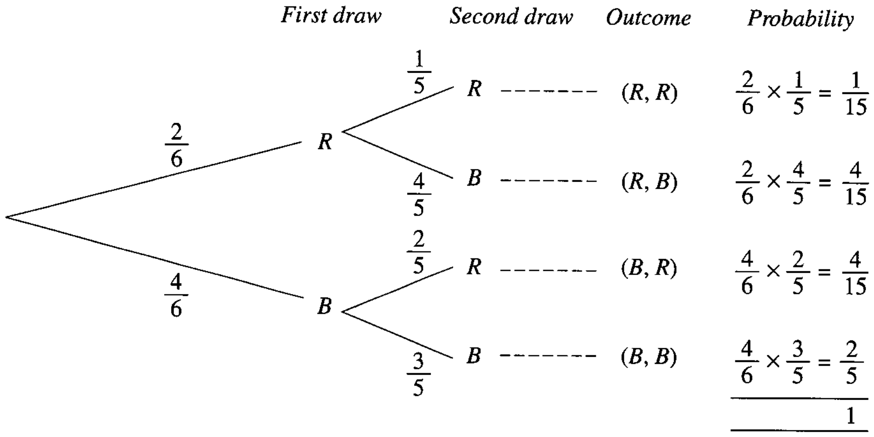


Fig. 6.28

As before, P(*A*) and P(*B*) can be found either by using the possibility diagram or probability tree. From Fig. 6.27, by counting,

$$P(A) = \frac{2}{30} = \frac{1}{15}; \quad P(B) = \frac{12}{30} = \frac{2}{5}.$$

From Fig. 6.28, by multiplying probabilities along appropriate 'branches',

$$P(A) = \frac{2}{6} \times \frac{1}{5} = \frac{1}{15}; \quad P(B) = \frac{4}{6} \times \frac{3}{5} = \frac{2}{5}.$$

Also, from Fig. 6.28,

$$P(C) = \frac{4}{15} + \frac{4}{15} = \frac{8}{15}.$$

In general, in a probability tree representing an experiment consisting of two **trials** (tossing two coins, tossing a coin twice, drawing two objects from a container together, drawing two objects from a container one after another with or without replacement, etc.),

1. **(a) each trial produces a number of mutually exclusive outcomes represented by the corresponding number of sub 'branches',**
   (b) total probabilities of these outcomes equals 1 and thus the probabilities appearing on the sub 'branches' must add to 1.

2. **(a) the main 'branches' represent all the mutually exclusive outcomes of the experiment and the probability of each outcome is obtained by multiplying the probabilities along the 'branch' leading to that outcome,**
   (b) the probabilities of all these outcomes must add to 1.

Drawing two objects from a container together is equivalent to drawing two objects from the container one after another without replacement.

We can also extend the same method to experiments with more than two trials.

Example 10

A box contains a large number of sweets, identical except for the colours. One quarter of the sweets are green and two thirds are red. The remainder of the sweets are yellow.

(a) A sweet is taken at random from the box.

- *(i) Explain why the probability of picking a yellow sweet is $\frac{1}{12}$.*
- *(ii) Find the probability ofobtaining a sweet that is not green.*

(b) Two sweets are taken at random from the box.

- *(i) Draw a probability tree to show the possible outcomes and their probabilities.*
- *(ii) Find the probability that*
  - *(a) both sweets picked are red,*
  - *(b) one sweet picked is yellow and the other is red,*
  - *(c) at least one sweet taken is red,*
  - *(d) neither sweets taken is green.*

Solution

(a) (i) Since $1 - \frac{1}{4} - \frac{2}{3} = \frac{1}{12}$ of the sweets in the box is yellow, the probability of picking a yellow sweet at random from the box is $\frac{1}{12}$.

(ii) P(sweet picked is not green) = 1 – P(sweet picked is green)

$$= 1 - \frac{1}{4} = \frac{3}{4}.$$

(b) (i) The probability tree is as shown in Fig. 6.29.

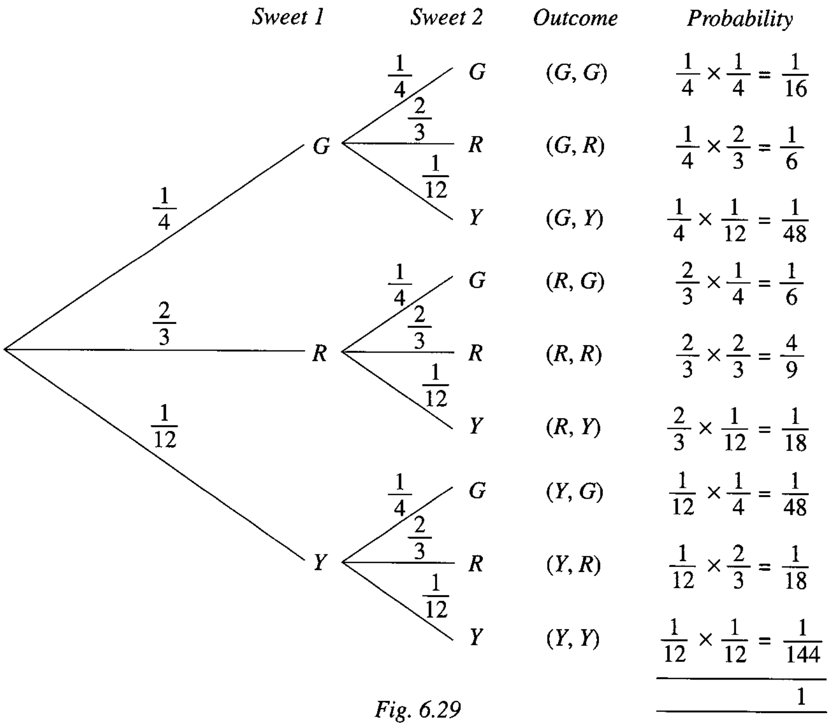
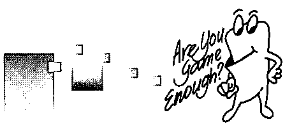


Fig. 6.29



Toss a coin 5, 10, 15, 20, 30, 50, 80, 100 times. You may work with a partner so that your partner can help in recording the number of times heads and tails turn up for each set of tosses. Enter your results in the table where m denotes the number of heads, l the number of tails, n the number of tosses.

n	*m*	*l*	$p = \frac{m}{n}$
5			
10			
15			
20			
30			
50			
80			
100			

Complete the table by

- *(a) calculating the values of $p = \frac{m}{n}$, giving your answers in decimals correct to 2 decimal places.*
- *(b) On a sheet of graph paper plot the points with coordinates given by (n, p). Add to your graph the horizontal line p = 0.5. What do you notice about the points plotted in relation to this line as n increases?*
- *(c) By combining the results for the set of 100 tosses for the whole class, i.e. using the values of m corresponding to n = 100 for the whole class, calculate the value*

$\frac{\text{total number of heads}}{\text{total number of tosses}}$, giving your answer in decimal correct to 2 decimal places.

(d) What conclusion can you draw concerning this experimental probability of obtaining a head when a coin is tossed once?

As mentioned earlier, the experiment of taking two sweets together from the box is the same as taking two sweets one after another from the box without replacement. However, the question indicates to us that we can assume the probabilities of drawing sweets of the three colours to remain the same from one draw to another by mentioning that the box contains a large number of sweets.

(ii) Define the following events:

A: both sweets picked are red
B: one sweet picked is yellow and the other is red
C: at least one sweet picked is red
D: neither sweets picked is green

Using the probability tree:

(a) $\text{P}(A) = \text{P}[(\text{R, R})]$
$= \frac{4}{9}.$

(b) $\text{P}(B) = \text{P}[(\text{R, Y}) \text{ or } (\text{Y, R})]$
$= \frac{1}{18} + \frac{1}{18}$
$= \frac{1}{9}.$

(c) $\text{P}(C) = \text{P}(A \text{ or } B \text{ or } [(\text{R, G}) \text{ or } (\text{G, R})]$
$= \frac{4}{9} + \frac{1}{9} + \left(\frac{1}{6} + \frac{1}{6}\right)$
$= \frac{8}{9}.$

(d) $\text{P}(D) = \text{P}[A \text{ or } B \text{ or } (\text{Y, Y})]$
$= \frac{4}{9} + \frac{1}{9} + \frac{1}{144}$
$= \frac{9}{16}.$

Alternatively, using the answer to (a)(ii),

$\text{P}(D) = \text{P(both sweets are not green)}$
$= \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}.$

Example 11

Twelve pupils in a group study either Chemistry or History but not both. 7 of them study Chemistry and 5 study History.

(a) A pupil is selected at random from the group. Write down the probability that the pupil studies Chemistry.

(b) If two pupils are chosen at random from the group on another occasion, find the probability that
  - *(i) they both study History,*
  - *(ii) they study the same subject,*
  - *(iii) the first pupil studies Chemistry and the second pupil studies History,*
  - *(iv) they study different subjects.*

Solution

(a) P(the selected pupil studies Chemistry) = $\frac{7}{12}$

(b) Choosing two pupils from the group can be done by first selecting one pupil followed by selecting another pupil from the remaining pupils.

This is a case of selection without replacement.

The probability tree for the selection is as shown in Fig. 6.30.

C denotes 'a pupil studies Chemistry'.

H denotes 'a pupil studies History'.

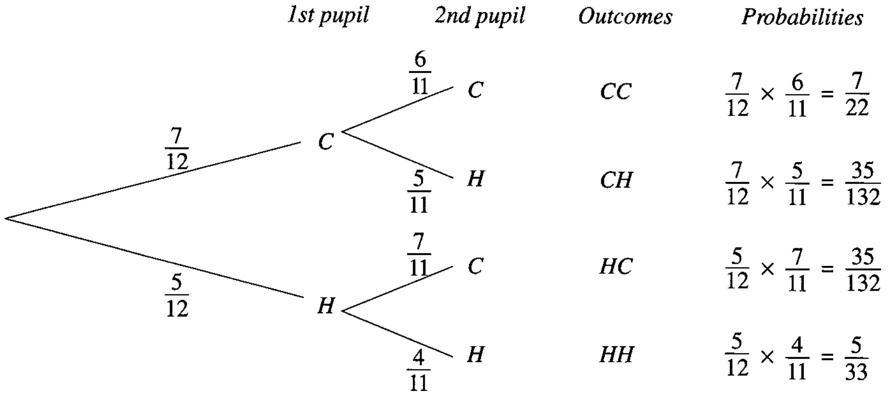


Fig. 6.30

Note: Check that $\frac{7}{22} + \frac{35}{132} + \frac{35}{132} + \frac{5}{33} = \frac{132}{132} = 1.$

From the probability tree,

(i) P(they study History) = $P(HH) = \frac{5}{33}$

(ii) P(they study the same subject) = $P(CC \text{ or } HH) = \frac{7}{22} + \frac{5}{33} = \frac{31}{66}$

(iii) P(1st pupil studies Chemistry and 2nd pupil studies History) = $P(CH) = \frac{35}{132}$

(iv) P(they study different subjects) = $P(CH \text{ or } HC) = \frac{35}{132} + \frac{35}{132} = \frac{35}{66}$

Alternatively,

P(they study different subjects) = 1 – P(they study the same subject) = $1 - \frac{31}{66} = \frac{35}{66}$

Example 12

Bag X contains 10 balls of which 3 are red and 7 are blue. Bag Y contains 10 balls of which 4 are red and 6 are blue. One ball is drawn at random from Bag X and placed in Bag Y. After thoroughly mixing, a ball is taken from Bag Y and placed in Bag X. With the help of a probability tree, calculate the probability that

(i) a red ball is drawn from Bag X and a blue ball is drawn from Bag Y,
(ii) two balls of different colours are drawn,
(iii) the ball drawn from Bag Y is red,
(iv) Bag X still contains exactly 3 red balls after the two draws.

Solution

Fig. 6.31 shows the probability tree.

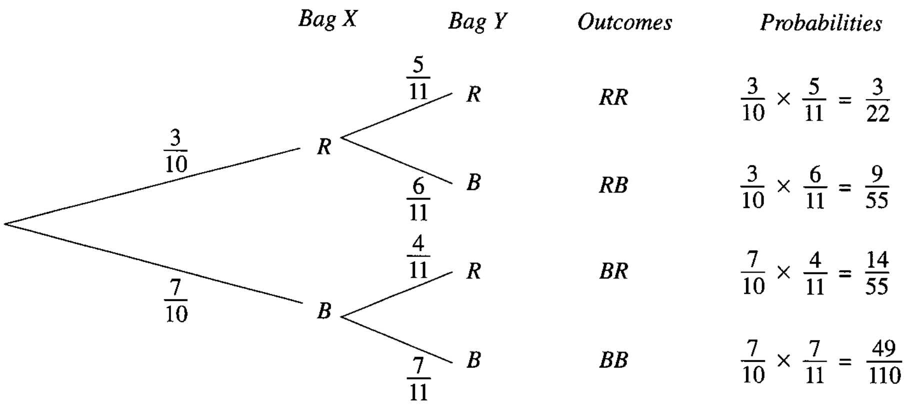


Fig. 6.31

Note that the number of balls in Bag Y is 11 after the ball drawn from Bag X is placed in it. The number of red and blue balls in Bag Y then depends on the result of the draw from Bag X.

Note: Check that $\frac{3}{22} + \frac{9}{55} + \frac{14}{55} + \frac{49}{110} = 1$

From the probability tree,

(a) P(a red from Bag X and a blue from Bag Y) = P(RB) = $\frac{9}{55}$

(b) P(two balls of different colours) = P(RB or BR)

$$= \frac{9}{55} + \frac{14}{55}$$

$$= \frac{23}{55}$$

(c) P(ball from Bag Y is red) = P(RR or BR)

$$= \frac{3}{22} + \frac{14}{55}$$

$$= \frac{43}{110}$$

(d) In order that Bag X still contains exactly 3 red balls, the balls drawn from Bag X and Bag Y must be of the same colour.

$\therefore$ P(Bag X still contains exactly 3 red balls) = P(RR or BB)

$$= \frac{3}{22} + \frac{49}{110}$$

$$= \frac{32}{55}$$

Exercise 6d

1. Peter has two bags each containing 5 black marbles and 4 red marbles. He takes one marble at random from each bag.
   - **(a)** Copy and complete the probability tree in Fig. 6.32.
   - **(b)** Calculate the probability that he takes
     - **(i)** a red marble from the first bag and a black marble from the second bag,
     - **(ii)** two marbles having different colours,
     - **(iii)** a black marble from the second bag.

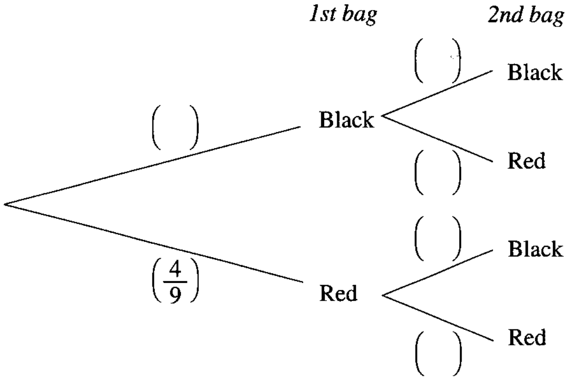


Fig. 6.32

2. A bag contains 6 red balls and 4 yellow balls. A ball is chosen at random and then put back into the bag. The process is carried out twice.
   - **(a)** Copy and complete the probability tree shown in Fig. 6.33.
   - **(b)** Find the probability of taking out
     - **(i)** two red balls,
     - **(ii)** one ball of each colour.

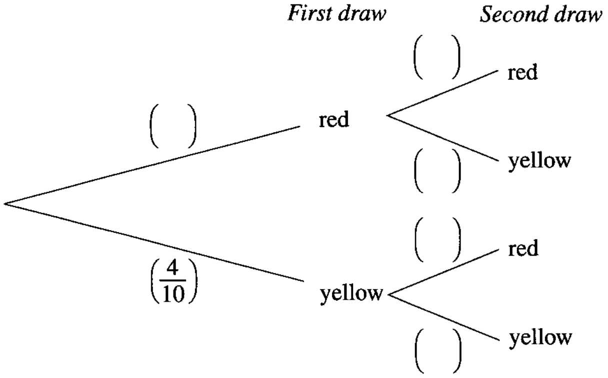


Fig. 6.33

3. Fig. 6.34 shows two discs each of which is divided into four equal sectors. Each disc has a pointer which, when spun, is equally likely to come to rest in any of the four equal sectors.

   In a game, the player spins each pointer once. His score is the sum of the numbers shown by the pointers.

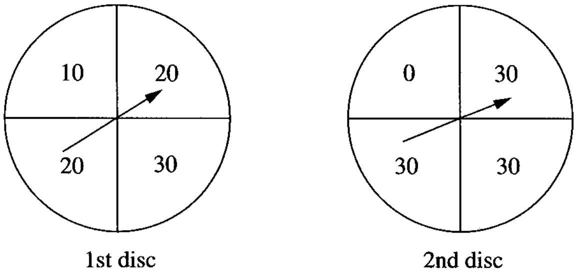


Fig. 6.34

   - **(a)** Copy and complete the probability tree in Fig. 6.35.
   - **(b)** With the help of the diagram, calculate the probability that
     - **(i)** the first score is less than or equal to the second score,
     - **(ii)** the second score is zero.
   - **(c)** If the player's score is between 10 and 50 but excluding 10 and 50, he receives \$2. If his score is more than 40, he receives \$5. Otherwise, he receives nothing. What is the probability that he receives
     - **(i)** \$2,
     - **(ii)** \$5,
     - **(iii)** \$2 or \$5,
     - **(iv)** nothing.

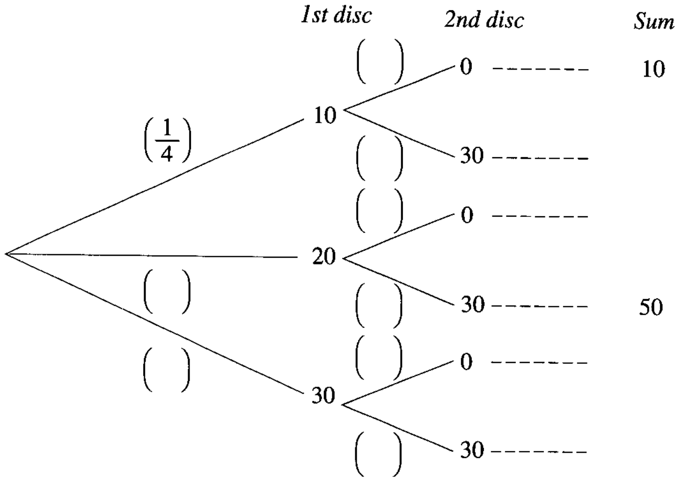


Fig. 6.35

4. (a) Box A contains 12 balls, 6 of which are blue, 2 are red and the remaining are yellow. Box B contains 10 balls, 4 of which are blue and the remaining are red. One ball is removed at random from each box.

   Find the probability that

   (i) the ball from Box A is red and the ball from Box B is blue,
   (ii) one red ball and one blue ball are removed,
   (iii) the ball removed from Box B is blue,
   (iv) balls of the same colour are removed,
   (v) balls of different colours are removed.

   (b) In Box A, three balls are labelled 2, four balls are labelled 3 and five balls are labelled 5. In Box B, two balls are labelled 1, three balls are labelled 4 and five balls are labelled 6. One ball is selected at random from each box.

   Copy and complete the probability tree in Fig. 6.36.

   (c) With the help of the probability tree, calculate the probability of obtaining

   (i) a sum which is a prime number,
   (ii) a sum which is less than 6,
   (iii) a product which is divisible by 3 and greater than 12,
   (iv) a product which is less than or equal to 5.

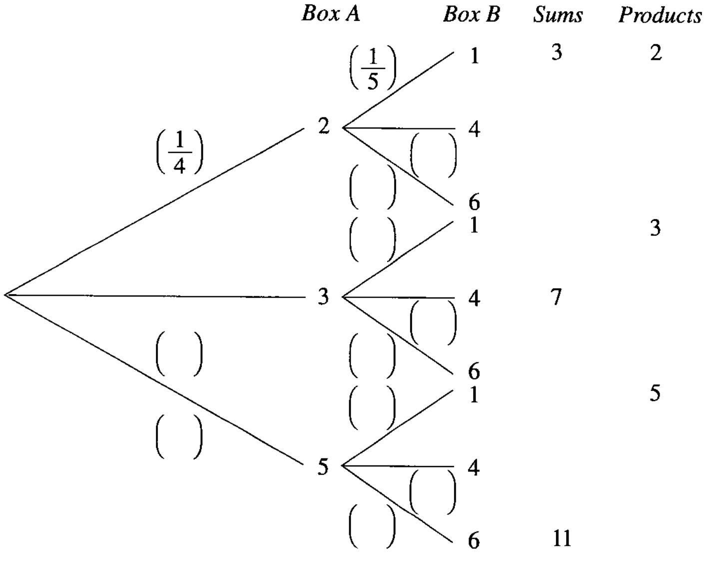


Fig. 6.36

5. A red die has the number 1 on one face, the number 2 on two faces and the number 3 on three faces. Two green dice each has the number 6 on one face and the number 5 on five faces. The three dice are thrown together.

(a) Copy and complete Fig. 6.37 by adding probabilities to the 'branches'.

(b) Using the probability tree, calculate the probability of obtaining
   - (i) 2 on the red die, 5 on the first green die and 6 on the second green die,
   - (ii) 3 on the red die and 6 on each of the two green die,
   - (iii) exactly 2 sixes,
   - (iv) a sum of 12,
   - (v) a sum which is divisible by 3.

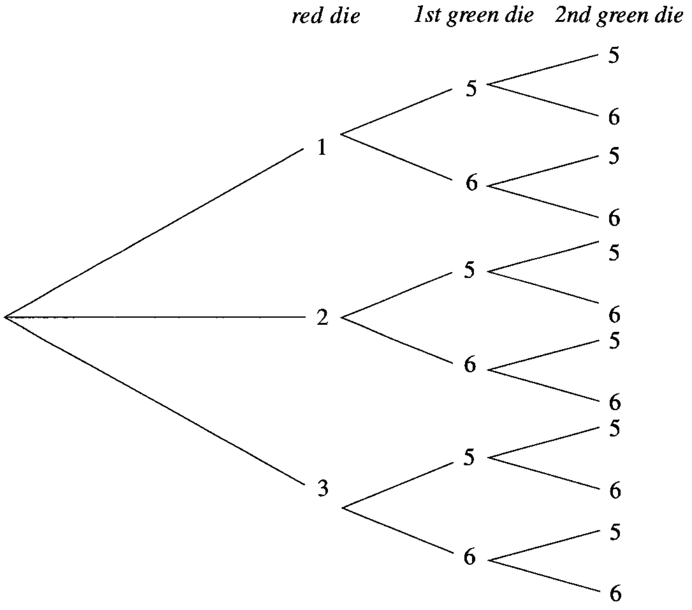


Fig. 6.37

6. In a group of 8 boys, 3 are left-handed. The remaining 5 boys are right-handed. If a boy is chosen at random from the group, state the probability that the boy chosen is left-handed.

(a) A second boy is then chosen at random from the remaining 7 boys. Given that the first boy chosen is left-handed, state the probability that the second boy chosen is also left-handed.

On another occasion, two boys are chosen at random from the same group of eight boys.

(b) Copy and complete the probability tree in Fig. 6.38.

(c) From your probability tree in (b), find the probability that
   - (i) both boys are left-handed,
   - (ii) both are right-handed,
   - (iii) only one boy is left-handed.

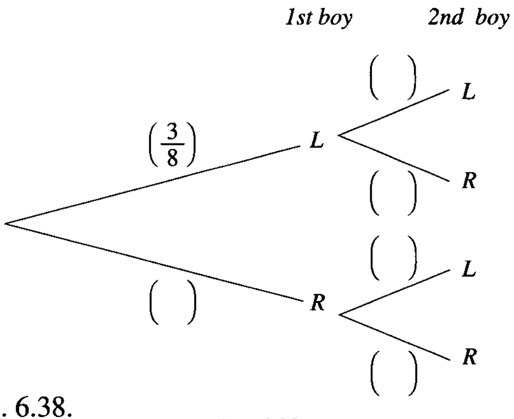


Fig. 6.38

7. A bag contains 6 black and 3 white counters. John takes two counters at random from the bag, one after the other.

(a) Copy and complete the probability tree in Fig. 6.39.

(b) Find the probability that he has taken out
   - (i) a white counter the first time and a black counter the second time,
   - (ii) one counter of each colour,
   - (iii) two black counters,
   - (iv) at least one black counter.

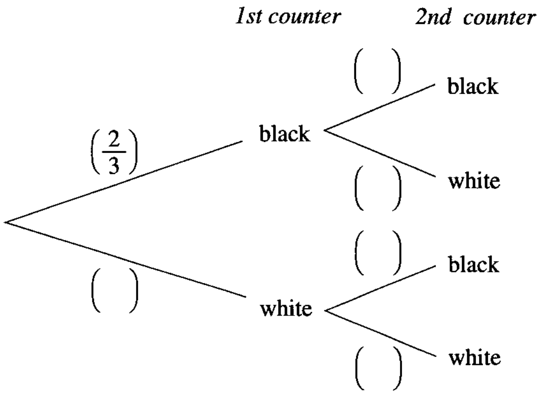


Fig. 6.39

8. A woman goes to the hairdresser once a week. The probability that she has a perm is $\frac{4}{9}$. Find the probability that

(a) she will not have a perm in a particular week,
(b) she will have a perm in each of two particular consecutive weeks,
(c) she will have a perm in just one of two particular consecutive weeks.

9. A bag contains 20 potatoes, 4 of which are rotten. Another bag contains 12 potatoes, 3 of which are rotten.

(a) If one potato is selected at random from the first bag, what is the probability that it is rotten?
(b) If one potato is selected at random from the second bag, what is the probability that it is good?
(c) If one potato is selected from each bag, what is the probability that
  (i) the potato from the first bag is good and the potato from the second bag is rotten,
  (ii) one potato is good and the other potato is rotten,
  (iii) either both potatoes are good or both are rotten,
  (iv) there is at least one good potato?

10. A bag contains 6 green and 4 blue cards.

(a) A card is drawn at random. Find the probability that it is green.
(b) The card drawn is returned to the bag and after mixing the cards thoroughly, Jane takes two cards at random from the bag, one after the other. Using a probability tree, or otherwise, calculate the probability that she has taken out
  (i) two green cards,
  (ii) one card of each colour,
  (iii) at least one blue card.

11. A class has 30 girls and 15 boys. Two representatives are to be selected at random from the class. What is the probability that

(a) the first representative selected is a girl,
(b) the first representative selected is a boy and the second one is a girl,
(c) 1 boy and 1 girl are selected as representatives?

12. In a drawer there are 16 pairs of socks, 8 black, 6 white and 2 grey.

(a) If two pairs of socks are taken out of the drawer, find the probability that
  (i) both pairs are black,
  (ii) one pair is black and the other is white,
  (iii) the two pairs are of the same colour.
(b) If a third pair of socks is drawn out, calculate the probability that all 3 pairs are black.

13. A box contains 15 components. This box was dropped in transit and five components became defective, but not visibly.

(a) If 2 components are selected at random without replacement from the box, what is the probability that
  (i) both are good,
  (ii) only one is good?
(b) If the components are taken at random from the box and tested until a good one is obtained, what is the probability that the first good component obtained is the
  (i) 2nd component tested,
  (ii) 3rd component tested?

14. Ten cards are marked with the letters P, R, O, P, O, R, T, I, O and N respectively. These cards are placed in a box. Two cards are drawn at random without replacement. Calculate the probability that

(a) the first card bears the letter O,

- **(b)** the two cards bear the letters P and O in that order,
- **(c)** the two cards bear the letters P and O in any order,
- **(d)** the two cards bear the same letter.

15. Five balls numbered 1, 2, 5, 8 and 9 are put in a bag.

- **(a)** One ball is selected at random from the bag. Write down the probability that it is a ball numbered 8.
- **(b)** On another occasion, two balls are selected at random from the bag.
  Find the probability that
  - **(i)** the number of each ball is even,
  - **(ii)** the sum of the numbers on the balls is more than 10,
  - **(iii)** the number on each ball is not prime,
  - **(iv)** only one ball bears an odd number.

16. Six cards are marked with the letters F, O, L, L, O and W respectively.

- **(a)** One card is chosen at random. State the probability that it is the card bearing the letter L or O.
- **(b)** The cards are put face down on the table and their positions are randomly mixed. The cards are turned over one at a time. In each of the following cases, find the probability that
  - **(i)** the first two cards turned over will each have the letter O marked on them,
  - **(ii)** the second card turned over will have the letter F marked on it,
  - **(iii)** the first three cards turned over are in the order L, O, and W.

17. **(a)** My dog Ben is given 11 biscuits for his breakfast. 7 of them are black, 3 are red and 1 is yellow.

- **(i)** He eats one of them. Assuming that he is equally fond of each sort of biscuit, what is the probability that the biscuit he eats is red?
- **(ii)** He then eats a second biscuit. What is the probability that the first biscuit is red and the second is black?

(b) On another day he is again given 7 black biscuits, 3 red biscuits and 1 yellow biscuit. He eats only 2 of them. What is the probability that one is yellow and one is black? (*C*)

Summary

1. In an experiment for which there are n equally likely outcomes, if m of these outcomes favour the occurrence of an event E, then the probability of the event E happening, written P(E), is defined as

$$\text{P}(E) = \frac{m}{n}, \text{ where } 0 \leqslant \frac{m}{n} \leqslant 1.$$

P(E) = 0 if and only if the event E cannot possibly occur.
P(E) = 1 if and only if the event E will certainly occur.

2. The **sample space** usually denoted by S refers to the collection of all the possible outcomes of an experiment.

3. An **event** E contains some or all of the outcomes from the sample space that favour the occurrence of the event.

4. The probability of an event E occurring is given by $\text{P}(E) = \frac{n(E)}{n(S)}$, where $n(E)$ and $n(S)$ represent the number of outcomes in E and the total number of possible outcomes in S.

5. An event E and its complementary event denoted by E' satisfy the probability equation

$$\text{P}(E) + \text{P}(E') = 1.$$

6. **Possibility diagrams** and **tree diagrams** are useful in solving probability problems. The diagrams are used to list all possible outcomes of an experiment.

7. **Addition of Probabilities:**
Two **mutually exclusive** events A and B cannot occur together. The probability that either A occurs **or** B occurs, written P(A or B), is given by

$$P(A \text{ or } B) = P(A) + P(B).$$

8. **Multiplication of Probabilities:**
A **probability tree**, a tree diagram with appropriate probabilities displayed on the 'branches', can be used to find the probability that event A and event B occur together written P(A and B). Each main 'branch' of the probability tree leads to an outcome representing 'A and B'. Probability of an outcome and hence P(A and B) is obtained by **multiplying** the probabilities displayed along the 'branches' leading to that outcome.

Review Questions 6

1. Two balanced dice are thrown together. Find the probability that they will show
- **(a)** the same number,
- **(b)** two even numbers,
- **(c)** two odd numbers,
- **(d)** one odd and one even number.

2. A man throws a die and a coin. Find the probability that he will get
- **(a)** the number 3 followed by a head,
- **(b)** an even number followed by a tail.

3. In an experiment, a card is drawn from a pack of playing cards and a coin is tossed. Find the probability of obtaining
- **(a)** a card which is a king and a 'head' on the coin,
- **(b)** the ace of diamonds and a 'tail' on the coin.

4. In an experiment consisting of throwing a die followed by drawing a card from a pack of playing cards, find the probability of obtaining
- **(a)** an odd number on the die and a card which is an ace,
- **(b)** a six on the die and a picture card,
- **(c)** a six on the die and a club.

✻**5.** On any day, the probability that I will miss my bus is $\frac{1}{7}$. Find the probability that
- **(a)** I will catch my bus on a particular day,
- **(b)** I will miss my bus on two particular consecutive days,
- **(c)** I will miss my bus on just one of two particular consecutive days,
- **(d)** I will catch my bus on three particular consecutive days.

✻**6.** Peter has five pairs of socks, one black, one green, one blue and two white. He has three pairs of shoes, one brown, one white and one black. He selects a pair of socks and a pair of shoes at random. Find the probability that Peter has selected
- **(a)** a pair of socks which is not green,
- **(b)** a pair of white socks and a pair of black shoes,
- **(c)** a pair of socks and a pair of shoes of the same colour.

✻**7.** In a class of 30 pupils, 20 are boys and 10 are girls. Of the 10 girls, 6 travel to school by bus and 4 travel by car.
- **(a)** If two pupils are selected at random, calculate the probability that
  - **(i)** one is a girl and one is a boy,
  - **(ii)** no girls are selected.
- **(b)** If two of the 10 girls are selected at random, calculate the probability that
  - **(i)** both travel to school by bus,
  - **(ii)** both travel to school by different means of transportation,
  - **(iii)** at least one travels to school by bus.

✲8. A bowl of sweets contains 2 fruit gums, 3 mints and 5 toffees. Three sweets are to be chosen at random and without replacement, from the bowl.

Calculate the probability that

(a) the first sweet chosen will be a mint,
(b) the first two sweets chosen will be different,
(c) the three sweets chosen will be the same,
(d) of the three sweets chosen, the first two will be the same and the third a toffee,
(e) a fruit gum, a mint and a toffee, in that order, will be chosen.

✲9. A garden has three flower beds. The first bed has 20 daffodils and 20 tulips, the second has 30 daffodils and 10 tulips and the third has 10 daffodils and 20 tulips.

A flower bed is to be chosen by throwing a die which has its six faces numbered 1, 1, 1, 2, 2, 3. If the die shows a '1', the first flower bed is chosen, if it shows a '2' the second flower bed is chosen and so on. A flower is then to be picked at random from the chosen bed.

(a) Copy and complete the probability tree in Fig. 6.40.

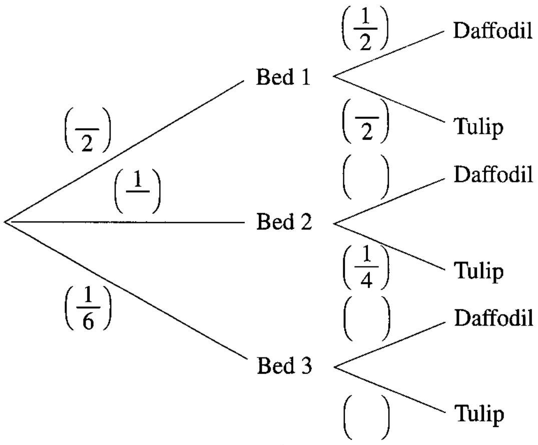


Fig. 6.40

(b) Calculate the probability of picking a daffodil.

✩10. A fair die is made from a regular eight-faced solid by numbering the faces from 1 to 8. The die is thrown twice. Draw a possibility diagram to represent the possible outcomes in the sample space. With the help of your possibility diagram, find the probability of obtaining

(a) the first score less than 5,
(b) the same score for each throw,
(c) the first score less than the second score,
(d) a total score of 8.

11. A regular octahedron which has its eight faces numbered 1, 1, 1, 1, 1, 2, 2 and 3 is to be used as a die.

(a) If the die is thrown once, write down the probability that the score on the die is
  (i) 1,
  (ii) a prime number.
(b) If two such octahedral die are thrown together, find the probability that
  (i) each die shows a score of 2,
  (ii) the sum of the two scores is 6,
  (iii) the sum of the two scores is 4,
  (iv) the two scores are not equal,
  (v) when the two scores are multiplied together, the result is an even number.

12. Suppose I had to find two light bulbs from a collection of 12, all but 3 of which are working. If I test each one in turn, what is the probability that I would find two working bulbs

(a) among the first three bulbs tested,
(b) when three bulbs have been tested, but not before?

Problem Solving

1. Two cards, one yellow on both sides and the other yellow on one side and green on the other, are placed in a box. One card is chosen at random from the box and placed on the table. Given that the upper side of the card on the table is yellow, find the probability that the underside is also yellow.

2. A point in the plane with coordinates (x, y), where x and y are integers having numerical values less than or equal to four, i.e., $|x| \leqslant 4$ and $|y| \leqslant 4$, is chosen at random. What is the probability that the distance of the point from the origin is at most two units?

3. Three fair dice are thrown. What is the probability of obtaining three consecutive numbers in any order?

4. A bag contains 7 green and 5 blue marbles. 6 marbles are drawn at random in succession from the bag without replacement. What is the probability that the colours appear alternately?

5. A number of identical purses are such that 2 purses each contains 50¢, 9 purses each contains 20¢ and 14 purses each contains 5¢. Calculate the probability that 3 purses, selected simultaneously, will together contain exactly 60¢.

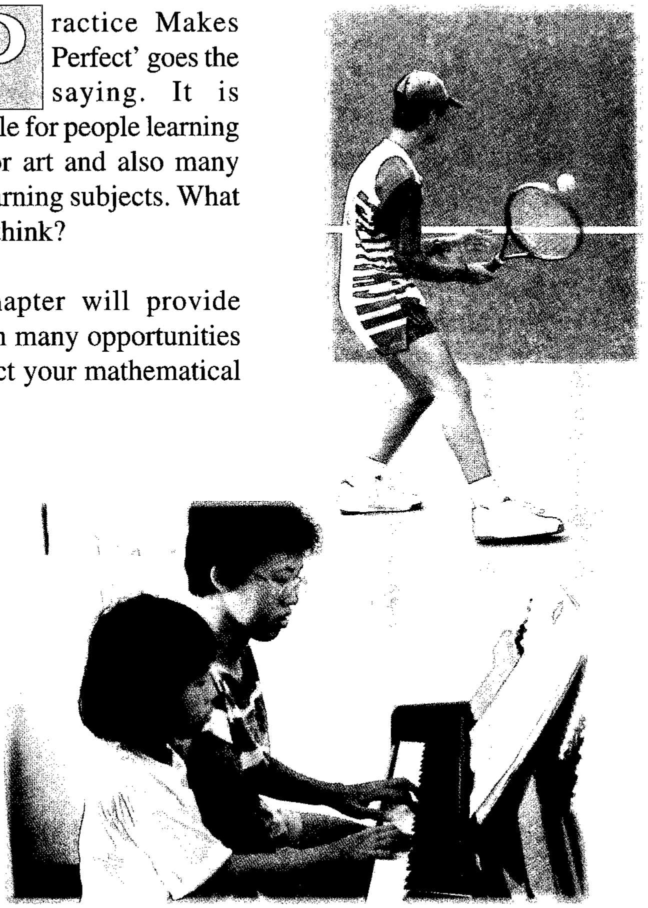
CHAPTER 7

Revision

Preliminary Problem

'Practice Makes Perfect' goes the saying. It is applicable for people learning games or art and also many other learning subjects. What do you think?

This chapter will provide you with many opportunities to perfect your mathematical skills.



7.1 Arithmetic

Example 1

A car travelled for 4 hours at 75 km/h. It then travelled another 280 km at 80 km/h.

Calculate

(a) the distance travelled in the first part of the journey,

(b) the time taken for the second part of the journey,

(c) the average speed of the car for the whole journey.

Solution

(a) The distance travelled = speed × time

$= 75 \times 4 = 300$ km.

(b) The time taken = distance ÷ speed

$= \frac{280}{80} = 3\frac{1}{2}$ hours.

(c) The total distance travelled = 300 + 280 = 580 km

The total time taken $= 4 + 3\frac{1}{2} = 7\frac{1}{2}$ hours

$\therefore$ the average speed of the car $= 580 \div 7\frac{1}{2} = 580 \times \frac{2}{15} = 77.5$ km/h.

Example 2

(a) When the petrol tank of a car is full, the weight of the petrol is $29\frac{1}{4}$ kg. Given that 1 cm³ of petrol weighs 0.78 g, calculate the volume of the tank in litres.

(b) Twenty-eight workers can finish a piece of work in 45 days.

(i) How many workers are needed if the work is to be finished in 36 days?

(ii) If after 33 days, seven of the twenty-eight workers leave, calculate the number of days that the rest of the workers require to finish the work.

(c) When a discount of 15% of the marked price of an air-cleaner is allowed, the air-cleaner is sold for \$323. Find the discount.

Solution

(a) The weight of the petrol is $29\frac{1}{4}$ kg = 29 250 g.

$\therefore$ the volume of the tank $= \frac{29\,250}{0.78} = 37\,500$ cm³ = 37.5 litres.

(b) (i) Number of workers needed $= \frac{45}{36} \times 28 = 35$.

(ii) 28 workers require 45 − 33 = 12 more days to finish the work.

$\therefore$ the number of days 21 workers require to finish the work $= \frac{28}{21} \times 12 = 16$

(c) \$323 is 85% of the marked price.

$\therefore$ the discount $= \frac{15}{85} \times \$323 = \$57$.

Example 3

(a) A map is drawn to a scale of 1 : 20 000

(i) Calculate the actual distance, in kilometres, between two towns which are represented on the map by points 15.5 cm apart.

(ii) On the map, a lake has an area of 150 cm^2. Calculate, in square kilometres, the actual area of the lake.

(b) A sum of money is divided among three men, A, B and C, in the ratio 9 : 5 : 3. If B has $12 more than C, calculate how much A has.

(c) A bank exchanged American dollars (US$) for Singapore dollars (S$) at a rate of US$1 to S$1.72 in May 2000.

(i) Calculate, in S$, the amount received in exchange for US$104.

(ii) Calculate, in US$, correct to the nearest cent, the amount received in exchange for S$110.

Solution

(a) (i) 1 cm on the map represents 20 000 cm or 200 m or $\frac{1}{5}$ km on the ground.

$\therefore$ 15.5 cm on the map represents $\left(15.5 \times \frac{1}{5}\right)$ km or 3.1 km between the two towns.

(ii) 1 cm on the map represents $\frac{1}{5}$ km on the ground.

$\therefore$ 1 cm^2 represents $\left(\frac{1}{5}\right)^2$ or $\frac{1}{25}$ km^2 on the ground.

$\therefore$ the actual area of the lake = $\left(150 \times \frac{1}{25}\right)$ km^2 = 6 km^2

(b) If $A : B : C = 9 : 5 : 3$, then B has 2 shares more than C and this is equivalent to $12.

$\therefore$ 1 share is equivalent to $6.

Thus, A has 9 × 6 = $54

(c) (i) US$1 is equivalent to S$1.72

Thus, US$104 is equivalent to 1.72 × 104 = S$178.88

(ii) S$1 is equivalent to US$$\frac{1}{1.72}$

Thus, S$110 is equivalent to $\frac{1}{1.72} \times 110$ = US$63.95 (correct to the nearest cent.)

Example 4

(a) Express the product of 4×10^6 and 7×10^{-3} in standard form.

(b) Express 2.4 m^3 in cubic centimetres, giving your answer in standard form.

Solution

(a) $(4 \times 10^6) \times (7 \times 10^{-3}) = (4 \times 7) \times 10^{6+(-3)} = 28 \times 10^3 = 2.8 \times 10^4$

(b) $2.4 \text{ m}^3 = 2.4 \times (100 \text{ cm})^3 = 2.4 \times (10^2 \text{ cm})^3 = 2.4 \times 10^6 \text{ cm}^3$

Revision Exercise 7.1a

(Neither mathematical tables, slide rules nor calculators may be used in this exercise.)

1. The volume of metal in a piece of metal pipe is calculated to be 0.034 56 m^3. Express this volume
   - **(a)** correct to 2 decimal places,
   - **(b)** correct to 3 significant figures,
   - **(c)** in standard form.

2. The average yield of a certain variety of wheat from seven plots of experimental land was 4.034 7 kg. Express this weight
   - **(a)** correct to 2 significant figures,
   - **(b)** correct to 2 decimal places.

3. Express 74 cm^3 in m^3, giving your answer
   - **(a)** as a decimal,
   - **(b)** in standard form.

4. **(a)** A bus leaves a terminal at 11.15 a.m. It arrives at the next terminal at 2.08 p.m. How many minutes does the journey take?
   (b) The last bus for the day leaves the terminal at 23 45. If the journey takes 1 hour 23 minutes, find the time of arrival of the bus at the next terminal.

5. If $a = 7.8 \times 10^5$ and $b = 3.9 \times 10^3$, find the value of each of the following, giving your answer in standard form.
   - **(a)** $2a - b$ **(b)** $3a + 12b$
   - **(c)** $a \div b$ **(d)** $4ab$
   - **(e)** $4ab^2$ **(f)** $4b \div a$

6. Estimate $\frac{11.9 \times 0.598}{23.6}$, giving your answer correct to one significant figure.

7. The diameter of the sun is 1 387 570 km. Express 1 387 570 in standard form, correct to 2 significant figures.

8. Evaluate each of the following, giving your answer as a decimal correct to 2 decimal places.
   - **(a)** $25 \div 13$ **(b)** 3.14×3.47
   - **(c)** $7.58 \div 0.12$

9. Use the fact that $453 \times 84 = 38\ 052$ to *write down* the exact value of
   - **(a)** 4.53×840
   - **(b)** $380.52 \div 0.84$

10. A man bought an article for $360 and sold it at a profit of 12%. What was the selling price of the article?

11. In a sale, prices are reduced by 22.5%. The price of a wrist blood pressure monitor in the sale is $170.50. Find its normal price.

12. In a sale, all prices are reduced by 20%. Calculate the original price of an article whose sale price is $68.

13. A man bought 20 books for $100, 15 of which were sold for $7 each and the rest for $4 each. What was his percentage gain?

14. An article was sold at a loss of 12%. If the cost price had been $350, what would the selling price have been?

15. There are 600 boys and 400 girls in a school. One day, 2% of the pupils were absent. If 1% of the boys were absent, how many girls were present?

16. Thirty-eight members from the youth club of Greenview Community Centre plan a day-trip to the zoo. The cost of hiring the bus is $62.50. The zoo tickets cost $3.20 each but one extra ticket is given free for every eight tickets bought.
    - **(a)** How many zoo tickets will have to be paid to enable the 38 members to enter the zoo?
    - **(b)** Calculate the total cost of the day-trip.

17. 240 boys and 180 girls sat for an examination. If 65% of the boys and 60% of the girls passed, what percentage of the total number of candidates passed?

18. A man buys a watch marked at $84. In addition, he has to pay a 6% sales tax. Calculate the total amount that he has to pay for the watch.

19. **(a)** Fabian and Martin agreed to meet each other at the MRT station at 11 45. Fabian was 12 minutes early and Martin only arrived at 12 14. How many minutes did Fabian have to wait?

(b) On another occasion, Martin went for a movie with Sheena at 6.15 p.m. Martin was at the meeting place by 5.38 p.m. and Sheena was late by 22 minutes. How many minutes had Martin waited for Sheena?

20. A man walks 9 km at 6 km/h and then jogs 18 km at 12 km/h. Find his average speed for the whole journey.

21. The average age of 8 boys is 10 years 9 months and that of 12 girls is 10 years 4 months. Find the average age of the whole group.

22. Forty-eight workers can build a hut in 60 hours. How many workers are needed if the hut is to be built in 32 hours?

23. The ratio of the number of boys to the number of girls in a school is 7 : 5. If the student population is 1 200, find the number of boys and girls in the school.

24. A set of bookshelves can hold 330 books of an average thickness of $2\frac{3}{4}$ cm. What is the maximum number of books of an average thickness of $3\frac{1}{2}$ cm that the same set of bookshelves could hold?

25. If 4 men can make 80 chairs in 12 days, how long will 24 men take to make 300 chairs?

26. **(a)** Find the value of $\sqrt{0.04} + \frac{1}{0.5}$.

(b) Arrange the following in order of size, starting with the greatest, $\frac{3}{20}$, $2^{-\pi}$, $\frac{1}{8}$, $\frac{2}{15}$.

27. A, B and C share $180 in such a way that A has $2\frac{1}{2}$ times as much as B, and B has 4 times as much as C. Find their shares.

28. A man invests $7 500 for 3 months and receives $62.50 as interest. What is the rate of the simple interest per annum?

29. A man invests $4 500 at 6% per annum simple interest. How long will it take to earn an interest of $405?

30. Find the principal that will earn an interest of $1 640 at 5% simple interest per annum for a period of 4 years.

Revision Exercise 7.1b

(You are allowed to use your calculator in the following exercises unless otherwise stated.)

1. The length of a rectangle is increased by 20% and its width is decreased by 20%. Find the percentage change, if any, in its area.

2. A man bought an article for $360. At what price must he mark the article so that he can allow a discount of 10% of the marked price and still make a profit of 10% on the cost price?

3. A man's petrol bill for 1999 was $3 000. In 2000, the price of petrol was increased by 20% and his petrol consumption increased by 15%. Find his petrol bill for 2000.

4. A shopkeeper allows a discount of 10 percent on the advertised price of a bag. At what price must he mark the bag, which costs him $60, in order to make a profit of 20 percent?

5. A fruitseller buys 138 oranges for $35. He sells 99 of them at 3 for a dollar and the remaining at 25 cents each. Find his profit percent.

6. Divide $7 200 into two parts such that the simple interest on one part at 6.5% per annum for $4\frac{1}{2}$ years would be equal to the simple interest on the other at 4.5% per annum for $5\frac{1}{2}$ years.

7. A shopkeeper buys a batch of goods. He fixes the price at 30% above his cost price. He manages to sell half of the stock at this price, one quarter of the stock at a discount of 20% on the marked price and the remainder at a discount of 40% on the marked price. Find his profit percent on the batch of goods.

8. 2 litres of paint containing 10% of turpentine are mixed with 5 litres of paint containing 8% of turpentine. 1 litre of turpentine is then added to the mixture. Find the percentage of turpentine in the final mixture.

9. The cost of manufacture of radio sets comprises the cost of materials and the cost of labour in the ratio 5 : 7. If the cost of materials increases by 20% and the cost of labour decreases by 4%, find the resulting percentage change in the cost of manufacture of radio sets.

10. John can do a piece of work in 8 days and Peter can do the same piece of work in 6 days. They work together on it for two days and then Peter stops working altogether. How long will it take for John to complete the remaining part of the work by himself?

11. Meiling can complete a piece of work in 12 days and Suling can complete it in 18 days. Find the number of days in which both, working together, will take to complete the work.

12. A householder pays the Public Utility Board (P.U.B.) 14.6 cents for every unit of electricity he uses and 73 cents for every cubic metre of water. If the water consumption exceeds 20 m^3 per month, the rate is 90 cents per cubic metre for any additional amount of water used.

    Calculate

    (a) his bill for a month in which he used 125 units of electricity and 26 m^3 of water,

    (b) his P.U.B. bill for a month in which he used 150 units of electricity and 18 m^3 of water.

    The P.U.B. revises the rate so that a unit of electricity now costs 13.8 cents while one cubic metre of water costs 87 cents if the number of cubic metres of water used is less than 20. For any additional amount of water used, the rate is 98 cents per cubic metre. Calculate the new P.U.B. bill for the month in which he used 125 units of electricity and 26 m^3 of water.

13. In 1999, a household spent $2 640 on housing, $3 000 on food and $1 020 on transportation. Express the amount spent on food as a percentage of his total expenditure for these three items. He estimates that in 2000, the cost of housing will be 10% more, that of food will be 12% more and that of transportation will be 5% more. Calculate his total expenditure on these items for 2000.

    Calculate the estimated percentage increase in his total expenditure between 1999 and 2000.

14. **(a)** The cost of posting a letter in 2000 was 20 cents. A company posted 1 750 letters and was allowed a 4% discount on the cost.

    Calculate the cost to the company of posting the 1 750 letters.

    (b) The cost of posting a letter in 2000 was 22 cents.

    Calculate the percentage increase in the cost of posting a letter.

(c) In 2000, the company paid \$373.45 for the cost of posting 1 750 letters.
Calculate the percentage discount that the company was allowed in 2000.

15. In the gravel-pump mining method for tin, the alluvial ore extracted contains 1.2% tin, 0.04% zinc and 0.002% tungsten. Given that 5 000 tons of alluvial ore are extracted each week and that the mine operates only 50 weeks in each year, calculate, in tonnes per year,

(a) the mass of alluvial ore extracted,
(b) the mass of each mineral present in the extracted ore.

As the gravel-pump mining method is not very efficient, only 80% of the three minerals present can be recovered and used.

(c) Calculate, in tonnes, the mass of each mineral that can be recovered. The three minerals are used in making an alloy. The masses of tin, zinc and tungsten are used in the ratio 241 : 8 : 1.

If 25 000 articles are made for each tonne of tungsten used, calculate

(d) the number of articles made in a year if all the tungsten recovered was used,
(e) the mass of tin and that of zinc recovered but not used.

16. In January 1995, Mr Abdul Rahman exchanged RM50 000 for Singapore dollars at a rate of RM10 to S\$6.20. He placed this money in a financial institution which paid a simple interest of 4% per annum for a period of 3 years. In January 1998, he withdrew all the money and exchanged them to Malaysian ringgit at a rate of RM10 to S\$4.45.

Calculate

(a) the sum in Singapore dollars that he received for the RM50 000 in 1995,
(b) the amount of money at the end of the 3-year period,
(c) the amount of Malaysian ringgit that he gets in 1998,
(d) the percentage gain or loss that he gets out of this investment.

17. In a small workshop, twenty-four workers and two supervisors are employed. Each worker receives \$324 for a working week of 45 hours and each supervisor is paid \$410 per week.
As a result of installing new machinery, it is possible to run the workshop with one supervisor and to reduce the number of workers by a quarter. The supervisor is given a rise of 15%. The workers now work a 40 hour week but receive \$1.20 an hour more than before. Find the reduction in the total weekly wages for the workshop and express it as a percentage of the original cost.

18. **(a)** A train travels 136 km at an average speed of 102 km/h. How many minutes does this journey take?
(b) On one part of the journey, the train takes 28 minutes to travel 49 km. Find the average speed in kilometres per hour.
(c) Each of the 390 passengers on the train is given a small gift. The gifts are in packs of 18. Find the least possible number of packs required and the number of gifts left over.
(d) On each of the 365 days of 1998, the train made three journeys. The average number of passengers on each journey was 390.
How many passengers did the train carry in the year? Give your answer in standard form correct to 2 significant figures.
(e) The number of passengers in 1997 was 8% higher than in 1996. The number in 1998 was 6% higher than in 1997. What was the total percentage increase in the number of passengers from 1996 to 1998?

19. **(a)** The cost of making an article is \$7.
  (i) Calculate the cost of making 32 such articles.
  (ii) The cost of making an article is divided between materials, wages and overheads in the ratio 3 : 5 : 6. Calculate the cost of material used in making each article.

(iii) A person is paid \$2.50 for making an article. Calculate how much he earns in one hour when he makes 4 articles in 12 minutes.

(iv) The articles are sold at a profit of 20%. Calculate the selling price of each article.

(b) In 1999, the cost of making an article increased and was divided between materials, wages and overheads in the ratio 1 : 4 : 5. An article was sold for \$9.20, making a profit of 15%. Calculate the cost of making an article. In 2000 the cost of materials increased by 25%, wages doubled, and overheads remained the same.
Calculate the total percentage increase in cost.

20. (a) On a given day, the rate of exchange between American dollars (US\$) and British pounds (£) was US\$1.50 = £1.

(i) Pat changed £240 into dollars. Calculate how many dollars she received.

(ii) On the same day, the rate of exchange between French francs (F) and pound was 7.65 F = £1. Calculate the rate of exchange between francs and dollars, giving the number of francs to the dollar.

(iii) Robert was planning a trip to American and received US\$900 in exchange for British pounds. Calculate how many pounds (£) he changed.

(iv) Robert was unable to make the trip so he changed the US\$900 back into pounds at a different rate of exchange.

(a) Given that he received £552, calculate the new rate of exchange, in dollars to the pound, giving your answer correct to the nearest cent.

(b) Calculate the resultant percentage loss that he made by changing his money twice (from pounds to dollars and back to pounds again).

(b) Adam bought a camera for £275. This price included a Sales Tax of 10%. Calculate the tax that was paid.

7.2 Algebra 7

Example 1

Peter is x years old. He is 6 years younger than his twin sisters, Jane and Jennifer.

(a) Write down Jane's age, in terms of x.

(b) Find, as simply as possible, the mean age of the three children, giving your answer in terms of x.

Solution

(a) Jane is $(x + 6)$ years old.

(b) The mean age of the three children $= \frac{1}{3}[x + 2(x + 6)]$

$$= \frac{1}{3}(3x + 12)$$

$$= (x + 4) \text{ years old}$$

Example 2

(a) Simplify

(i) $4(3x-2)-3(2x-7)$; *(ii)* $6-[(3x-7)-(7x-3)]$;

(iii) $(2x-3)^2-4x(x-5)$.

(b) Make a the subject of each of the following formula.

(i) $3x-5a=2y$ *(ii)* $5x-3a=7ax+k$

(iii) $y^2-a^2=2xy$

(c) Given that $a=-2$, $b=5$ *and* $c=-1$, *find the value of each of the following.*

(i) $2a+b$ *(ii)* a^2+2bc

(iii) $(a+b-c)^2$

Solution

(a) (i) $4(3x-2)-3(2x-7)=12x-8-6x+21$
$$=6x+13$$

(ii) $6-[(3x-7)-(7x-3)]=6-[3x-7-7x+3]$
$$=6-(-4x-4)$$
$$=6+4x+4$$
$$=4x+10$$

(iii) $(2x-3)^2-4x(x-5)=(2x)^2-2(2x)(3)+(3)^2-4x^2+20x$
$$=4x^2-12x+9-4x^2+20x$$
$$=8x+9$$

(b) (i) $3x-5a=2y \Rightarrow 5a=3x-2y$

$$\therefore \quad a=\frac{3x-2y}{5}$$

(ii) $5x-3a=7ax+k \Rightarrow 5x-k=7ax+3a$

$$\therefore \quad a(7x+3)=5x-k, \text{ and } a=\frac{5x-k}{7x+3}$$

(iii) $y^2-a^2=2xy \Rightarrow a^2=y^2-2xy$ and $a=\pm\sqrt{y^2-2xy}$

(c) (i) $2a+b=2(-2)+5$
$$=-4+5$$
$$=1$$

(ii) $a^2+2bc=(-2)^2+2(5)(-1)$
$$=4-10$$
$$=-6$$

(iii) $(a+b-c)^2=[-2+5-(-1)]^2$
$$=(-2+5+1)^2$$
$$=4^2$$
$$=16$$

Example 3

Factorise each of the following completely.

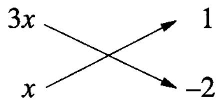
(a) $4x - 12y$ (b) $12x^2 - 20x - 8$

(c) $5x^2 - 20y^2$ (d) $3xy + 2y - 12x - 8$

Solution

(a) $4x - 12y = 4(x - 3y)$

(b) $$\begin{aligned} 12x^2 - 20x - 8 &= 4(3x^2 - 5x - 2) \\ &= 4(3x + 1)(x - 2) \end{aligned}$$



(c) $$\begin{aligned} 5x^2 - 20y^2 &= 5(x^2 - 4y^2) \\ &= 5[x^2 - (2y)^2] \\ &= 5(x + 2y)(x - 2y) \end{aligned}$$

(d) $$\begin{aligned} 3xy + 2y - 12x - 8 &= y(3x + 2) - 4(3x + 2) \\ &= (3x + 2)(y - 4) \end{aligned}$$

Example 4

(a) *Solve the following equations:*

(i) $5x - 4 = x + 12$ (ii) $2(5x + 1) = 32$ (iii) $3(x - 1) = 5(4 - 7x)$

(b) *Solve the simultaneous equations $5x + 3y = 4$, $2y - 3x = 9$.*

Solution

(a) (i) $$\begin{aligned} 5x - 4 &= x + 12 \\ 5x - x &= 12 + 4 \\ 4x &= 16 \\ \therefore\quad x &= 4 \end{aligned}$$

(ii) $$\begin{aligned} 2(5x + 1) &= 32 \\ 5x + 1 &= 16 \\ 5x &= 15 \\ \therefore\quad x &= 3 \end{aligned}$$

(iii) $$\begin{aligned} 3(x - 1) &= 5(4 - 7x) \\ 3x - 3 &= 20 - 35x \\ 3x + 35x &= 20 + 3 \\ 38x &= 23 \\ \therefore\quad x &= \frac{23}{38} \end{aligned}$$

(b) $5x + 3y = 4$ ———— (1)

$2y - 3x = 9$ ———— (2)

From (2): $2y = 3x + 9$, i.e. $y = \frac{3x + 9}{2}$ ———— (3)

Substitute (3) into (1): $$\begin{aligned} 5x + 3\left(\frac{3x + 9}{2}\right) &= 4 \\ 5x + 4\frac{1}{2}x + 13\frac{1}{2} &= 4 \\ 9\frac{1}{2}x &= -9\frac{1}{2} \\ x &= -1 \end{aligned}$$

Substitute $x = -1$ into (3): $y = \frac{3(-1) + 9}{2} = 3$

Therefore $x = -1$, $y = 3$ is the solution.

Example 5

Solve the following equations.

(a) $(7 - 2x)(4 + 3x) = 0$

(b) $6x^2 - 19x + 15 = 0$

Solution

(a) $(7 - 2x)(4 + 3x) = 0$

$7 - 2x = 0$ or $4 + 3x = 0$

$\therefore \quad x = 3\frac{1}{2}$ or $x = -1\frac{1}{3}$

(b) $6x^2 - 19x + 15 = 0$

$(2x - 3)(3x - 5) = 0$

$(2x - 3) = 0$ or $(3x - 5) = 0$

$\therefore \quad x = 1\frac{1}{2}$ or $x = 1\frac{2}{3}$

Revision Exercise 7.2

1. Devi is x years old and Shanti is twice as old as Devi.

- **(a)** Find the sum of the ages of Devi and Shanti in
  - **(i)** two years' time,
  - **(ii)** k years' time.
- **(b)** How old will Shanti be when Devi is $2x$ years old?
- **(c)** How old will Devi be when Shanti is $6x$ years old?

2. The cost of unleaded petrol is \$$x$ per litre while the cost of leaded petrol is 10 cents more per litre.

- **(a)** Find the cost of buying
  - **(i)** 5 litres,
  - **(ii)** k litres of unleaded petrol.
- **(b)** Find the cost of buying
  - **(i)** 8 litres,
  - **(ii)** h litres of leaded petrol.
- **(c)** Find the total cost of buying y litres of unleaded petrol and z litres of leaded petrol.

3. Peter is 7 years old and Jane is 3 years younger.

- **(a)** How old will Peter be in
  - **(i)** 5 years' time,
  - **(ii)** x years' time?
- **(b)** How old will Jane be in
  - **(i)** 7 years' time,
  - **(ii)** y years' time?
- **(c)** Find the sum of the ages of Peter and Jane in
  - **(i)** 12 years' time,
  - **(ii)** z years' time.

4. Aaron is x years old and Adrian is y years older than Aaron.

- **(a)** How old will Aaron be in
  - **(i)** 4 years' time,
  - **(ii)** h years' time?
- **(b)** How old will Adrian be in
  - **(i)** 8 years' time,
  - **(ii)** k years' time?
- **(c)** Find the sum of the ages of Aaron and Adrian in
  - **(i)** 9 years' time,
  - **(ii)** z years' time.

5. A car is moving at an average speed of 60 km/h. How far would it travel in

(a) 2 hours, **(b)** x hours,
(c) 5 minutes, **(d)** y minutes?

6. A train is moving at an average speed of x km/h. How far would it travel in

(a) 3 hours, **(b)** y hours,
(c) 7 minutes, **(d)** z minutes?

7. A cyclist can travel at an average speed of v km/h. How long would he take to travel a distance of

(a) 10 km, **(b)** x km,
(c) y m?
(Give your answer in hours.)

8. A man can paint a house in x days while his apprentice takes 5 days longer to paint the same house. Write down an expression for the part of the house that

(a) the man can paint in one day,
(b) his apprentice can paint in two days,
(c) he and his apprentice can paint in three days.

9. A swimming pool can be filled by a large pipe operating alone in x hours. If the pool is to be filled by a small pipe alone, it will take 6 hours longer than the larger pipe filling it alone. Write an expression for the part of the pool that

(a) the large pipe can fill in one hour,
(b) the small pipe can fill in four hours,
(c) both pipes can fill together in one hour.

10. Simplify each of the following:

(a) $3(2x - 1) - 4(x - 7)$
(b) $5(3x + 4) - 2(7x - 4)$
(c) $6(2x - 3) + 5(3 - 7x)$
(d) $14 - 3(5 - 4x) + 6x$
(e) $7(2a + 3) - 4(3 - a)$
(f) $9(5k - 6) + 4(7 - 13k)$
(g) $5 - 3(c + a) - 6(3a - 2c)$
(h) $15p - 3(p - q) + 4(q - 3p)$
(i) $(x + 2y)^2 - 4x(x + y)$
(j) $(a + 2b)^2 - (a - 2b)^2$
(k) $(3x - y)^2 - (x - 3y)^2$
(l) $3x(x - 4y) - 4x(y - x)$

11. Simplify each of the following below by removing the brackets.

(a) $2[3a - 2(3a - 1) + 4(a + 1)]$
(b) $x - (x - y) - [x - y - z - 2(y + z)]$
(c) $8(x - y) - [x - y - 3(y - z - x)]$
(d) $-5(a + b) - [3(a - b) - 2(3b - c)]$
(e) $2b - [5a - 2(4b - 3a)]$
(f) $2x - (3y - 5z) - [2x - (5y + z) - (2z - 7y)]$
(g) $a(2b - c) - c(a + 3b) + 5a(c - a)$
(h) $2a(5b - 2c) - [2(a - b + c) - b(a - c)]$
(i) $2b(c - a) - [3c(a - b) - 3a(b + c)]$
(j) $3(a - c) - \{5(2a - 3b) - [5a - 7(a - b)]\}$
(k) $a + b - \{c + a - (c + a) - [b + c - (c + a - b)]\}$
(l) $5\{x - 3[x - 2(x + y)]\} - 3\{x - 2[x - 3(x + y)]\}$

12. Given that $a = 2$, $b = -3$, and $c = -1$, find the value of each of the following:

(a) $2a + 3b - c$ **(b)** $a^2 - c^2$
(c) $(a + b)(2a - c)$

13. Given that $x = 3$, $y = -4$ and $z = -2$, evaluate

(a) $x - y + 3z$, **(b)** $(x - y - z)^2$,
(c) $(x - z)^2(y - z)^3$.

14. Given that $x = 7$, $y = 5$ and $z = -1$, find the value of each of the following:

(a) $x - 2y$ **(b)** $3y - z$
(c) $(x - z)(z - y)(y - x)$

15. Given that $y = \sqrt{2x - a}$,

(a) find the value of y when $x = 4$ and $a = -1$,
(b) express x in terms of a and y.

16. Given that $y^2 = 2k(x + 1)$,

(a) find the two possible values of y when $k = \frac{1}{16}$ and $x = 3$,
(b) express k in terms of x and y.

17. Given that $V = \frac{1}{3}\pi r^2 h$,

(a) find the value of V when $\pi = \frac{22}{7}$, $r = 14$ and $h = 20$,
(b) express r in terms of V, π and h, on condition that $r \geqslant 0$.

18. Given that $y = \frac{x - k}{2x}$,

(a) find the value of y when $x = 4$ and $k = -2$,
(b) express x in terms of k and y.

19. Given that $y + 2 = \frac{3y + k}{a}$, express y in terms of a and k.

20. Given that $x = 2t - 1$ and $y = \frac{2}{3}t + 2$,

(a) express $3x - y + 2$ in terms of t,
(b) find the value of t, x and y when $3x - y + 2 = 0$.

21. Given that $a = 3x + 7$ and $b = 2x - 5$,

(a) express $2a + 3b + 1$ in terms of x,
(b) find the value of x, a and b when $2a + 3b + 1 = 3$.

22. Given that $a = 3t - 1$ and $b = 2t^2 + 3$,

(a) find the value of a and of b when $t = -4$,
(b) express $2a + 3b + 7$ in terms of t.

23. Given that $y = \frac{1 - x}{1 + x}$,

(a) find the value of y when $x = -4$, giving your answer as a fraction in its lowest terms,
(b) express x in terms of y.

24. Given that $\frac{3a + b}{2a - b} = \frac{2}{5}$, express a in terms of b.

25. Given that $\frac{3}{a} = \frac{2}{b} + \frac{1}{c}$, express b in terms of a and c.

26. Solve the following equations:

(a) $5(x - 2) = 20$
(b) $2(3x - 5) = 8$
(c) $3x - 4 = 2x + 1$
(d) $9x - 4 = 14$
(e) $7a - 3 = 5 - 7a$
(f) $2k - 2(3k + 1) = k + 4$
(g) $x + 1\frac{1}{2}x = 7\frac{1}{2}$
(h) $2 - \frac{1}{3}(y + 2) = \frac{1}{6}y$

27. Solve the following equations:

(a) $3(5 - 3x) = 25$
(b) $3(2x - 3) - 2(5x + 4) = 5$
(c) $\frac{1}{2}x + 3 = \frac{3}{4}x$
(d) $\frac{3x + 5}{6} = \frac{2x + 7}{7}$
(e) $\frac{1}{x + 3} - \frac{7}{2x - 5} = 0$
(f) $\frac{2x + 3}{5} + \frac{1 - x}{3} = 0$
(g) $7 - 3x = 2(5x - 1)$
(h) $2(5y - 7) - 2(3y - 1) = 8$
(i) $5(3x - 7) - 8 = 2(7x + 1)$
(j) $9(2x - 4) = 8 - 2(x - 7)$
(k) $\frac{3}{x + 7} = \frac{5}{3x + 4}$
(l) $\frac{2x + 17}{3} - \frac{7x - 4}{5} = 0$

28. Make the letter in the brackets the subject of the formula.

(a) $2a + b = 4b - c$ (b)
(b) $4x - 5y = x - 3$ (y)
(c) $ax - ky = hy - c$ (y)
(d) $y = \sqrt{\frac{x + 2}{x - a}}$ (x)
(e) $\frac{a}{x + 2b} = \frac{x - 2b}{k}$ (x)
(f) $\frac{a}{y} + \frac{b}{y} = \frac{2k}{h}$ (y)
(g) $x = \frac{1}{2}(y^2 + h)$ (y)
(h) $A = \frac{1}{3}\pi r^2 h$ (r)

29. Solve the following pairs of simultaneous equations.

(a) $x + 2y = 8$
$3x + 2y = 12$
(b) $2x - 3y = 7$
$x + 3y = -1$
(c) $x + y = \frac{5}{6}$
$x - y = \frac{1}{6}$
(d) $5x + 3y = 2$
$2y - 5x = -7$
(e) $3a - 2b = 1$
$5a + 3b = -11$
(f) $a + 2b = 3$
$3a + 4b = 11$
(g) $2p + 3q = 0$
$4p - 5q = 22$
(h) $3p - 4q - 24 = 0$
$5p - 6q - 38 = 0$
(i) $6x - 3y - 15 = 0$
$5x - 4y + 1 = 0$

30. Factorise the following expressions completely.

(a) $9x^2 - 36$
(b) $8x^2 + 16x$
(c) $4x^2 - 9y^2$
(d) $10x^2 - 9(x^2 + 16)$
(e) $x^2 - 4xy + 4y^2$
(f) $3x^3y - 9xy^2$
(g) $9x^2 - y^2$
(h) $9 - x^2$
(i) $25 - 64k^2$
(j) $(a - b)^2 - c^2$
(k) $(3x + 4y)^2 - 9z^2$
(l) $(a + 2b)^2 - 4c^2$
(m) $(a - b)^2 - (x - y)^2$
(n) $x^2 - 4x - 12$
(o) $x^2 - 11x - 152$
(p) $x^2 - 14x - 51$
(q) $2a^2 + 3a - 2$
(r) $3a^2 + 7a - 6$
(s) $4k^2 + k - 14$
(t) $5x^2 + 11x + 2$
(u) $6a^2 - 31a + 35$
(v) $7a^2 + 69a - 10$
(w) $8a^2 - 38a + 35$
(x) $9x^2 + 30x - 11$
(y) $10x^2 - 73x + 21$
(z) $11x^2 + 75x - 14$

31. Factorise the following expressions completely.

- **(a)** $x^2 + 3y + xy + 3x$
- **(b)** $ab - bc - ac + c^2$
- **(c)** $ax - ab + kx - kb$
- **(d)** $3x + cx + 3c + c^2$
- **(e)** $ax - kx - ah + kh$
- **(f)** $5ax + ay + 5bx + by$
- **(g)** $3xy + 2y - 12x - 8$
- **(h)** $20ac - 4ad - 15kc + 3kd$
- **(i)** $6a^2 + 3ab - 8ka - 4kb$
- **(j)** $2a^4 - a^3 + 4a - 2$
- **(k)** $x^3 - x^2 + x - 1$
- **(l)** $5x^3 + 7x^2 + 5x + 7$

32. Solve the following equations.

- **(a)** $(x - 5)(x + 14) = 0$
- **(b)** $(x - 12)(x + 13) = 0$
- **(c)** $(12 - 3x)(14 - 2x) = 0$
- **(d)** $(7 - 3x)(23 - 2x) = 0$
- **(e)** $x^2 + 4x - 21 = 0$
- **(f)** $x^2 - 8x + 15 = 0$
- **(g)** $x^2 - 19x + 48 = 0$
- **(h)** $x^2 - 32x + 175 = 0$
- **(i)** $7 - 8x + x^2 = 0$ **(j)** $9x + x^2 + 20 = 0$
- **(k)** $2x^2 - 9x + 4 = 0$ **(l)** $3x^2 - 10x + 3 = 0$
- **(m)** $4x^2 + 7x - 2 = 0$
- **(n)** $5x^2 + 26x + 5 = 0$
- **(o)** $6x^2 - 17x + 12 = 0$
- **(p)** $9x^2 - 17x - 2 = 0$

33. The cost of two cups of coffee and three bowls of noodles is \$6.80 while the cost of five cups of coffee and seven bowls of noodles is \$16.10. Find the cost of a cup of coffee and that of a bowl of noodles.

34. Solve the following simultaneous equations:

- **(a)** $\frac{1}{2}y + 2x = 36,\ \frac{1}{2}x - \frac{1}{4}y + 3 = 0$
- **(b)** $3x + 5y = 4,\ 5x = 16 + y$
- **(c)** $3x - 4y = 73,\ 5x + y = 22$
- **(d)** $3x - 2y = 3,\ 4x - 5y + 10 = 0$
- **(e)** $x + 2y = 2,\ 2x - 3y = 7\frac{1}{2}$
- **(f)** $x + 3y = 5,\ 4x + 13y = 3$
- **(g)** $\frac{1}{4}x + \frac{3}{5}y = -4,\ \frac{1}{5}x + \frac{1}{4}y = -\frac{9}{10}$
- **(h)** $x = 3 + 4y,\ y = 2 + 3x$

35. If x books cost \$$y$ and z cents, find the cost of one book in cents. Also write down an expression for the number of books that could be bought for \$$k$.

36. **(a)** Solve the simultaneous equations

$$8x - 3y = 21,\ 5x + 2y = 17.$$

(b) Explain briefly why the following pair of simultaneous equations has no solutions.

$$8x - 3y = 21,\ 24x - 9y = 17$$

37. Given that $x = 2$, $y = -4$ and $z = 3$, find the values of

- **(a)** $(x + y + z)^2 + (x - y - z)^2$,
- **(b)** $(3x - 4y)(z - 2y)$,
- **(c)** $\sqrt{y + z^2 + x^2 + y^2}$,
- **(d)** $\frac{x}{y} - z^2$,
- **(e)** $(x + y^3) \div (y^2 - z^2)$.

38. If $5 : 7 = (x - y) : (x + 2y)$, find the value of $\frac{4x}{5y}$.

7.3 Algebra 77

Example 1

(a) Simplify (i) $3a^2 \times 4a^5$, (ii) $8x^{2\frac{1}{2}} \div 2x^{\frac{1}{2}}$.

(b) Evaluate (i) $8^{-\frac{2}{3}}$, (ii) $\left(\frac{2}{3}\right)^{-2}$, (iii) $3^0 + 4^{\frac{1}{2}} \times 4^{1\frac{1}{2}}$.

(c) If $3^{14} \div 9^2 = 3^x$, *find the value of* x.

Solution

(a) (i) $3a^2 \times 4a^5 = (3 \times 4)a^{2+5}$
$= 12a^7$

(ii) $8x^{2\frac{1}{2}} \div 2x^{\frac{1}{2}} = (8 \div 2)x^{2\frac{1}{2}-\frac{1}{2}}$
$= 4x^2$

(b) (i) $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}}$
$= \frac{1}{(\sqrt[3]{8})^2}$
$= \frac{1}{2^2}$
$= \frac{1}{4}$

Alternatively, $8^{-\frac{2}{3}} = \left(2^3\right)^{-\frac{2}{3}}$
$= 2^{-2}$
$= \frac{1}{2^2}$
$= \frac{1}{4}$

(ii) $\left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2}$
$= \frac{1}{\frac{4}{9}}$
$= \frac{9}{4}$
$= 2\frac{1}{4}$

(iii) $3^0 + 4^{\frac{1}{2}} \times 4^{1\frac{1}{2}} = 1 + 4^{\frac{1}{2}+1\frac{1}{2}}$
$= 1 + 4^2$
$= 1 + 16$
$= 17$

(c) $3^{14} \div 9^2 = 3^{14} \div (3^2)^2$
$= 3^{14} \div 3^4$
$= 3^{14-4}$
$= 3^{10}$
i.e. $3^{10} = 3^x$
$\therefore \quad x = 10$

Example 2

Express each of the following as a fraction in its lowest terms.

(a) $\frac{2-7x}{3} - \frac{3x-1}{5}$

(b) $\frac{4}{5} + \frac{a-3}{4a}$

(c) $\frac{a}{a-5} - \frac{2}{a+1}$

(d) $\frac{5}{x-3} - \frac{2x-3}{x^2-x-6} - \frac{4}{x+2}$

Solution

(a) $\frac{2-7x}{3} - \frac{3x-1}{5} = \frac{5(2-7x)-3(3x-1)}{3 \times 5}$
$= \frac{10-35x-9x+3}{15}$
$= \frac{13-44x}{15}$

(b) $\frac{4}{5} + \frac{a-3}{4a} = \frac{4(4a)+5(a-3)}{5(4a)}$
$= \frac{16a+5a-15}{20a}$
$= \frac{21a-15}{20a}$

(c) $\frac{a}{a-5} - \frac{2}{a+1} = \frac{a(a+1)-2(a-5)}{(a-5)(a+1)}$
$= \frac{a^2+a-2a+10}{(a-5)(a+1)}$
$= \frac{a^2-a+10}{(a-5)(a+1)}$

(d) $$\frac{5}{x-3} - \frac{2x-3}{x^2-x-6} - \frac{4}{x+2} = \frac{5}{x-3} - \frac{2x-3}{(x-3)(x+2)} - \frac{4}{x+2}$$
$$= \frac{5(x+2)-(2x-3)-4(x-3)}{(x-3)(x+2)}$$
$$= \frac{5x+10-2x+3-4x+12}{(x-3)(x+2)}$$
$$= \frac{-x+25}{(x-3)(x+2)}$$

Example 3

Solve the following equations:

(a) $\frac{5}{x} = 3$

(b) $\frac{2}{x} = \frac{7}{x+4}$

(c) $\frac{a}{3} = \frac{27}{a}$

(d) $\sqrt{k} = 4$

(e) $\frac{8}{x} - \frac{5}{2x+1} = 3$

Solution

(a)
$$\frac{5}{x} = \frac{3}{1}$$
$$3x = 5$$
$$x = \frac{5}{3}$$
$$= 1\frac{2}{3}$$

(b)
$$\frac{2}{x} = \frac{7}{x+4}$$
$$7x = 2(x+4)$$
$$7x = 2x + 8$$
$$7x - 2x = 8$$
$$5x = 8$$
$$x = \frac{8}{5}$$
$$= 1\frac{3}{5}$$

(c)
$$\frac{a}{3} = \frac{27}{a}$$
$$a^2 = 81$$
$$a = \pm\sqrt{81}$$
$$= \pm 9$$

(d)
$$\sqrt{k} = 4$$
$$(\sqrt{k})^2 = (4)^2$$
$$\therefore \quad k = 16$$

(e)
$$\frac{8}{x} - \frac{5}{2x+1} = 3$$
$$\frac{8(2x+1)-5x}{x(2x+1)} = 3$$
$$16x + 8 - 5x = 3x(2x+1)$$
$$11x + 8 = 6x^2 + 3x$$
$$6x^2 - 8x - 8 = 0$$
$$3x^2 - 4x - 4 = 0$$
$$(3x+2)(x-2) = 0$$
$$\therefore \quad 3x + 2 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -\frac{2}{3} \quad \text{or} \quad x = 2.$$

Example 4

Given that y is directly proportional to x^3 and that $y = 24$ when $x = 2$. Express y in terms of x and

(i) find the value of y when $x = 5$,

(ii) find the value of x when $y = 10\frac{1}{8}$.

Solution

$y \propto x^3$ i.e. $y = kx^3$ where k is a constant.

When $y = 24, x = 2$

$$24 = k(2^3)$$
$$k = 3$$
$$\therefore \quad y = 3x^3$$

(i) When $x = 5$,

$$y = 3(5^3)$$
$$= 3(125)$$
$$= 375$$

(ii) When $y = 10\frac{1}{8}$

$$10\frac{1}{8} = 3x^3$$
$$x^3 = 10\frac{1}{8} \div 3 = \frac{27}{8}$$
$$\therefore \quad x = \sqrt[3]{\frac{27}{8}} = 1\frac{1}{2}$$

Example 5

Fig. 7.1 shows the lengths of the sides of a right-angled triangle ABC. Given that BC = (x + 1) cm, AB = (x + 2) cm and AC = (2x + 2) cm, form an equation in x. Solve this equation for values of x correct to 2 decimal places. Hence or otherwise, find the perimeter and area of △ABC.

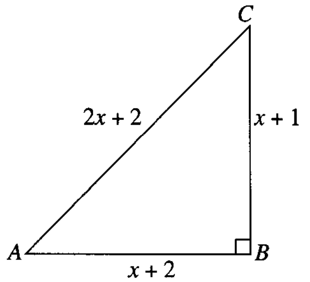


Fig. 7.1

Solution

$$(x + 1)^2 + (x + 2)^2 = (2x + 2)^2$$
$$x^2 + 2x + 1 + x^2 + 4x + 4 = 4x^2 + 8x + 4$$
$$2x^2 + 2x - 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)}$$
$$= \frac{-2 \pm \sqrt{12}}{4}$$
$$= 0.37 \quad \text{or} \quad -1.37 \quad \text{(correct to 2 decimal places)}$$
$$\therefore \quad x = 0.37 \text{ (since } BC > 0)$$

Hence $AB = 2.37$ cm, $BC = 1.37$ cm and $AC = 2.74$ cm.

The perimeter of $\triangle ABC = 2.37 + 1.37 + 2.74 = 6.48$ cm.

The area of $\triangle ABC = \frac{1}{2}(1.37)(2.37) = 1.62 \text{ cm}^2$.

Revision Exercise 7.3

1. Simplify each of the following.

(a) $3x^5 \times 7x^2$ **(b)** $2x^{\frac{1}{2}} \times 3x^{\frac{1}{3}}$

(c) $(2x^2)^3 \times (x^{\frac{1}{2}})^3$ **(d)** $8x^8 \div 2x^4$

(e) $9x^5 \div 3x^{\frac{1}{2}}$ **(f)** $(2x^7)^3 \div (4x^2)^4$

(g) $(7x^0)^4 \div 49x^2$ **(h)** $(5x)^0 \div 8x^{-4}$

(i) $3x^{-2} \times 7x^5$ **(j)** $x^{\frac{2}{3}}y^{\frac{3}{4}} \div x^{\frac{1}{3}}y^{\frac{1}{2}}$

(k) $(x^{\frac{1}{3}}y^2)^3 \div (x^{\frac{1}{3}}y^{\frac{3}{4}})^2$

(l) $(2x^3y^{\frac{1}{2}})^3 \div 4x^2y^{\frac{1}{2}}$

2. Find the value of each of the following without using a calculator.

(a) 3^{-2} **(b)** $4^{1.5}$

(c) $64^{-\frac{2}{3}}$ **(d)** $9^{-2.5}$

(e) $\frac{1}{5^0}$ **(f)** $625^{\frac{3}{4}}$

(g) $\left(\frac{3}{4}\right)^{-3}$ **(h)** $8^{-\frac{4}{3}}$

(i) $\frac{1}{27^{-\frac{2}{3}}}$ **(j)** $\frac{1}{36^{\frac{3}{2}}}$

(k) $\frac{2}{8^{-\frac{1}{3}}}$ **(l)** $\frac{5}{2^5}$

(m) $\frac{26}{2^{-4}}$ **(n)** $\frac{9}{4^0}$

(o) $27^{\frac{2}{3}} \times \left(\frac{1}{8}\right)^{\frac{1}{3}}$ **(p)** $\frac{2^0 - 2^{-1}}{4 - 3(4)^{-1}}$

(q) $\frac{8^2 + (4)^3}{7^0 - 3^{-1}}$ **(r)** $\frac{3^{-2} - 5 \times 2^0}{6^{-1} + 2 \div 4^{-1}}$

3. Evaluate each of the following, giving your answer in the standard form.

(a) $2.5 \times 10^5 + 6 \times 10^6$

(b) $3.2 \times 10^7 + 6.3 \times 10^8$

(c) $4.5 \times 10^{10} - 7.8 \times 10^9$

(d) $6.7 \times 10^{15} - 9.8 \times 10^{13}$

(e) $5.89 \times 10^{-2} + 7.9 \times 10^{-3}$

(f) $6.45 \times 10^{-7} - 8.8 \times 10^{-8}$

(g) $2.14 \times 10^{-2} \times 7.8 \times 10^{-4}$

(h) $6.9 \times 10^{-14} \times 3.2 \times 10^7$

(i) $(3.98 \times 10^{15}) \div (7.96 \times 10^{-1})$

(j) $(3.51 \times 10^{-4}) \div (3.9 \times 10^3)$

4. (a) Given that y varies directly as x^3 and that $y = 10$ when $x = 1$, find the value of y when $x = 3$.

(b) Given that y is directly proportional to $\sqrt{x}$ and that $y = 5$ when $x = 4$, find the value of y when $x = 9$.

(c) Given that y is directly proportional to $x^{\frac{2}{3}}$ and that $y = 14$ when $x = 8$, express y in terms of x and find the value of y when $x = 27$.

5. Express each of the following as a fraction in its simplest form.

(a) $\frac{x}{2} + \frac{5x}{7}$ **(b)** $\frac{x-2}{3} - \frac{2x-5}{4}$

(c) $\frac{x+7}{5} + \frac{3-x}{4}$ **(d)** $\frac{3}{4} + \frac{x-3}{2x}$

(e) $3 - \frac{x-2}{2x-3}$ **(f)** $\frac{2a}{5a-1} + \frac{2}{5}$

(g) $\frac{2}{x-1} + \frac{3}{x+2}$ **(h)** $\frac{5}{x+1} - \frac{3}{2x-1}$

(i) $\frac{4}{3a-1} - \frac{3}{2a-3}$ **(j)** $\frac{a}{1-2a} + \frac{3}{4+a}$

(k) $\frac{x}{x+1} - \frac{5}{x-2}$ **(l)** $\frac{3x}{x-3} + \frac{2}{x+4}$

(m) $\frac{5p}{p-3} - \frac{4}{p+1}$ **(n)** $\frac{1}{2a-3} + \frac{a}{a+1}$

(o) $\frac{2}{x-y} - \frac{3}{x}$ **(p)** $\frac{5x}{2x-y} + \frac{y}{3x-y}$

6. Solve the following equations:

(a) $\frac{3}{a} - 2 = 3$ **(b)** $\frac{5}{x} + 1\frac{1}{2} = \frac{3}{x} + 4$

(c) $\frac{2}{x} + \frac{3}{5x} = \frac{2}{3}$ **(d)** $\frac{t}{4} = \frac{1}{9t}$

(e) $\frac{x}{27} = \frac{125}{x^2}$ **(f)** $\frac{1}{x-1} + \frac{1}{x+2} = \frac{1}{2}$

(g) $\sqrt{x-1} = 2$ **(h)** $\frac{5}{x} = \frac{7}{x-2}$

(i) $\frac{2}{x-3} = \frac{3}{3x-5}$ **(j)** $\frac{3}{x+1} + \frac{5}{x-3} = 3$

(k) $\frac{x+1}{3x-2} = \frac{3x}{x+6}$ **(l)** $2 + \frac{2a-1}{a+3} = 2\frac{1}{3}$

7. Given that y is inversely proportional to x^2 and that $y = 2$ when $x = 3$, find the value of y when $x = 2$.

8. Given that y varies inversely as z^3 and that $y = 7$ when $z = 1$, find the value of y when $z = 2$ and express y in terms of z.

9. If y varies inversely as x^2, copy and complete the table below and express y in terms of x.

y	16		4
x	1	3	

10. Given that y is inversely proportional to $(x + 7)$ and that $y = 2$ when $x = 3$, find the value of y when $x = 1$ and the value of x when $y = 5$.

11. Given that y is directly proportional to $(x + 1)^2$ and that the difference between the value of y when $x = 2$ and $x = 5$ is 32, express y in terms of x and find the value of y when $x = 3$.

12. Solve the following equations.

(a) $5^x - 25^{3x-1} = 0$
(b) $3^{2x} - 27^{x+1} = 0$
(c) $16^{3x+1} = 32^{1-x}$

13. A piece of elastic 12 cm long, hangs from a nail N, as shown in Fig. 7.2(a).

When a mass of m grams is attached to the lower end, the length of the elastic increases to L cm, as shown in Fig. 7.2(b).

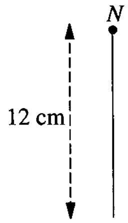


Fig. 7.2(a)

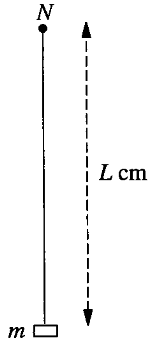


Fig. 7.2(b)

For every 100 grams which is attached, the length of the elastic increases by 3 cm.

(a) Calculate the length of the piece of elastic when a mass of 700 grams is attached to it.
(b) If the length of the elastic is 48 cm, calculate the mass that is attached to it.
(c) Write down a formula connecting the length of the elastic L, and the mass m, which is attached to it.
(d) Use your formula to check your answers to parts **(a)** and **(b)**. *(C)*

14. Solve the following equations:

(a) $25^{x-4} = 0.2$
(b) $\sqrt{3} \times \sqrt[x]{9} = \frac{1}{27}$
(c) $49^{x-1} \div \sqrt{7} = \frac{1}{343}$
(d) $9^{x+4} = 2^6 \div 4^3$
(e) $(3x - 4)^3 - \frac{1}{8} = 0$
(f) $25^{x+3} \div 125^{x-4} = 0.04$

15. Make the letter in the brackets the subject of the formula below.

(a) $ax^2 + bx + c = 0 \quad (b)$
(b) $t = \frac{\pi}{4}\sqrt{\frac{2l}{g}} \quad (l)$
(c) $\frac{1}{a} + \frac{b}{2} + \frac{3}{c} = k \quad (c)$
(d) $\sqrt{4x^2 - 5k} = 2x + 3 \quad (x)$
(e) $v^2 = u^2 + 2as \quad (u)$
(f) $x = \sqrt[3]{\frac{a}{b - a}} \quad (a)$

16. Simplify each of the following:

(a) $\frac{1}{1 - x} + \frac{2}{1 + x} + \frac{2x}{x^2 - 1}$
(b) $\frac{3}{x + 2} - \frac{x - 5}{x^2 - 4} + \frac{1}{x - 2}$
(c) $\frac{1}{2x - 3} - \frac{2}{x + 2} - \frac{2x - x^2}{2x^2 + x - 6}$
(d) $\frac{5}{x - 2} - \frac{3x + x^2}{x^2 - x - 2} + \frac{x}{x + 1}$

17. Solve the following equations, giving your answers correct to 2 decimal places where necessary.

(a) $2x - 7 = \frac{2}{x}$

(b) $x + 2 = \frac{3}{5 - 3x}$

(c) $4x - 7 = \frac{2}{x - 1}$

(d) $3x - 2 = \frac{1}{x - 1}$

(e) $x + 2 = \frac{4}{4 - 3x}$

(f) $\frac{x}{x + 1} = \frac{4}{3x + 2}$

18. The diagonal of a rectangle exceeds its length by 4 cm and its width by 6.4 cm. Find the length of the diagonal.

19. If the side of a cube is increased by 2 cm, its volume will be increased by 54 cm^3. Find the original length of the cube.

20. Solve the following equations, giving your answers correct to 2 decimal places where possible.

(a) $2x^2 - 5x - 1 = 0$

(b) $3x^2 - 7x - 14 = 0$

(c) $5x^2 + 4x - 7 = 0$

(d) $7x^2 + 14x - 20 = 20$

(e) $5 + 4x - 3x^2 = 0$

(f) $5x - 2 + 7x^2 = 0$

(g) $11 + 2x - 3x^2 = 0$

(h) $46x + 7 + x^2 = 0$

(i) $7 - 4x - 9x^2 = 0$

(j) $11x^2 + 14x - 1 = 0$

21. The length of a page of a book is 3.4 cm more than the width and the area is 125 cm^2. If the width is x cm, form an equation in x and hence, find the length in centimetres, giving your answer correct to 2 decimal places.

22. A stone is thrown down from a high building, and the formula $d = 7t + 5t^2$ gives the distance, d m, that it has fallen after t seconds. How long does it take to reach a distance of 100 m?

23. The area of a rectangular picture is 156 cm^2 and the width is 2.4 cm less than the length. Find the perimeter of the picture, giving your answer correct to 2 decimal places.

24. A man swims at a speed of 50 m/min in still water, swims 100 m *against* the current and 100 m *with* the current. If the difference between the two times is 3 min 45 s, find the speed of the current.

25. Two trains A and B travel between two stations 80 km apart. If train A travels at an average speed of 5 km/h faster than train B and completes the journey 20 minutes earlier, find the average speeds of the two trains.

26. Two numbers differ by 5 and the sum of their reciprocals is $\frac{13}{36}$. Find the numbers.

27. The numerator of a fraction is 2 less than the denominator. When both numerator and denominator are increased by 3, the fraction is increased by $\frac{3}{20}$. Find the original fraction.

28. A motorist makes a journey from Singapore to Kuala Lumpur, a journey of 380 km, at an average speed of x km/h. Write down an expression for the number of hours taken for the journey.

On his return journey from Kuala Lumpur to Singapore, his average speed for the journey is reduced by 5 km/h due to heavy traffic along the stretch of road from Ayer Hitam to Kulai. Write down an expression for the number of hours taken for the journey.

If the return journey takes 25 minutes longer, form an equation in x and solve it. Hence, find the average speed for each journey.

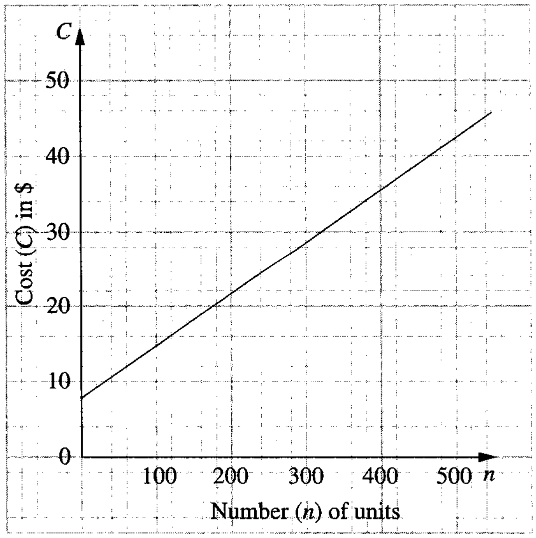
29. A water tank is filled by two pipes in 48 minutes. If the larger pipe can fill it alone in 40 minutes less time than the smaller pipe, find the time it takes for the smaller pipe to fill the tank alone.

30. A curve has equation $x^2 + y^2 + 8x - 16y + 15 = 0$. Find the coordinates of the points where the curve cuts

(a) the x-axis,
(b) the y-axis.

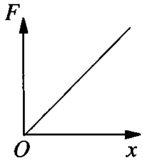
31. The graph shows the relation between the number (n) of units of electricity used and the total cost (C) of an electricity bill.

(a) Use the graph to find
  (i) the cost of the bill if 300 units are used,
  (ii) the number of units used when the bill is \$32.50.
(b) Given that the relation is $C = pn + q$,
  (i) state the value of q and explain its significance,
  (ii) find the value of p and explain its significance,
  (iii) find the total cost of the bill if 1 100 units are used. *(C)*



32. It is given that the force (F units) between two particles is **inversely** proportional to the square of the distance (x units) between them.

(a) Which one of the graphs below could represent the relation between the force and the distance?



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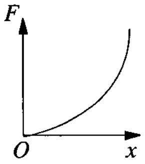

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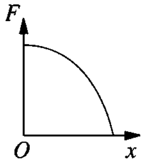

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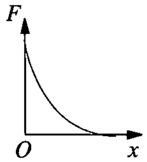

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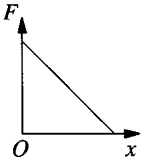
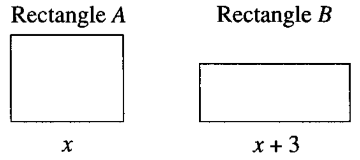


V

(b) What happens to the force when the distance between the two particles is doubled?
(c) Given that $F = 4$ when $x = 3$, find
  (i) the equation connecting F and x,
  (ii) the value of F when $x = 10$. *(C)*

33. Two rectangles, A and B, each has an area of 11 cm^2. The length of rectangle A is x cm. The length of rectangle B is $(x + 3)$ cm.



(a) Find, in terms of x, an expression for the width of
  (i) rectangle A,
  (ii) rectangle B.
(b) Given that the width of rectangle A is 2 cm greater than the width of rectangle B, form an equation in x and show that it simplifies to

$$2x^2 + 6x - 33 = 0.$$

(c) Solve the equation $2x^2 + 6x - 33 = 0$, giving both answers correct to 2 decimal places.
(d) Hence, find the width of rectangle B. *(C)*

7.4 Mensuration

Example 1

Fig. 7.3 shows a circular pattern of radius 21 cm, O is the centre of the circle. AD, BC and PQ are diameters of the circle. $A\hat{O}P = 60°$ and $B\hat{O}Q = 30°$.

(a) Calculate

(i) the length of arc of a blue sector,

(ii) the area of a blue sector,

(iii) the area of a red segment.

(b) Given that the circular pattern is the top face of a cake of thickness 4 cm, calculate the volume of a yellow slice.

$\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$

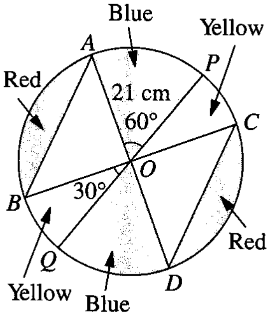


Fig. 7.3

Solution

(a) (i) Length of arc of a blue sector = $\frac{60}{360} \times 2 \times \frac{22}{7} \times 21 = 22$ cm

(ii) Area of blue sector = $\frac{60}{360} \times \frac{22}{7} \times 21^2 = 231 \text{ cm}^2$

(iii) $A\hat{O}B = 180° - 60° - 30° = 90°$

Area of sector $AOB = \frac{90}{360} \times \frac{22}{7} \times 21^2 = 346.5 \text{ cm}^2$

Area of $\triangle AOB = \frac{1}{2} \times 21 \times 21 = 220.5 \text{ cm}^2$

$\therefore$ Area of a red segment = $346.5 - 220.5 = 126 \text{ cm}^2$

(b) Area of a yellow sector = $\frac{30}{360} \times \frac{22}{7} \times 21^2 = 115.5 \text{ cm}^2$

$\therefore$ the volume of a yellow slice = $115.5 \times 4 = 462 \text{ cm}^3$

Example 2

A solid is cylindrical with hemispherical ends as shown in Fig. 7.4. The height of the cylinder is 56 cm and the area of its base is 1 386 cm².

(a) Calculate the volume of the cylinder.

(b) Calculate the radius of the base of the cylinder.

(c) Calculate the volume of the solid.

(d) Given that the solid is made from material of density 0.05 g/cm³, calculate its mass.

(e) Calculate the total surface area of the solid.

(f) Given that the solid is melted down and made into a cone having a base radius of 28 cm, calculate the height of the cone.

$\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$

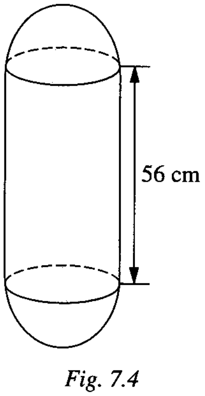


Fig. 7.4

Solution

(a) Volume of the cylinder = area of base × height = 1 386 × 56 = 77 616 cm^3

(b) Let r be the radius of the base of the cylinder.
Area of base = πr^2

i.e. $\pi r^2 = 1\ 386$ or $\frac{22}{7}r^2 = 1\ 386$

$r^2 = 1\ 386 \times \frac{7}{22} = 441 \qquad \therefore \quad r = 21$ cm

(c) Volume of the two hemispheres = volume of a sphere = $\frac{4}{3} \times \frac{22}{7} \times 21^3 = 38\ 808\ \text{cm}^3$

$\therefore$ the volume of the solid = 77 616 + 38 808 = 116 424 cm^3

(d) The mass of the solid = 116 424 × 0.05 = 5 821.2 g

(e) The curved surface area of the cylinder = $2 \times \frac{22}{7} \times 21 \times 56 = 7\ 392\ \text{cm}^2$

The surface area of a sphere of radius 21 cm = $4 \times \frac{22}{7} \times 21^2 = 5\ 544\ \text{cm}^2$

$\therefore$ the total surface area of the solid = 7 392 + 5 544 = 12 936 cm^2

(f) Let h be the height of the cone.

Volume of the cone = $\frac{1}{3} \times \pi \times (28)^2 \times h$

i.e. $\frac{1}{3} \times \frac{22}{7} \times 28^2 \times h = 116\ 424$

$$h = \frac{116\ 424 \times 3 \times 7}{22 \times 28^2} = 141.75 \text{ cm}$$

Example 3

A model consists of a solid cuboid attached to a solid pyramid as shown in Fig. 7.5. The height of the cuboid is 24 cm and the area of its base is 96 cm².

(a) Calculate the volume of the cuboid.
(b) Given that the volume of the pyramid is 144 cm³, calculate the height of the pyramid.
(c) Given that the model is made from material of density 0.5 g/cm³, calculate its mass.
(d) Given that the width and length of the base of the cuboid are in the ratio 2 : 3, calculate the width and length of the cuboid.

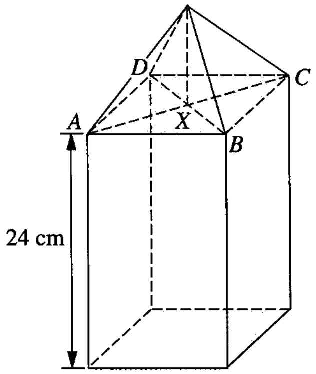


Fig. 7.5

Solution

(a) The volume of the cuboid = base area × height = 96 × 24 = 2 304 cm^3

(b) The volume of the pyramid = $\frac{1}{3}$ × base area × height

i.e. $\frac{1}{3} \times 96 \times \text{height} = 144$

$\therefore \quad \text{height} = \frac{144 \times 3}{96} = 4.5$ cm

(c) The volume of model = volume of the cuboid + volume of the pyramid

$= 2\,304 + 144$

$= 2\,448 \text{ cm}^3$

The mass of the model = 2 448 × 0.5 = 1 224 g

(d) Let the width of the base = $2x$

$\therefore$ the length of the base = $3x$

Area of the base = $2x \times 3x = 6x^2$

i.e. $6x^2 = 96$

$x^2 = 16$

$\therefore x = 4 (x = -4$ not applicable here)

$\therefore$ the width of the base = 8 cm and the length of the base = 12 cm

Revision Exercise 7.4

1. The circumference of a circle, centre O, is 25 cm.

(a) Taking π to be 3.14, calculate the diameter of the circle.

(b) Given that P and Q are points on the circumference of the circle such that $P\hat{O}Q = 75°$, calculate the length of the major arc PQ.

2. (a) In Fig. 7.6, the area of the shaded sector POQ is $\frac{3}{20}$ of the area of the whole circle. Calculate $P\hat{O}Q$.

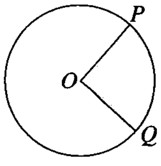


Fig. 7.6

(b) Given that the area of the circle is 616 cm², calculate the

(i) radius of the circle,

(ii) area of the shaded sector POQ.

$\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$

3. The radius of the chain wheel of a bicycle is 7 cm. Given that the portion of chain in contact with the wheel is of length 11 cm and taking π to be $\frac{22}{7}$, calculate the angle subtended at the centre of the wheel by this portion of chain.

4. The square $ABCD$ is inscribed in the circle of radius 8 cm. Calculate the following to the nearest cm², taking π to be 3.14.

(a) The area of the circle.

(b) The area of the shaded portion.

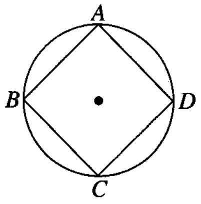


Fig. 7.7

5. Fig. 7.8 shows the top face of a concrete logo of a firm. It consists of a quadrant $OABC$ with a blue segment ABC and a red sector OAD of a circle of radius 64 cm. The red sector has an angle of 120° at the centre of the circle.

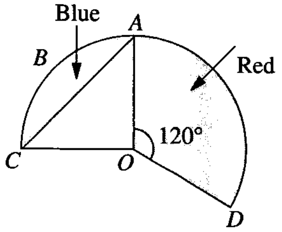


Fig. 7.8

(a) Calculate

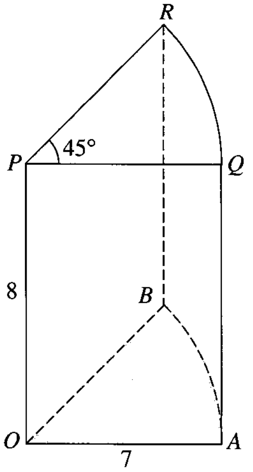
(i) the length of the arc of the red sector,

(ii) the area of the red sector,

(iii) the area of the blue segment.

(b) Given that the logo has a uniform thickness of 5 cm, find the volume of concrete used. (Take π to be 3.142.)

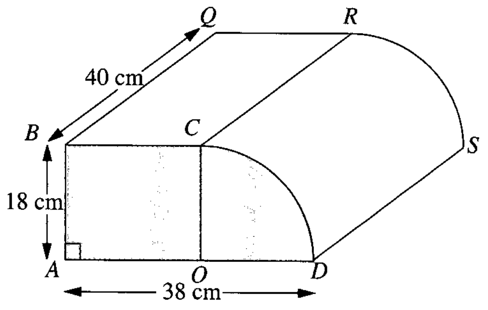
6. The diagram represents a prism in which each cross-section of the prism is a sector of a circle of radius 7 cm, with angle at centre equal to 45°. Two cross-sections are OAB and PQR where A, B, Q, R lie on the curved surface of the prism. The cross-sections OAB and PQR are horizontal and 8 cm apart. The vertical planes $OAQP$ and $OBRP$ are rectangular.



Calculate, taking π to be $\frac{22}{7}$,

(a) the length of the arc AB,
(b) the area of sector PQR,
(c) the volume of the prism,
(d) the total surface area of the prism.

7. The dimensions of a closed container for holding bread are marked on the diagram shown. The container is of uniform cross-section and the shaded side is made up of a rectangle $ABCO$ and a quadrant OCD of a circle centre O. Taking π to be 3.14, calculate, giving your answers to 3 significant figures,



(a) the area of the cross-section $ABCD$,
(b) the volume of the container,
(c) the area of the lid $BCDSRQ$.

8. Calculate the volume of each of the following figures. $\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$

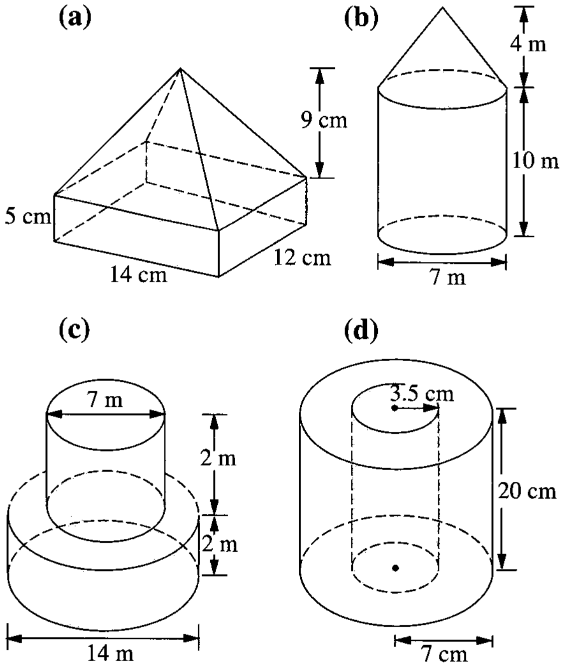
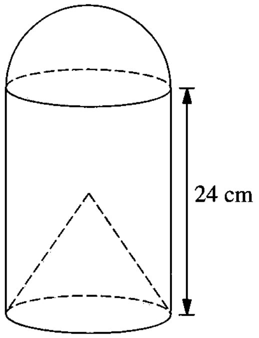


Fig. 7.9

9. A model consists of a solid hemisphere attached to a solid cylinder. Part of the cylinder in the shape of a cone is removed as shown in the diagram. The height of the cylinder is 24 cm and the area of its base is 154 cm². $\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$



(a) Calculate the volume of
  (i) the cylinder,
  (ii) the hemisphere.
(b) Given that the volume of the cone removed is 462 cm³, calculate the height of the cone.

$\left(\text{Volume of cone} = \frac{1}{3} \times \text{area of base} \times \text{height.}\right)$

(c) Given that the model is made from material of density 0.5 g/cm^3, calculate its mass.

(d) Taking the area of the curved surface of the cone to be 251 cm^2, calculate the total surface area of the model.

10. A solid metal cylinder A has a volume of 176 cm^3.

(a) Calculate the volume of another cylinder B which has the same height as the cylinder A but a base radius three times that of A.

(b) Given that the cylinder A is melted down and made into a cone of height 3.5 cm, calculate the radius of the base of the cone.

(c) Given that the cylinder B is melted down and made into a pyramid with the area of its base equal to 528 cm^2, calculate the height of the pyramid.

11. (a) A metal sphere of radius 8 cm is dropped into a cylindrical vessel of diameter 20 cm containing water to a depth of 25 cm. Find the rise in the water level.

(b) A uniform cylindrical pipe has an external diameter of 3.6 cm and an internal diameter of 3.2 cm. Find the volume of material required to construct a metre length of the pipe, giving your answer correct to the nearest cm^3.

If the material used has a density of 2.8 g/cm^3, calculate the mass of 25 m of such a pipe, giving your answer correct to the nearest kg.

12. (a) A closed cylindrical container with the area of the base equal to 38.5 cm^2 has the same volume as the closed rectangular container of length 11 cm, width 7 cm and depth 4 cm. Calculate the height of the cylindrical container.

(b) Given that both containers are made of thin sheet metal, calculate in cm^2, the total area of sheet metal required to make one container of each type, assuming that there is no wastage.

13. Fig. 7.10 shows a container made of metal sheets and closed at both ends. Both the base $PQVU$ and the top $SRWT$ are horizontal and rectangular. Each of the vertical sides $PQRS$ and $UVWT$ is a trapezium. The two sloping ends $SPUT$ and $RQVW$ are rectangular and are inclined at the same angle to the horizontal. $SR = TW = 2$ m, $PQ = UV = 1.5$ m, $ST = RW = PU = QV = 1.2$ m and the perpendicular height of $SRWT$ above the base is 1.2 m.

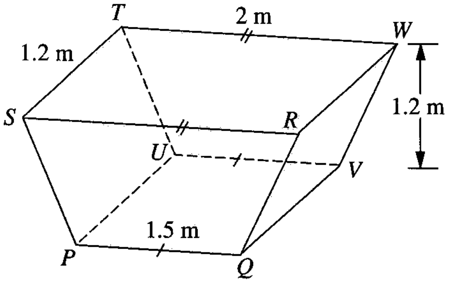


Fig. 7.10

(a) Calculate

(i) the cross-sectional area $PQRS$,

(ii) the weight of the container in kg, correct to 1 decimal place, given that the metal sheet weighs 30 kg per square metre.

(b) The container is completely filled with paint. Find the volume of paint in the container in m^3.

(c) The paint is sold in cylindrical tins of radius 4.3 cm and volume 500 cm^3. Taking π to be 3.142, calculate the height of one of these tins.

(d) Assuming that each tin is completely filled and that no paint is wasted, how many tins can be filled from the paint in the container?

(e) A shopkeeper buys 70 tins of paint for \$112 and marks each tin at the price which gives him a 27.5% profit. In addition, a customer has to pay 3% GST (goods and services tax) which is charged on the marked price. Calculate, to the nearest cent, the total amount the customer must pay for a tin of paint.

14. A tank has a square base of side 2.2 m and height of 96 cm. It is used to store a certain liquid in a factory. This liquid is sold in cylindrical tins with radii of their bases 10 cm and heights 28 cm. Taking π to be $\frac{22}{7}$, calculate

(a) the volume, in cm^3, of the tank,

(b) the volume, in cm^3, of the cylindrical tin,

(c) the number of tins which can be filled from a full tank assuming that each tin is completely filled and that no liquid is wasted.

15. 80 spheres, each of radius 35 mm are packed in a rectangular wooden box, 8 to a layer, as shown in Fig. 7.11. Calculate

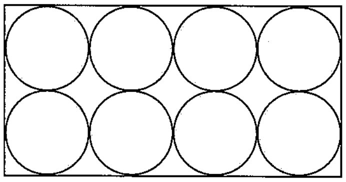


Fig. 7.11

(a) the volume of the smallest box required in cm^3,

(b) the percentage of the total volume of the box filled by the spheres.

When unpacked, each sphere is coated with paint 0.002 mm thick.

(c) How many boxes of spheres can be painted with 1 litre of paint?

✶16. (a) A solid sphere of diameter 6 cm is made from material of density 11.3 g/cm^3. Calculate the mass of the sphere in kg, giving your answer correct to two significant figures.

(b) A cylindrical container has a diameter of 14 cm and its base is horizontal. The sphere is placed in the container. It rests on the base of the container and is just covered by water. Calculate the volume of water in the container.

The sphere is then removed. Calculate the depth of water in the container.

(c) A long solid rod with a uniform cross-section of 16 cm^2 is next placed vertically in the container with one end in contact with the base. Calculate the length of the rod which is in contact with the water.

The rod is then removed, melted and cast into a cone of base diameter 10 cm. Given the length of the rod is 18 cm, find the height of the cone.

(Take π to be 3.142.)

✶17. A model consists of a solid cone and a solid hemisphere attached to a solid cylinder as shown in Fig. 7.12.

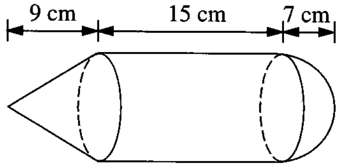


Fig. 7.12

(a) Calculate the volume of the model.

(b) Given that the model is made from material of density 0.4 g/cm^3, calculate its mass.

(c) Calculate the total surface area of the model.

(d) If the model is to be plated with material costing \$24 per cm^2, find the cost of the plating.

✶18. Fig. 7.13(a) represents a vertical cross-section of a rectangular tank which stands on a horizontal table represented by XY. The tank is 12 cm high, has a square base of side 20 cm and contains 3000 cm^3 of water. Calculate

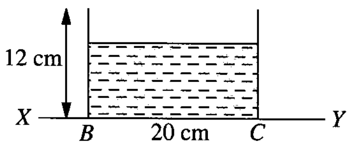


Fig. 7.13(a)

(a) the volume of the tank,

(b) the depth of the water.

The tank is now tilted about a base edge through C, so that some of the water spills out, until the position shown in Fig. 7.13(b) is reached.

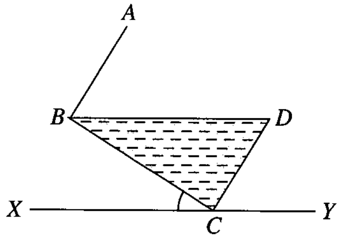


Fig. 7.13(b)

Calculate

(c) the volume of water remaining in the tank,

(d) $B\hat{C}X$,

(e) the vertical height of B above the table.

19. A hollow sphere is made of metal 1.5 cm thick and has an external diameter of 22 cm. Calculate the volume of metal used to make the hollow sphere. If the metal weighs 10.7 g/cm^3, calculate the weight of the sphere. (Take $\pi = 3.14$.)

★20. In Fig. 7.14, the rocket model consists of three parts. Parts A and C can be joined together to form a right circular cone. Part B is a right cylinder. Find,

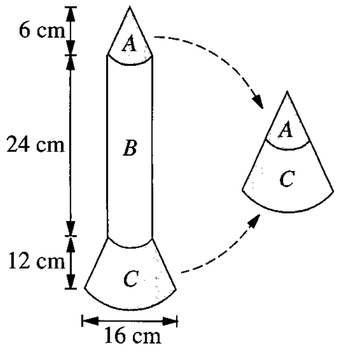


Fig. 7.14

(a) the volume of the rocket model,

(b) the total curved surface area of the rocket model.

(Take $\pi = 3.14$.)

7.5 Geometry

Example 1

(a) Each interior angle of a regular polygon is 168°. Find the number of sides of the polygon.

(b) In Fig. 7.15, AB, BC and CD are three adjacent sides of a regular polygon of 12 sides. Calculate the value of (i) $A\hat{B}C$, (ii) $A\hat{C}D$.

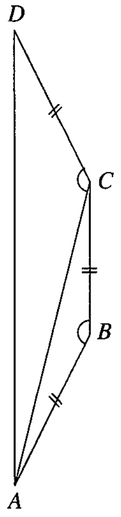


Fig. 7.15

Solution

(a) Each exterior angle = 180° − 168° = 12°

$\therefore$ number of sides of the polygon = $\dfrac{360}{12} = 30$

(b) (i) Each exterior angle of the polygon = $\dfrac{360°}{12} = 30°$

$\therefore$ $A\hat{B}C = 180° - 30° = 150°$

(ii) Now $B\hat{C}D = 150°$ and $B\hat{C}A = \frac{180 - 150}{2} = 15°$ ($\triangle ABC$ is isosceles)

$\therefore$ $A\hat{C}D = 150° - 15° = 135°$.

Example 2

A cone K has a volume of 400 cm³. Calculate the volume of

(a) a cone similar to K but with a height twice that of K,

(b) a cone with a height four times that of K and a base radius one quarter that of K.

Solution

(a) $\frac{\text{Vol. of } K}{\text{Vol. of new cone}} = \left(\frac{1}{2}\right)^3$ or $\frac{400}{\text{Vol. of new cone}} = \frac{1}{8}$

$\therefore$ Vol. of new cone = $8 \times 400 = 3\ 200$ cm³

(b) Let the height of K be h and its base radius be r.

$\therefore$ $\frac{1}{3}\pi r^2 h = 400$ cm³.

The new cone has a height of $4h$ and a base radius of $\frac{1}{4}r$.

Volume of new cone = $\frac{1}{3}\pi\left(\frac{1}{4}r\right)^2(4h) = \frac{1}{4}\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{4}(400) = 100$ cm³.

Example 3

Fig. 7.17(a) shows a parallelogram PQRS. C is a point on PQ such that 2PC = 3CQ. A is a point on PS such that SA = 3AP and AB is parallel to PQ. Given that the area of △SPC = 24 cm², calculate the area of

(a) parallelogram PQRS, *(b) △SAB.*

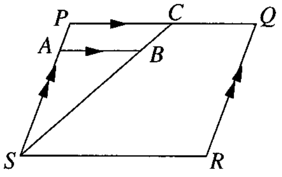


Fig. 7.17(a)

Solution

Let $PC = 3x$, then $CQ = 2x$;
$AP = y$, then $SA = 3y$;
the height of the parallelogram CK be h.
(See Fig. 7.17(b).)

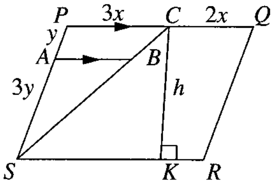


Fig. 7.17(b)

(a) Area of $\triangle SPC = 24 = \frac{1}{2}(3x)h$

i.e. $xh = 16$ cm²

Area of parallelogram $PQRS = 5x \times h = 5 \times 16 = 80$ cm²

(b) $\triangle SAB$ is similar to $\triangle SPC$ (AAS).

$\frac{\text{area of } \triangle SAB}{\text{area of } \triangle SPC} = \left(\frac{SA}{SP}\right)^2$ or $\frac{\text{area of } \triangle SAB}{24} = \left(\frac{3y}{4y}\right)^2 = \frac{9}{16}$

$\therefore$ area of $\triangle SAB = \frac{24 \times 9}{16} = 13.5$ cm²

Example 4

In the diagram, the line PAHD is parallel to QBC, AB = BC, $P\hat{A}B = 122°$, $A\hat{C}D = 90°$ *and BH is the angle bisector of* $A\hat{B}C$. *Calculate*

(a) $A\hat{B}H$,

(b) $B\hat{K}C$,

(c) $A\hat{D}C$.

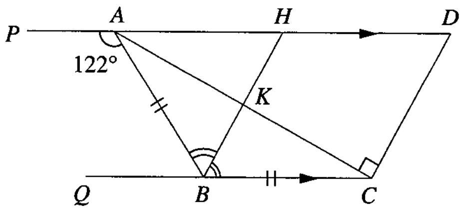


Fig. 7.16

Solution

(a) $A\hat{B}C = 122°$ (alt ∡, *PD//QC*)

$A\hat{B}H = \frac{122°}{2} = 61°$

(b) $B\hat{C}A = \frac{180° - 122°}{2}$ (base ∡ of isoceles △)

$= 29°$

$\therefore\ B\hat{K}C = 180° - 29° - 61° = 90°$ (∡ sum of a △)

(c) $A\hat{D}C + 90° + A\hat{C}B = 180°$ (interior ∡, *PD//QC*)

$\therefore\ A\hat{D}C = 180° - 90° - 29°$

$= 61°$

Revision Exercise 7.5

1. Find the exterior angle of a regular polygon with

(a) 8 sides, **(b)** 12 sides, **(c)** 24 sides.

2. Find the value of an interior angle of a regular polygon with

(a) 6 sides, **(b)** 10 sides, **(c)** 15 sides.

3. Find the number of sides of a regular polygon whose exterior angles are each

(a) 5°, **(b)** 8°, **(c)** 12°.

4. Find the number of sides of a regular polygon whose interior angles are each

(a) 170°, **(b)** 176°, **(c)** 162°.

5. *AB*, *BC* and *CD* are three adjacent sides of a regular polygon of 18 sides. Calculate the value of the following angles.

(a) $A\hat{B}C$, **(b)** $A\hat{C}D$.

6. The exterior angles of a six-sided polygon are in the ratio 4 : 5 : 6 : 7 : 7 : 7. Calculate the largest interior angle of the polygon.

7. A polygon has *n* sides and three of its exterior angles are 85°, 76° and 46°. The remaining $(n - 3)$ exterior angles are each 17°. Calculate the value of *n*.

8. In a regular polygon, each interior angle is 160° greater than each exterior angle. Calculate the number of sides of the polygon.

9. *ABCD* is a quadrilateral in which $\hat{A} = 112°$, $\hat{B} = 86°$,$\hat{C} = 72°$ and$A\hat{D}B = 28°$. Calculate the following angles.

(a) $A\hat{D}C$ **(b)** $A\hat{B}D$ **(c)** $B\hat{D}C$

10. One interior angle of a 7-sided polygon is 126°, and each of the other 6 angles is $x°$. Find *x*.

11. The interior angles of a quadrilateral $ABCD$ taken in order are in the ratio 1 : 2 : 3 : 4. Prove that the quadrilateral is a trapezium.

12. $ABCDE$ is a pentagon in which AB is parallel to ED. If $\hat{B} = 155°, \hat{C} = 3x°, \hat{D} = 2x°$ and $\hat{E} = 75°$. Calculate the following.

(a) x **(b)** $\hat{A}$

13. Find the number of sides of the polygon whose sum of interior angles is

(a) 1 080° **(b)** equal to 28 right angles.

14. In the diagram $HK//PQ//AB$, $C\hat{A}T = 154°$ and $A\hat{B}C = 52°$. Calculate

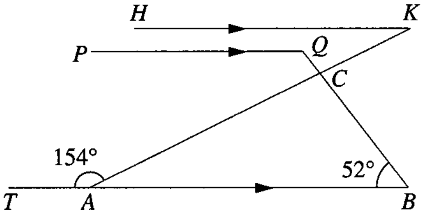


Fig. 7.18

(a) $H\hat{K}C$, **(b)** $Q\hat{C}K$, **(c)** $P\hat{Q}B$.

15. In the diagram, $AB//PQ, BP//RQ$, $A\hat{B}T = 72°$ and $P\hat{Q}T = 42°$. Calculate

(a) $B\hat{P}Q$,

(b) $P\hat{T}Q$,

(c) $R\hat{Q}T$.

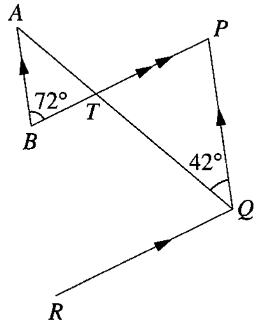


Fig. 7.19

16. In the diagram, $ABCDE$ is parallel to RS, $PC//QD$, $Q\hat{D}E = 156°$ and $R\hat{B}A = 107°$. Calculate

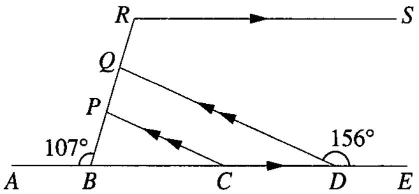


Fig. 7.20

(a) $D\hat{Q}R$,

(b) the reflex angle $B\hat{R}S$.

17. In the diagram $BT//HK$, $BP//CQ$, $A\hat{B}T = 16°$, $C\hat{B}H = 46°$ and $B\hat{C}Q = 72°$. Calculate

(a) $P\hat{B}T$,

(b) $B\hat{H}K$.

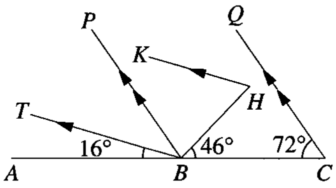


Fig. 7.21

18. In the diagram $BP//ER$, $CD//EQ$, $Q\hat{E}B = 103°$, $A\hat{B}P = 43°$ and $C\hat{B}D = 78°$. Calculate

(a) $B\hat{D}C$,

(b) $Q\hat{E}R$.

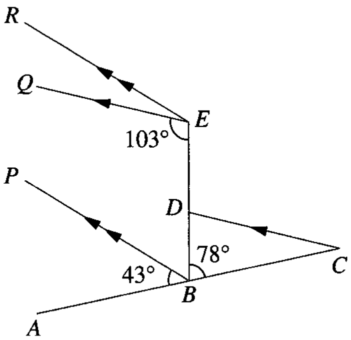


Fig. 7.22

19. In the diagram, $AP = AB$, $\triangle ABQ$ is equilateral and $PAB = 82°$, calculate

(a) $A\hat{C}B$,

(b) $P\hat{Q}R$,

(c) $A\hat{R}B$.

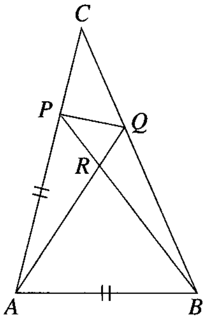


Fig. 7.23

20. In the diagram, $ABCDEF$ is a regular hexagon. PCA is produced to meet EF produced at T. Calculate

(a) $A\hat{T}E$,

(b) $B\hat{C}P$.

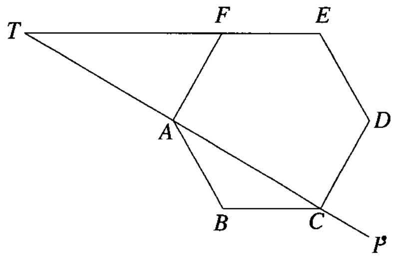


Fig. 7.24

21. In the diagram, $ABCDEF$ is a regular hexagon and $APQRB$ is a regular pentagon. Calculate

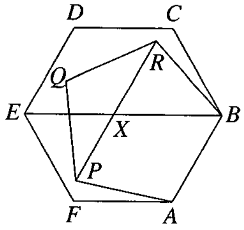


Fig. 7.25

(a) $B\hat{A}P$,

(b) $A\hat{B}X$,

(c) $E\hat{X}R$.

22. In a garden, there are two ponds which are similar. The depth of the larger pond is twice the depth of the smaller pond.

(a) Write down the ratio of their surface areas.

(b) Given that the capacity of the larger pond is 3 360 litres, find the capacity of the smaller pond.

23. Two solid spheres have surface areas in the ratio 9 : 16. If the smaller sphere has a radius of 12 cm and a mass of 5 kg, calculate

(a) the radius of the larger sphere,

(b) the mass of the larger sphere.

24. The ratio of the surface areas of two similar cones is 4 : 25. If the smaller cone has a height of 6.8 cm and a volume of 500 cm^3, calculate

(a) the height of the larger cone,

(b) the volume of the larger cone.

25. A conical flask has a surface area of 50 cm^2 and a capacity of 845 cm^3. Find the volume of a similar conical flask which has a surface area of 32 cm^2.

26. A cone with radius of r cm and a height of h cm has a volume of 420 cm^3. Find the volume of a cone whose height is $\frac{1}{2}h$ cm and whose radius is $3r$ cm.

27. The curved surface area of a cylindrical can of radius r cm and height h cm is 540 cm^2. Find the curved surface area of another cylinder whose height is $\frac{1}{4}h$ cm and whose radius is $5r$ cm.

28. The volume of a solid stone statue 3 m high is V_1 and its surface area is A_1. A wooden model of the statue, 20 cm high has a volume of V_2 and a surface area of A_2. The wooden model weighs 4 kg. Calculate the ratio of

(a) $V_1 : V_2$, **(b)** $A_1 : A_2$.

Given that the density of the stone is 3 000 kg/m^3 and the density of the wood is 540 kg/m^3, calculate the weight of the stone statue.

29. Two solid spheres have diameters of 35 cm and 14 cm respectively. Find the total surface area of each sphere. Compare the areas in the form of a ratio. If 200 cm^2 of the first sphere is to be painted in red, find the area of the second sphere to be painted in red also so that the two spheres will appear similar in design.

30. A suspension bridge is 2 500 metres long and a model is made of it on a scale of 1 : 600. If the supporting towers are 34 cm high in the model, find their actual height in metres.

If it costs $4 to paint the model, how much will it cost to paint the actual bridge?

If a steel section weighs 432 tonnes, how much would the section weigh in the model assuming that it was constructed from the same material?

31. [The value of π is 3.142, correct to 3 decimal places.]

A large Traffic Marker consists of a solid cone, of height 40 cm and radius 9 cm, with a solid cylindrical base of diameter 30 cm and thickness 2 cm.

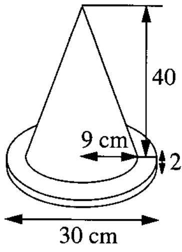


Fig. 7.26

(a) **(i)** Calculate the volume of the cone.

$$\left[\text{Volume of a cone } = \frac{1}{3}\pi r^2 h.\right]$$

(ii) Calculate the **total** volume of the Marker.

(b) Every part of the surface of the Marker is painted orange.

  (i) Calculate the slant height of the cone and hence, the area of the painted part of the cone.
  [Curved surface area of a cone = πrl, where l is the slant height.]

  (ii) Calculate the area of the painted part of the base.

(c) A small Traffic Marker is geometrically similar to a large one, and the diameter of its base is 15 cm.

  (i) Write down the ratio of the volume of a small Marker to that of a large Marker.

  (ii) Hence, calculate the volume of a small Marker. *(C)*

32. Mr Yang buys two cylindrical cans of tonic food drink from a shopkeeper. The cans are similar geometrically. The diameter of one can is 10 cm and that of the other is 15 cm.

(a) The height of the large can is 18 cm. Calculate the height of the small one.

(b) Calculate the following ratios, giving your answer in its lowest term.

  (i) Base area of small can : Base area of large can

  (ii) Volume of small can : Volume of large can

(c) The shopkeeper's profit for selling tonic food drink is 96 cents for a small can. Calculate his profit from selling a large can, assuming that there is no reduction in profit for doing so.

33. In the diagram, the points P and Q lie on the sides AB and AC respectively of triangle ABC. The line PQ is parallel to BC. Given that $AB = 8$ cm, $AP = 2$ cm and $AC = 12$ cm,

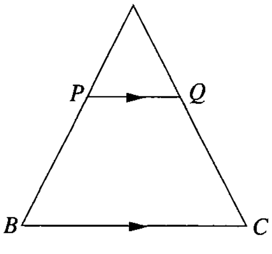


Fig. 7.27

(a) calculate the length of QC,

(b) write down the values of

  (i) $\dfrac{\text{the area of triangle } APQ}{\text{the area of triangle } PQB}$,

  (ii) $\dfrac{\text{the area of triangle } ABC}{\text{the area of triangle } APQ}$.

34. In Fig. 7.28, BC is parallel to PQ. If $AB = 6$ cm, $BP = 3$ cm and the area of $\triangle ABC = 20$ cm^2, find the area of

(a) $\triangle APQ$,

(b) trapezium $BPQC$.

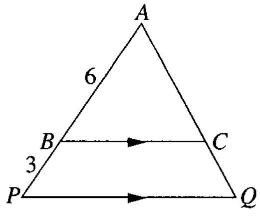


Fig. 7.28

35. In Fig. 7.29, $BP : PC = 2 : 3$ and the area of $\triangle APC = 36$ cm^2. Find the area of $\triangle ABP$. If PQ is parallel to BA, find the area of $\triangle CPQ$.

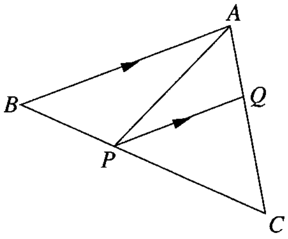


Fig. 7.29

36. In Fig. 7.30, $ABCD$ is a parallelogram. If $BC = 4$ cm, $CP = 6$ cm and the area of $ABCD = 40$ cm^2, find the area of $\triangle ABP$.

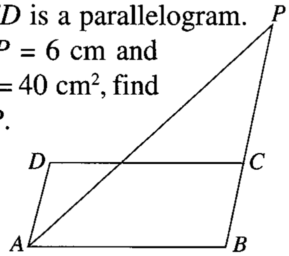


Fig. 7.30

37. In Fig. 7.31, $AP = 3$ cm, $PC = 2$ cm, $BC = 4$ cm and PQ is parallel to CB. Find

(a) the length of PQ,

(b) the ratio of the area of $\triangle ABP$ to the area of $\triangle BPC$,

(c) the ratio of the area of $\triangle APQ$ to the area of $PQBC$.

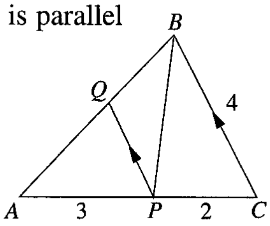


Fig. 7.31

38. In Fig. 7.32, PQ is parallel to BA, P is on BC such that $BP : PC = 2 : 5$ and R is on BA such that $BR : RA = 4 : 3$. If the area of $\triangle BCR = 98$ cm^2, calculate the area of the following.

(a) $\triangle ACR$,

(b) $\triangle CPQ$.

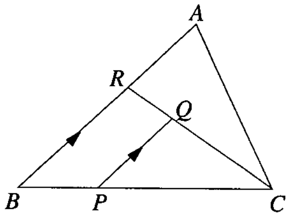
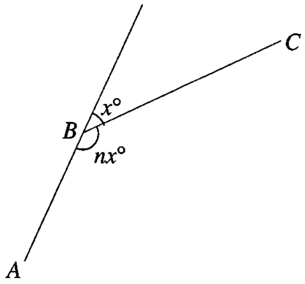


Fig. 7.32

39. A, B, C, ... are some of the vertices of a regular polygon with N sides. Each interior angle of the polygon is n times as large as each exterior angle, n being an integer.



When $n = 1$, $x + x = 180$ (or $x = 90$) and

$N = \frac{360}{90} = 4.$

When $n = 2$, $3x = 180$ (or $x = 60$) and

$N = \frac{360}{60} = 6.$

When $n = 3$, $4x = 180$ (or $x = 45$) and

$N = \frac{360}{45} = 8.$

(a) Complete the table in the answer space, showing the first 5 sets of values of n, x and N.

n	x	N
1	90	$4 = 2(1 + 1)$
2	60	$6 = 2(2 + 1)$
3	45	$8 = 2(3 + 1)$
4		
5		

(b) A regular polygon has each interior angle 49 times as large as each exterior angle. How many sides does this polygon have?

(c) Another regular polygon has 180 sides. What is the size of each of its exterior angle?

40. (a) Find the number of sides of a polygon if the sum of its interior angles is 2 700°.

(b) A polygon has n sides. Three of its exterior angles are 36°, 55°, 65° and the remaining $(n - 3)$ exterior angles are each equal to $8\frac{1}{2}^{\circ}$. Find the value of n.

41. The diagram shows four kitchen containers with circular lids. Each container is a cuboid with a square base. The containers are geometrically similar in all respects. The bases of the FLOUR, SUGAR and TEA containers have sides of lengths 14 cm, 10 cm and 6 cm respectively.

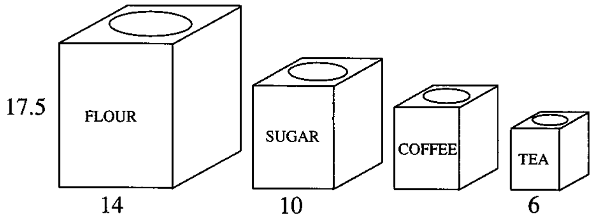


Fig. 7.33

(a) The height of the FLOUR container is 17.5 cm. Calculate the height of the SUGAR container.

(b) The area of the lid of the COFFEE container is $\frac{9}{16}$ of the area of the lid of the SUGAR container. Calculate the length of the side of the base of the COFFEE container.

(c) Calculate the ratio, volume of the TEA container : volume of the SUGAR container. *(C)*

7.6 Coordinate Geometry and Inequalities

Example 1

The coordinates of $\triangle ABC$ in Fig. 7.34(a) are $A(-2, 1)$, $B(4, 4)$ and $C(6, 1)$.

(a) Calculate the length of AB.
(b) Find the equation of AB.
(c) Given that AC is the axis of symmetry of the quadrilateral ABCD, find the coordinates of D.
(d) Find the mid-point of AB.
(e) Given that AOBK is a parallelogram where O is the origin (0, 0), find the coordinates of K.

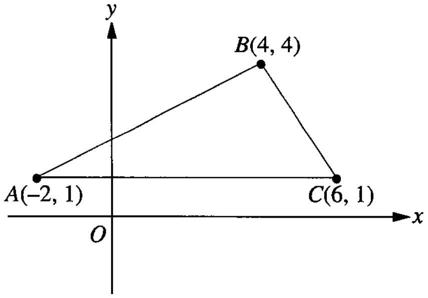


Fig. 7.34(a)

Solution

(a) $AB = \sqrt{[4 - (-2)]^2 + (4 - 1)^2}$

$= \sqrt{6^2 + 3^2}$

$= \sqrt{45}$

$= 6.71$ units.

(b) Gradient of $AB = \frac{4 - 1}{4 - (-2)}$

$= \frac{3}{6}$

$= \frac{1}{2}$

$\therefore$ equation of AB is $y = \frac{1}{2}x + c$.

Since (4, 4) lies on AB, $4 = \frac{1}{2}(4) + c$,

i.e. $c = 2$

$\therefore$ equation of AB is $y = \frac{1}{2}x + 2$.

(c) AC is the axis of symmetry $\Rightarrow BH = HD$ where BH is perpendicular to AC. y coordinate of D is $1 - 3 = -2$.

$\therefore$ point D is $(4, -2)$ (refer to Fig. 7.34(b))

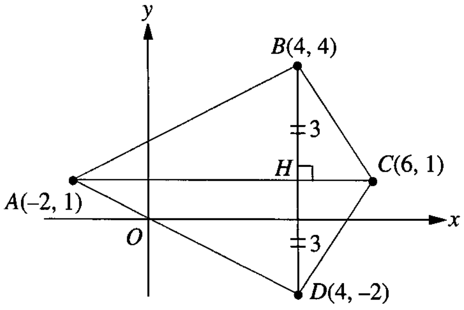


Fig. 7.34(b)

(d) Mid point of $AB = \left(\frac{4 + (-2)}{2}, \frac{4 + 1}{2}\right)$

$= \left(1, 2\frac{1}{2}\right)$

(e) Let the point K be (a, b).

The mid-point of AB = mid-point of OK

We have $\left(1, 2\frac{1}{2}\right) = \left(\frac{0 + a}{2}, \frac{0 + b}{2}\right)$

$\therefore$ $a = 2$ and $b = 5$

$\therefore$ the coordinates of K is $(2, 5)$.

Example 2

Find the integer values of x for which $x - 3 \leqslant 7$ and $4x - 5 > 22$.

Solution

$$\begin{aligned} x - 3 &\leqslant 7 \\ \therefore \quad x &\leqslant 7 + 3 \\ \text{i.e.} \quad x &\leqslant 10 \end{aligned}$$

$$\begin{aligned} 4x - 5 &> 22 \\ \therefore \quad 4x &> 22 + 5 \\ \text{i.e.} \quad 4x &> 27 \\ x &> \frac{27}{4} \\ \text{i.e.} \quad x &> 6\frac{3}{4} \end{aligned}$$

The integer values satisfying $x \leqslant 10$ and $x > 6\frac{3}{4}$ are 7, 8, 9 and 10.

Example 3

Given that x and y are integers such that $-8 \leqslant x \leqslant 4$ and $-2 \leqslant y \leqslant 3$. Calculate

(a) the greatest value of $x - y$,

(b) the least value of $2x + y^2$,

(c) the greatest value of xy,

(d) the least value of $\frac{x}{y}$.

Solution

(a) the greatest value of $x - y = 4 - (-2) = 6$

(b) the least value of $2x + y^2 = 2(-8) + 0^2 = -16$

(c) the greatest value of $xy = (-8) \times (-2) = 16$

(d) the least value of $\frac{x}{y} = \frac{-8}{1} = -8$

Revision Exercise 7.6

1. Given that $-2 \leqslant x \leqslant 7$, write down
- **(a)** the largest integer value of x,
- **(b)** the smallest integer value of x,
- **(c)** the largest prime number in the given range.

2. Given that $x \geqslant 7\frac{3}{4}$, write down the smallest possible value of x if
- **(a)** x is an integer,
- **(b)** x is a prime number,
- **(c)** x is a natural number.

3. List the integer value of x for which $-5 < 12 - 3x < -1$.

4. List the integer value of a for which $2a + 1 < 14 \leqslant 5a - 2$.

5. List the integer value of p for which $2p - 3 < 21 \leqslant 4p - 3$.

6. List the prime numbers of x for which $x - 1 \leqslant 25 \leqslant 3x - 1$.

7. Find the integer x for which $9 < 2x - 1 < 13$.

8. List the possible values of (x, y) for which x and y are positive integers such that $x + y = 3$.

9. List the possible values of (x, y) for which x and y are positive integers such that $y + 2x = 5$.

10. Given that $1 \leqslant x \leqslant 5$ and $-1 \leqslant y \leqslant 7$, find
- **(a)** the greatest possible value of $2x - y$,

(b) the least possible value of $2xy$,

(c) the greatest possible value of $\frac{y}{x}$.

11. Solve the following inequalities:

(a) $9x - 7 \leqslant 12$ **(b)** $7 - 2x > 2$

(c) $3 + 5x \geqslant 32$ **(d)** $3x - 4 \geqslant \frac{1}{3}x - 2$

12. Given that $-5 \leqslant x \leqslant 3$ and $-8 \leqslant y \leqslant 6$ where both x and y are integers, calculate

(a) the greatest value of $2x + 3y$,

(b) the least value of $3xy$,

(c) the greatest value of $x^2 + y^2$,

(d) the least value of $x^2 - y^2$.

13. Find the odd integer values of x for which $x - 5 \leqslant 7$ and $3x - 2 \geqslant 11$.

14. **(a)** Find the gradient of the straight line passing through the points (1, 2) and (9, 10).

(b) Find the equation of the straight line passing through the point (2, 3) and having a gradient of 5.

15. Given that the gradient of the line joining (5, k) and (4, –3) is 2, find k. Find the equation of the straight line having a gradient of $-\frac{1}{2}$ and passing through the point (1, –5).

16. The straight line $3x + 4y = 24$ cuts the axes at the points P and Q. Calculate the length of PQ.

17. The line $2x + 3y = 18$ intersects the x-axis at P and the y-axis at Q.

(a) Find the coordinates of M, the mid-point of PQ.

(b) Find the length of PQ.

(c) Find the equation of the straight line which passes through M and has a gradient of 3.

18. The mid-point of $A(3, 2)$ and $B(-1, 10)$ is $M(h, k)$. Find the value of h and of k. Find the equation of the straight line which passes through M and is parallel to OA where O is the origin.

19. The line $\frac{x}{4} + \frac{y}{6} = 1$ cuts the x-axis at H and the y-axis at K. Find the coordinates of the mid-point of HK.

20. Given that $3 \leqslant x \leqslant 8$ and $-2 \leqslant y \leqslant 5$, find

(a) the largest possible value of $3x - y$,

(b) the smallest possible value of $x^2 + y^2$,

(c) the smallest possible value of $\frac{x^2}{y^2}$.

21. It is given that $-1 \leqslant x \leqslant 6$, $2 \leqslant y \leqslant 13$ and $0.1 \leqslant z \leqslant 2$. Calculate

(a) the smallest possible value of $x + y + z$,

(b) the largest possible value of $y - 2x^2$,

(c) the largest possible value of $\frac{x}{z}$,

(d) the least possible value of $x^2 - \frac{y}{z}$.

22. The sides of a rectangle are given as x cm and y cm where $8.5 \leqslant x \leqslant 9.5$ and $5.5 \leqslant y \leqslant 6.5$. Calculate

(a) the smallest possible value of the perimeter of the rectangle,

(b) the largest possible value of the area of the rectangle.

23. Find the integer values of x for which $12 < 3x - 1 < 27$.

24. Find the integer values of x which satisfy all the conditions below.

$$3x + 15 \geqslant 0, x \neq -4, x < -1$$

25. **(a)** Find the smallest prime number x such that $5x - 1 > 58$.

(b) Find the largest integer k for which $4k - 3 < 24$.

(c) Solve the inequality $2x^2 \leqslant 32$.

26. **(a)** The point (t, $5t - 7$) lies on the line $y = 2x - 8$. Calculate the value of t.

(b) A line passes through the point (1, 7) and is parallel to $2x + y = 12$. Find the equation of the line.

27. The coordinates of $\triangle ABC$ are $A(-2, -2)$, $B(7, -2)$ and $C(10, 4)$. Calculate

(a) the gradient of AC,

(b) the equation of AC,

(c) the coordinates of the point where AC cuts the x-axis,

(d) the area of $\triangle ABC$,

(e) the length of AC, giving your answer correct to 2 decimal places,

(f) the length of the perpendicular from B to AC.

28. Find the equation of the line parallel to $\frac{x}{2} + \frac{y}{3} = 1$ and passing through the point of intersection of the lines $5x + 3y = 2$ and $x - y = 6$.

29. The coordinates of P, Q and R are $P(2, -5)$, $Q(3, -2)$ and $R(5, k)$ respectively.

(a) If P, Q and R lie on a straight line, find the value of k.

(b) With this value of k, find the length of PR.

(c) Find the equation of the line passing through the point Q and parallel to the line $2x - 3y = 14$.

30. $PQRS$ is a rhombus with points $P(-1, 1)$, $Q(0, 8)$, $R(5, 5)$ and $S(h, k)$ respectively. Given that X is the point of intersection of the diagonals PR and QS, calculate

(a) the values of h and k,

(b) the lengths of diagonals PR and QS in surd form,

(c) the area of the rhombus $PQRS$.

31. Given that the points $(3, -7)$, $(-1, k)$ and $(6, -1)$ are collinear, find the value of k.

32. Find the equation of the line joining the point $(3, 7)$ to the point of intersection of the lines $x + y = 4$ and $x - y + 3 = 0$.

33. The vertices of a triangle ABC are $A(-3, 7)$, $B(11, 3)$ and $C(2, -2)$. By finding the lengths of AB, AC and BC, prove that $\triangle ABC$ is an isosceles right-angled triangle.

34. In the figure, $A(0, -3)$, $B(4, -2)$ and $C(2, 6)$ are the vertices of $\triangle ABC$.

(a) Calculate the lengths of AB, BC and AC and hence show that $\triangle ABC$ is a right-angled triangle.

(b) Find the area of $\triangle ABC$.

(c) Calculate the length of the perpendicular from B to AC.

(d) If a circle is to be drawn so that it will pass through points A, B and C, find the coordinates of the centre of this circle.

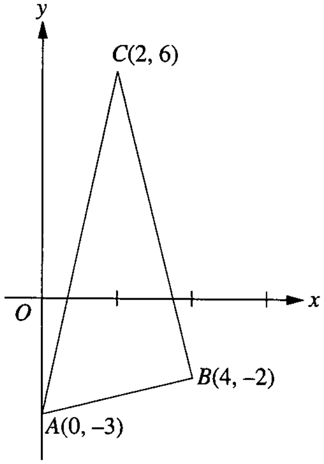


Fig. 7.35

35. Given that the coordinates of the points A, B and C are $(1, -4)$, $(-3, 2)$ and $(7, 9)$ respectively and M is the mid-point of BC,

(a) write down the coordinates of M,

(b) find the equation of the line AM.

36. The coordinates of the points A and B are $(1 - k, k)$ and $(2, 1 + k)$ respectively. If $AB^2 = 26$ units, find the possible values of k.

37. In the figure, P, Q, R and S are the mid-points of AB, BC, CD and AD respectively where the coordinates of the quadrilateral $ABCD$ are $A(2, -10)$, $B(10, -2)$, $C(0, 8)$ and $D(-6, 6)$.

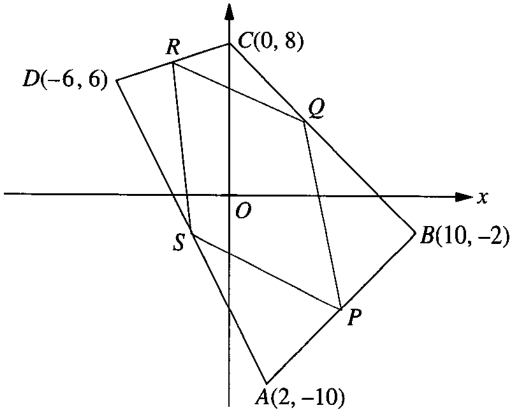


Fig. 7.36

(a) Find the coordinates of P, Q, R and S.

(b) Prove that $PQRS$ is a parallelogram.

38. The coordinates of triangle ABC in Fig. 7.37 are $A(-7, 0)$, $B(0, t)$ and $C(-4, 9)$. Given that DC is parallel to the x-axis and $DC = CB = 5$ units, find

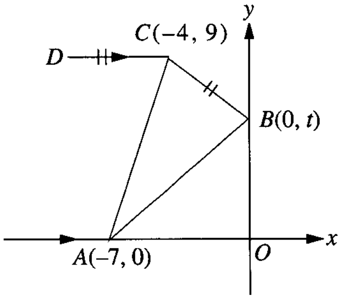


Fig. 7.37

- **(a)** the coordinates of D,
- **(b)** the value of t,
- **(c)** the length of AB,
- **(d)** the equation of AC,
- **(e)** the area of $\triangle ABC$.

39. In Fig. 7.38, it is given that the equation of the line $ABCD$ is $2y = 3x - 14$, the coordinates of B and D are $(2, h)$ and $(k, 8)$.

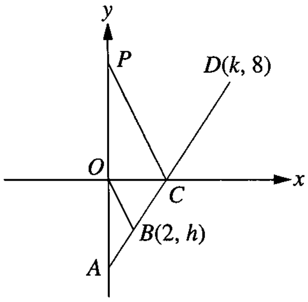


Fig. 7.38

- **(a)** Find the values of h and k.
- **(b)** Find the equation of CP given that OB is parallel to PC.
- **(c)** Calculate the area of $\triangle OCP$.

40. In the diagram, OAB is a straight line such that $OB = 3OA$ and the coordinates of A are $(-4, 3)$. Calculate

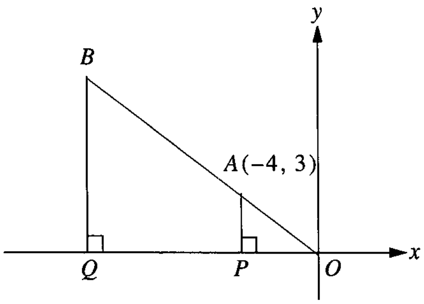


Fig. 7.39

- **(a)** the length of OB,
- **(b)** the coordinates of B,
- **(c)** the area of $ABQP$,
- **(d)** the length of AQ.

41. In the diagram, $OA//BC$ and the coordinates of the points A, P and C are $A(4, 6)$, $P(10, -4)$ and $C(15, t)$. Calculate

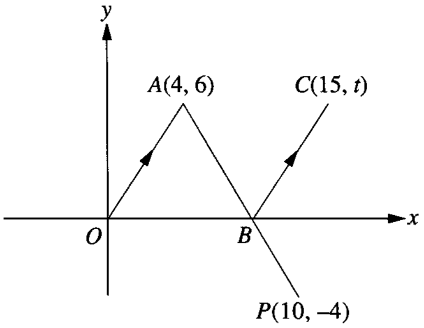


Fig. 7.40

- **(a)** the coordinates of B,
- **(b)** the equation of BC,
- **(c)** the value of t,
- **(d)** the area of $\triangle OAB$.

42. In the diagram, A is the point $(15, 0)$, $OA = OP$, $OB = BQ$ and AB has a length of 17 units. Calculate

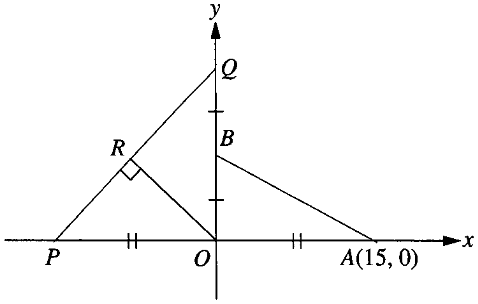


Fig. 7.41

- **(a)** the coordinates of B, Q and P,
- **(b)** the equation of the line joining PQ,
- **(c)** the length of PQ,
- **(d)** the length of OR where OR is perpendicular to PQ.

7.7 Graphs and Kinematic Problems

Example 1

Fig. 7.42 shows part of the curve of $y = \frac{6}{x^2}$.

Write down the equation of the line of symmetry of the curve.

The point $\left(h, 1\frac{1}{2}\right)$ *lies on the curve. Find the possible values of h.*

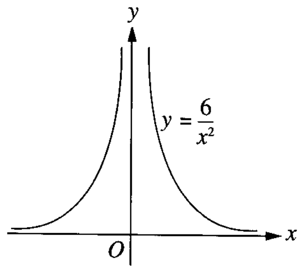


Fig. 7.42

Solution

$x = 0$ is the equation of the line of symmetry.

$\left(h, 1\frac{1}{2}\right)$ lies on the curve.

i.e. $1\frac{1}{2} = \frac{6}{h^2} \Rightarrow h^2 = \frac{6}{1\frac{1}{2}} = 4$

$\therefore \quad h = \pm 2$

Example 2

Fig. 7.43 is the speed-time graph of an M.R.T. train. Given that the distance travelled in the first 50 seconds is 850 m, calculate

(a) the maximum speed, V m/s,
(b) the acceleration of the train during the first 15 seconds,
(c) the further distance travelled before the train comes to a stop.

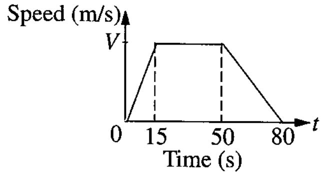


Fig. 7.43

Solution

(a) The distance moved is given by the area under the speed-time graph

$$850 = \frac{1}{2}(15)V + (50-15)V$$
$$850 = 7.5V + 35V$$
$$\therefore \quad V = 850 \div 42.5 = 20$$

(b) The acceleration of the train during the first 15 seconds is given by the gradient of the graph for that period.

$$\therefore \quad \text{acceleration} = \frac{20}{15} = 1\frac{1}{3} \text{ m/s}^2.$$

(c) The train slows down for 30 seconds before it comes to rest.

The further distance moved = $\frac{1}{2} \times 20 \times 30 = 300$ m.

Example 3

The following is a table of values for the graph of $y = 2(x + 1)(x - 3)$.

x	-2	-1	0	1	2	3	4
y	h	0	-6	-8	k	0	10

Calculate the value of h and of k.

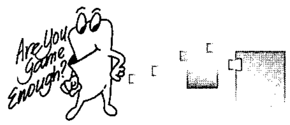
Using a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 4 units on the y-axis, plot the graph of $y = 2(x + 1)(x - 3)$ for $-2 \leqslant x \leqslant 4$.

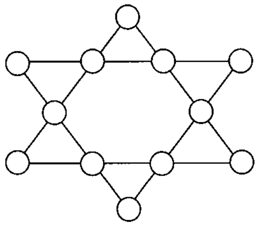
(a) Use your graph to solve the equation $2(x + 1)(x - 3) = 4$.

(b) By drawing a tangent, find the gradient of the graph at the point $x = 2$.

(c) Using the same scale and axes, draw the graph of $y = 2x - 4$ and use it to solve the equation $2(x + 1)(x - 3) = 2x - 4$.

(d) State the range of values of x for which $2(x + 1)(x - 3) \leqslant 2x - 4$.





Fill the numbers 1, 2, 3, ..., 12 in the circles so that the sum of the numbers on each side will be equal to 26.

Solution

When $x = -2$, $y = 2(-2 + 1)(-2 - 3) = 10$

When $x = 2$, $y = 2(2 + 1)(2 - 3) = -6$,

$\therefore$ $h = 10$ and $k = -6$

(a) The graph of $y = 2(x + 1)(x - 3)$ is plotted in Fig. 7.44. From the graph, when $y = 4$, $x = -1.45$ or 3.45. Therefore, the solution of $2(x + 1)(x - 3) = 4$ is -1.45 or 3.45.

(b) A tangent is drawn at the point $x = 2$. The gradient of the tangent is approximately $\frac{8}{2}$ or 4. Therefore, the gradient of the curve at 2 is 4.

(c) The table of values for $y = 2x - 4$ is shown below.

x	0	2	4
y	-4	0	4

The line $y = 2x - 4$ cuts the curve at the points where $x = -0.3$ and $x = 3.3$.

$\therefore$ the solution of the equation $2(x + 1)(x - 3) = 2x - 4$ is $x = -0.3$ or $x = 3.3$.

(d) The range of values of x for which $2(x+1)(x-3) \leqslant 2x-4$ is $-0.3 \leqslant x \leqslant 3.3$.

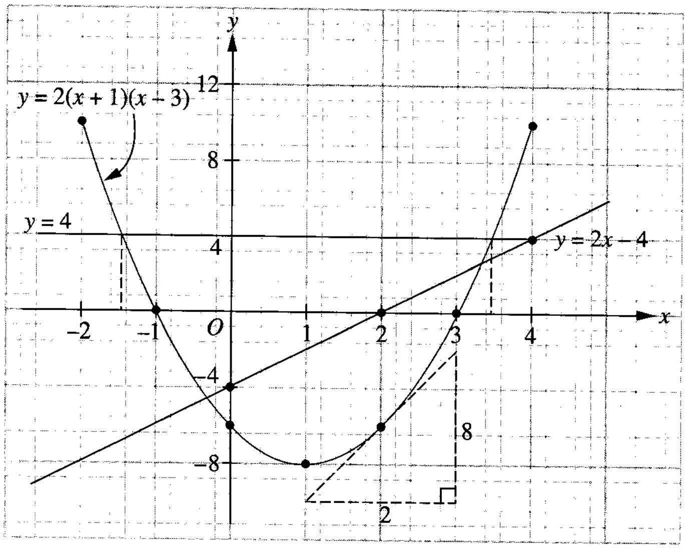


Fig. 7.44

Revision Exercise 7.7

1. In Fig. 7.45, the curve $y = (x+1)(2-x)$ cuts the x-axis at points A and C and the y-axis at B.
   (a) Find the coordinates of the points A, B and C.
   (b) Find the equation of the line of symmetry of the curve.

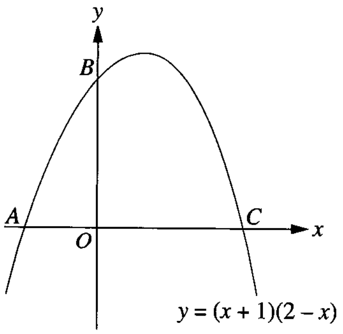


Fig. 7.45

2. On separate diagrams, sketch the graph of each of the following functions:
   (a) $y = x^2 + 2$
   (b) $x + y = 2$
   (c) $y = x^2$
   (d) $y = 4 - x^2$
   (e) $y = x^3 + 2$
   (f) $y = \frac{2}{x}$
   (g) $y = \frac{3}{x^2}$
   (h) $y = 3 - x^3$

3. The curve $y = x^2 + kx - 5$ cuts the y-axis at A and passes through the point (1, −2).
   (a) Find the coordinates of A.
   (b) Calculate the value of k.

4. Fig. 7.46 shows part of the graph of $y = x^2 + kx + p$. Find the value of k and of p.

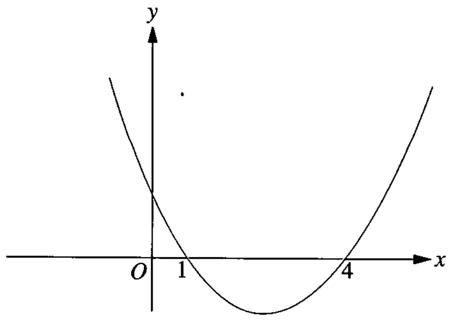


Fig. 7.46

5. Fig. 7.47 shows part of the graph $y = \frac{1}{x}$.

(a) Write down the equations of the lines of symmetry of the graph.

(b) Describe the single transformation which will map the part of the curve P_1 onto P_2.

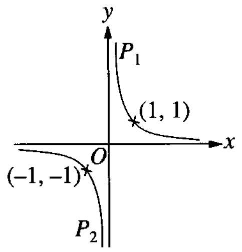


Fig. 7.47

6. Fig. 7.48 shows the speed-time graph of a particle over a period of 50 seconds. If the total distance moved is 300 m, find

(a) the value of v,

(b) the acceleration of the particle,

(c) the distance moved in the first 10 seconds.

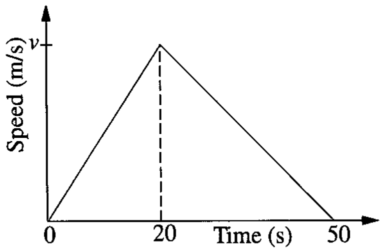


Fig. 7.48

7. Fig. 7.49 shows the speed-time graph of a body over a period of 90 seconds. Given that the total distance moved is 1.84 km, find

(a) the value of v,

(b) the acceleration during the first 10 seconds,

(c) the distance moved during the first 10 seconds of its motion.

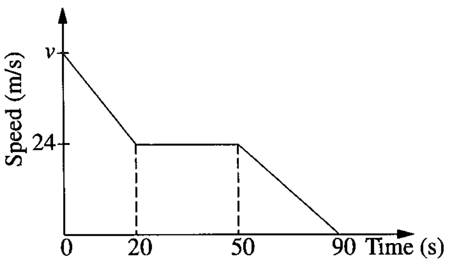


Fig. 7.49

8. Fig. 7.50 shows the speed-time graph of a body moving in a straight line.

(a) Calculate the acceleration of the body during the first 15 seconds.

(b) Calculate the distance moved by the body in the first 40 seconds of its motion.

(c) Given that the body decelerates at 1.25 m/s², find the value of t.

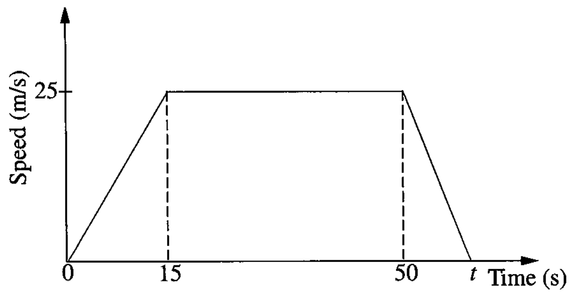


Fig. 7.50

9. Fig. 7.51 shows the speed-time graph of a particle which accelerates at 4 m/s² for 5 seconds, thereafter maintaining the maximum speed for a further 5 seconds before it is brought to rest at a rate of 2 m/s². Calculate

(a) the speed of the particle at the end of 5 seconds,

(b) the value of t,

(c) the average speed of the particle during the whole journey.

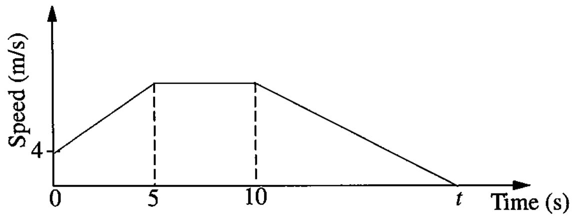


Fig. 7.51

10. The diagram shows the speed-time graph of a moving particle. The graph consists of 3 line segments BC, CD, DE and a quadrant with centre A.

(a) Write down the acceleration of the particle at time $t = 10$.

(b) Calculate the acceleration of the particle during the last 5 seconds.

(c) Calculate the total distance covered in the first 20 seconds of the journey. (Take $\pi = 3.14$)

(d) Calculate the average speed for the whole journey.

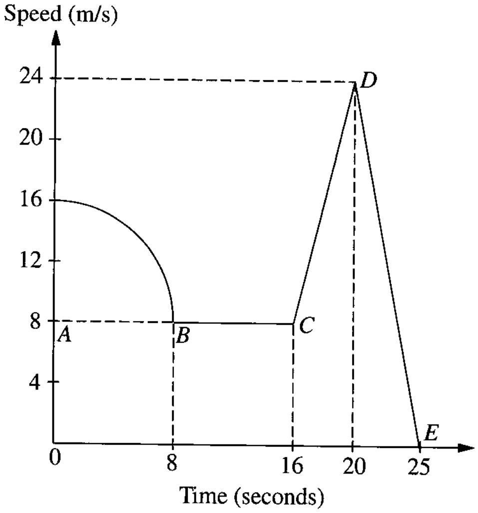


Fig. 7.52

11. The diagram (Fig. 7.53(a)) shows the speed-time graph for the first 20 seconds of a cyclist's journey.

(a) Calculate the distance travelled in the first 10 seconds.

(b) Calculate the total distance travelled in 20 seconds.

(c) On the axes in Fig. 7.53(b), draw the distance-time graph for this part of the journey. (*C*)

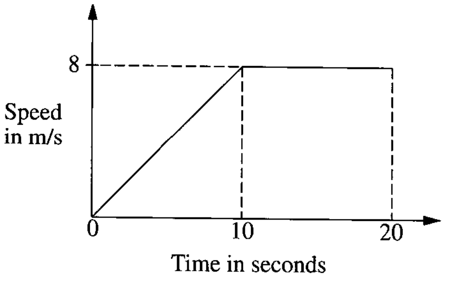


Fig. 7.53(a)

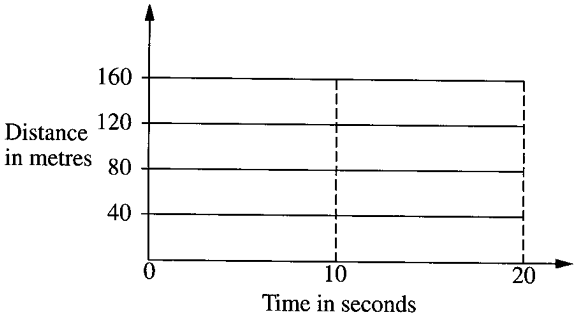


Fig. 7.53(b)

12. Fig. 7.54(a) shows the speed-time graphs of two objects A and B for the first 10 seconds of their motion.

Object A travels at a constant speed of 12 m/s throughout the 10 seconds.

Object B starts from rest, attains a speed of 18 m/s after 4 seconds and then travels at constant speed.

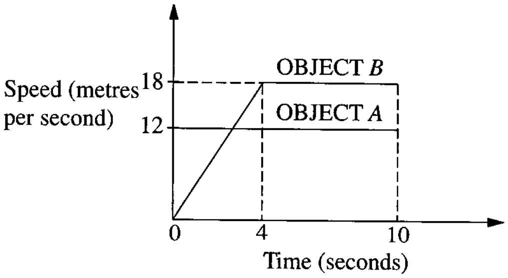


Fig. 7.54(a)

(a) Calculate

(i) the distance travelled by object B during the first 4 seconds of its motion,

(ii) the average speed of object B for the first 10 seconds of its motion,

(iii) how long it takes before object A is travelling at the same speed as object B,

(iv) how long it takes before both objects have travelled the same distance.

(b) After 10 seconds, both objects slow down at the same rate until they come to rest. Given that object A comes to rest after a further 6 seconds, calculate

(i) the deceleration of object A,
(ii) how long it takes from the start of its motion before object B comes to rest.

(c) In Fig. 7.54(b), the straight line OP and the curve PQ form the distance-time graph of the object A for the whole of its motion.
(i) State the values of T_1 and T_2
(ii) What does the gradient of OP represent? (C)

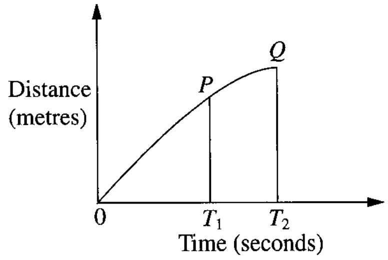


Fig. 7.54(b)

13. The variables x and y are connected by the equation $y = x^2 + 3x - 3$.
Some corresponding values of x and y are given in the following table.

x	–5	–4	–3	–2	–1	0	1	2
y	7	1	–3	–5	–5	–3	1	7

(a) Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 2 units on the y-axis, draw the graph of $y = x^2 + 3x - 3$ for $-5 \leqslant x \leqslant 2$.
(b) Use your graph to find the values of x for which
(i) $x^2 + 3x = 8$, **(ii)** $x^2 + 3x - 3 \leqslant x$, **(iii)** $x^2 + 4x = 2$.
(c) By drawing a suitable tangent to your curve, find the coordinates of the point at which the gradient of the tangent is equal to 1.

14. Copy and complete the following table for $y = 2x^2 - 15x + 50$.

x	0	1	2	3	4	5	6
y	50		28		22		32

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to 10 units on the y-axis, plot the graph of $y = 2x^2 - 15x + 50$ for $0 \leqslant x \leqslant 6$.
Use the graph to find the least value of y and the corresponding value of x.
By drawing a suitable straight line, find the gradient of the graph at the point $x = 5$.

15. The table below shows the speed of a car over a period of 8 seconds.

Time (s)	0	1	2	3	4	5	6	7	8
Speed (m/s)	0	4.5	7.8	10.1	11.6	12.4	13	13.4	14

After the 8th second, the car moves with a constant speed of 14.0 m/s. Using a scale of 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 2 m/s on the vertical axis, plot the graph of the motion of the car.

A sports car starts from rest at $t = 2$ and moves with an acceleration of 3 m/s^2. Plot the graph of the motion of the sports car for $2 \leqslant t \leqslant 8$ and find, graphically,

(a) the time at which the two cars have the same speed,
(b) the acceleration of the car at the time $t = 5$.

★**16.** Copy and complete the following table of values for $y = 25 + 4x - 3x^2$.

x	–3	–2	–1	0	1	2	3	4	5
y	–14	5		25		21		–7	–30

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis, draw the graph of $y = 25 + 4x - 3x^2$ for $-3 \leqslant x \leqslant 5$.

Use your graph to estimate

(a) the greatest value of $y = 25 + 4x - 3x^2$,
(b) the solution of $25 + 4x - 3x^2 = 0$,
(c) the range of values of x for which $25 + 4x - 3x^2 \geqslant 10$,
(d) the gradient of the graph at the point where $x = 3$ by drawing a tangent.

17. Given that $y = x^2(x - 4)$, copy and complete the following table.

x	–2	–1	0	1	2	3	4	5
y	–24		0		–8		0	

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis, draw the graph of $y = x^2(x - 4)$ for $-2 \leqslant x \leqslant 5$.

(a) Use your graph to estimate
  (i) the solution of the equation $x^2(x - 4) = -10$,
  (ii) the range of values of x for which $x^2(x - 4) \geqslant -10$.
(b) By drawing a tangent, find the gradient of the graph at the point $x = 3$.
(c) By drawing a suitable straight line on the graph, solve the equation $x^2(x - 4) = 10 - 5x$.

18. The following is a table of values for the graph of $y = 3x - 2 + \frac{10}{x + 3}$.

x	–2.25	–2	–1.5	–1	–0.5	0	0.5	1
y	4.58	2	h	0	0.5	k	2.36	3.5

(a) Calculate the value of h and of k correct to 2 decimal places.
(b) Using a scale of 4 cm to 1 unit on the x-axis and a scale of 2 cm to 1 unit on the y-axis, draw the graph of $y = 3x - 2 + \frac{10}{x + 3}$ for $-2.25 \leqslant x \leqslant 1$.
(c) Use the graph to find
  (i) the values of x for which $3x + \frac{10}{x + 3} = 4$,
  (ii) the gradient of the curve at the point $x = 0$,
  (iii) the range of values of x for which $3x - 2 + \frac{10}{x + 3} \geqslant 2.8$.

19. The following is an incomplete table of values for the graph of $y = 3 + \frac{12}{x} - x$. Copy and complete the table.

x	1	1.5	2	2.5	3	4	5	6
y	14	9.5	7			2		–1

(a) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm for 1 unit on the y-axis, draw the graph of $y = 3 + \frac{12}{x} - x$ for $1 \leqslant x \leqslant 6$.

(b) By drawing a suitable straight line, find the gradient of the curve at the point $x = 2$.

(c) Draw the graph of $y = 18 - 2x$ on the same axes. Write down and simplify the equation in x which determines the points of intersection of the two graphs.

(d) By drawing another straight line, determine the values of x in this range for which $\frac{12}{x} = 4x + 3$.

20. **Answer the whole of this question on a sheet of graph paper.** The volume of an **open** rectangular box, made of thin metal, is 7 500 cm^3.

The lengths of the edges of the base of the box are 30 cm and x cm.

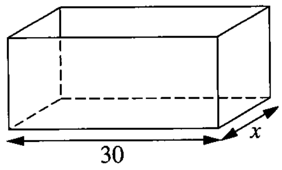


Fig. 7.55

(a) Find, in terms of x, an expression for

(i) the area of the base of the box, **(ii)** the height of the box.

(b) The total external area, of the base and the four sides, is A cm^2. Show that

$$A = 500 + 30x + \frac{15\ 000}{x}.$$

(c) The table below shows some values of x and the corresponding values of A. The values of A are given correct to the nearest integer, where appropriate.

x	10	12.5	15	20	25	30	35	40	45
A	2 300	2 075	1 950	1 850	1 850	1 900	1 979	2 075	2 183

Using a scale of 2 cm to 5 units, draw a horizontal x axis for $10 \leqslant x \leqslant 45$.

Using a scale of 2 cm to 100 units, draw a vertical A axis for $1800 \leqslant A \leqslant 2\ 300$.

Plot the points represented by the values in the table and join them with a smooth curve.

(d) Use your graph to find

(i) the range of values of x for which $A \leqslant 2\ 000$,

(ii) the minimum value of A,

(iii) the height of the box for which the least amount of metal is used. *(C)*

21. **Answer the whole of this question on a sheet of graph paper.** In the diagram, $ABCD$ represents a rectangular plot of area 40 square metres. $PQRS$ represents the rectangular base of a shed positioned symmetrically on the plot. The distances, in metres, of the sides of the shed from the edges of the plot are as shown.

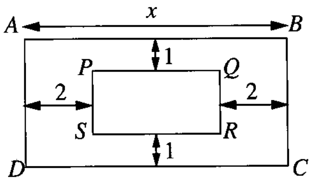


Fig. 7.56

(a) Taking the length of AB to be x metres, write down expressions, in terms of x, for the lengths

(i) BC, **(ii)** PQ, **(iii)** QR.

(b) Show that the area, y square metres, of $PQRS$ is given by

$$y = 48 - 2x - \frac{160}{x}.$$

(c) The table below shows some values of x and the corresponding values of y. The values of y are given correct to two decimal places where appropriate.

x (metres)	6	7	8	9	10	11	12	13
y (square metres)	9.33	11.14	12	12.22	12	11.45	10.67	9.69

Using a scale of 2 cm to 1 unit on each axis, draw x- and y-axes for $6 \leqslant x \leqslant 13$ and $9 \leqslant y \leqslant 13$.

On your axes plot the points given in the table and join them with a smooth curve.

(d) By using your graph, find

(i) the smaller value of x for which the area of $PQRS$ is 10 m^2,

(ii) the value of x for which the area of $PQRS$ is maximum,

(iii) the dimensions of the base of the shed for which it has maximum area. *(C)*

22. **(a)** Given that $y = 2^{2-x}$, copy and complete the following table.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
y			4	2.8			1		

(b) Using a scale of 4 cm to represent 1 unit on the x-axis, and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 2^{2-x}$ for $-1 \leqslant x \leqslant 3$.

(c) Use your graph to solve the equation $2^{2-x} = 5.6$.

(d) Find the gradient of the curve $y = 2^{2-x}$ at the point where $x = 1.5$ by drawing a suitable tangent.

(e) Solve the equation $2^{2-x} = 3 + 2x$.

23. The following table gives corresponding values of x and y which are connected by the equation $y = 8(0.6)^x$.

x	0	0.5	1	1.5	2	3	4	5	6	7
y	8	6.2	4.8	h	2.9	k	1.0	0.6	0.4	0.2

(a) Calculate the value of h and of k correct to 1 decimal place.

(b) Using a scale of 2 cm for 1 unit on both axes, plot the graph of $y = 8(0.6)^x$ for $0 \leqslant x \leqslant 7$.

(c) By drawing a tangent, find the gradient of the curve at the point $x = 1.5$.

(d) Use your graph to find

(i) the value of y when $x = 2.5$, **(ii)** the solution of the equation $(0.6)^x = 0.5$,

(iii) the solution of the equation $8(0.6)^x = x$,

(iv) the range of values of x for which $8(0.6)^x > x + 3$.

24. **Answer the whole of this question on a sheet of graph paper.**

The number of bacteria in a colony doubles every hour. The colony starts with 50 bacteria. The table below shows the number of bacteria in the colony after time t.

Time in hours (t)	0	1	2	3	4	5	6	7
Number of bacteria (n)	50	100	200	400	800	1 600	3 200	6 400

(a) Using a horizontal scale of 2 cm to represent 1 hour and vertical scale of 2 cm to represent 1 000 bacteria, draw the graph of n against t.

(b) Use your graph to find the value of n when $t = 6.5$.

(c) **(i)** By drawing a tangent, find the gradient of the graph when $t = 5.5$.

(ii) State briefly what this gradient represents.

(d) The number of bacteria in another colony is given by the equation

$$n = 4\ 000 - 500t.$$

(i) On the same axes, draw a graph to represent the number of bacteria in this colony.

(ii) Find the value of t when the numbers in the colonies are equal.

(e) Given that the equation of the first graph is $n = k2^t$, find the value of k. *(C)*

25. Answer the whole of this question on a sheet of graph paper.

(a) Table I below gives some values of x and the corresponding values of y, correct to two decimal places, where $y = x\,(1 + x)(3 - x)$.

x	0	0.5	1	1.5	2	2.5	3
y	0	1.88	4	5.63	6	4.37	p

Table I

(i) Find the value of p.

(ii) Using a scale of 4 cm to represent 1 unit, draw a horizontal x-axis for $0 \leqslant x \leqslant 3$.
Using a scale of 2 cm to represent 1 unit, draw a vertical y-axis for $0 \leqslant y \leqslant 8$.

On your axes plot the points given in the table and join them with a smooth curve.

(iii) Use your graph to find the greatest value of $x(1 + x)(3 - x)$ in the interval $0 \leqslant x \leqslant 3$.

(b) Table II shows some corresponding values of x and y where $y = 2^x$.

x	0	0.5	1	1.5	2	2.5	3
y	1	1.41	2	q	4	5.66	8

Table II

(i) Find the value of q, correct to 2 decimal places.

(ii) On the axes used in part **(a)(ii)**, draw the graph of $y = 2^x$.

(c) From your graphs, find the values of x in the interval $0 \leqslant x \leqslant 3$, for which

(i) $x(1 + x)(3 - x) = 3$, **(ii)** $x(1 + x)(3 - x) > 2^x$. (*C*)

26. Answer the whole of this question on a sheet of graph paper.

When x copies of a book are produced, the cost, \$$y$, of each copy is given by the formula

$$y = 10 + \frac{2\,400}{x}.$$

(a) The table below gives some values of x and the corresponding values of y.

x	100	200	300	400	600	800	1 200
y	34	22	18	16	14	13	p

(i) Find the value of p.

(ii) Using a scale of 2 cm to represent 200 books, draw a horizontal x-axis for $0 \leqslant x \leqslant 1\,200$. Using a scale of 2 cm to represent \$5, draw a vertical y-axis for $0 \leqslant y \leqslant 40$.

On your axes plot the points given in the table and join them with a smooth curve.

(b) Use your graph to estimate the number of books to be printed if the cost of producing each book is \$15.

(c) **(i)** By drawing a tangent, find the gradient of the curve at the point where $x = 300$.

(ii) Describe briefly what this gradient represents.

(d) In order to sell x books, the selling price of each book must be $\left(25 - \frac{x}{60}\right)$ dollars.

(i) On the axes used in part **(a)**, draw the graph of

$$y = 25 - \frac{x}{60}, \quad \text{for values of } x \text{ from 0 to 1 200.}$$

(ii) Use your graphs to find the range of the number of books that should be printed if no loss is to be made, assuming that all of the books will be sold. (*C*)

27. Answer the whole of this question on a sheet of graph paper.

A particle was projected directly up a slope. Its distance, d metres, from the bottom of the slope, t seconds after it was projected, is given in the table below.

t	0	0.5	1	2	3	4	5	6	6.5	7
d	0	1.58	3.02	5.48	7.38	8.72	9.5	9.72	9.72	9.72

(a) Using a horizontal scale of 2 cm to represent 1 second, and a vertical scale of 1 cm to represent 1 metre, draw a graph of d against t.

(b) Use your graph to find the distance of the particle from the bottom of the slope when $t = 2.5$.

(c) What happened to the particle after approximately 6 seconds?

(d) **(i)** By drawing a tangent, find the gradient of your curve when $t = 4$.

(ii) State briefly what this gradient represents.

(e) Calculate the average speed of the particle during the first 6 seconds.

(f) At the instant the particle was projected, another particle was projected down the slope from a point 10 metres from the bottom.

This particle moved directly down the slope at a constant speed of 2 m/s.

(i) On the same axes, draw the graph which shows the position of this particle.

(ii) Use your graph to find when the particles passed each other. *(C)*

28. Answer the whole of this question on a sheet of graph paper.

The variables x and y are connected by the equation

$$y = \frac{x^2}{6} + \frac{12}{x} - 6.$$

The table below shows some corresponding values of x and y. The values of y are given correct to one decimal place where appropriate.

x	1	1.5	2	3	4	5	6	7
y	6.2	2.4	0.7	–0.5	–0.3	0.6	2	k

(a) Calculate the value of k, correct to one decimal place.

(b) Using a scale of 2 cm to 1 unit on each axis, draw a horizontal x-axis for $0 \leqslant x \leqslant 8$ and a vertical y-axis for $-1 \leqslant y \leqslant 7$. On your axes, plot the points given in the table and join them with a smooth curve.

(c) By drawing a tangent, find the gradient of the curve at the point (1.5, 2.4).

(d) Showing your method clearly, use your graph to find the values of x in the range $1 \leqslant x \leqslant 7$ for which

$$\frac{x^2}{6} + \frac{12}{x} = 7.$$

(e) **(i)** On the same axes, draw the graph of the straight line $y = \frac{x}{4}$.

(ii) Using your graphs, find the values of x in the range $1 \leqslant x \leqslant 7$ for which

$$\frac{x^2}{6} + \frac{12}{x} - 6 \leqslant \frac{x}{4}.$$ *(C)*

7.8 Angle Properties of Circles

Example 1

In Fig. 7.57, AB is a diameter of the circle, centre O, $P\hat{A}B = 32°$ and $A\hat{Q}P = 28°$. Calculate the following angles.

(a) $P\hat{O}B$, *(b)* $B\hat{A}Q$, *(c)* $R\hat{B}Q$.

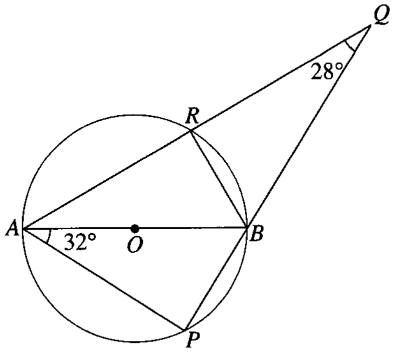


Fig. 7.57

Solution

(a) $P\hat{O}B = 2 \times 32°$ ($\measuredangle$ at centre = 2 $\measuredangle$ at $\odot^{ce}$)
$= 64°$

(b) $A\hat{P}B = 90°$ (rt. $\measuredangle$ in semicircle)
$\therefore$ $B\hat{A}Q = 180° - 90° - 28° - 32°$ ($\measuredangle$ sum of $\triangle$)
$= 30°$

(c) $A\hat{R}B = 90°$ (rt. $\measuredangle$ in semicircle)
$\therefore$ $R\hat{B}Q = 180° - 90° - 28°$ ($\measuredangle$ sum of $\triangle$)
$= 62°$

Example 2

A chord of length 15 cm is drawn in a circle of radius 12 cm as shown in Fig. 7.58. Calculate the perpendicular distance from the centre of the circle to the chord.

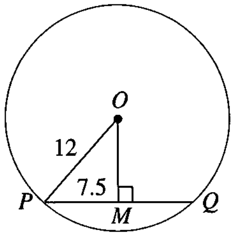


Fig. 7.58

Solution

The perpendicular from the centre cuts the chord PQ at its mid-point, i.e. $PM = 7.5$ cm. (See Fig. 7.58.)

$OP^2 = PM^2 + OM^2$

$12^2 = 7.5^2 + OM^2 \Rightarrow OM = \sqrt{12^2 - 7.5^2}$

$\therefore$ the perpendicular distance is approximately 9.37 cm.

Example 3

In Fig. 7.59, TAR is a tangent to the circle whose centre is O. The chord AC intersects the diameter BD at X. Given that $A\hat{E}D = 122°$ and $B\hat{D}C = 28°$, calculate the following angles:

(a) $A\hat{C}D$ *(b)* $A\hat{O}B$ *(c)* $A\hat{D}O$

(d) $B\hat{A}R$ *(e)* $A\hat{X}B$

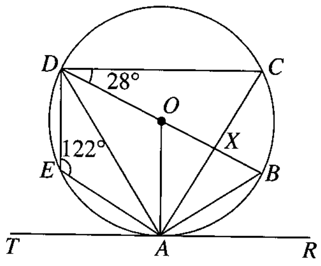


Fig. 7.59

Solution

(a) $A\hat{C}D = 180° - 122°$ (opp. $\measuredangle$ of a cyclic quad. are supplementary)
$= 58°$

(b) $A\hat{B}D = 58°$ ($\measuredangle$ in the same segment)
$O\hat{A}B = 58°$ (base $\measuredangle$ of an isos. $\triangle$)
$\therefore\ A\hat{O}B = 180° - 58° - 58°$
$= 64°$

(c) $A\hat{D}O = \frac{1}{2}(64°)$ ($\measuredangle$ at centre = 2 $\measuredangle$ at $\odot^{ce}$)
$= 32°$

(d) $B\hat{A}R = A\hat{D}O$ ($\measuredangle$ in alt. segment)
$= 32°$

(e) $B\hat{A}C = B\hat{D}C$ ($\measuredangle$ in the same segment)
$= 28°$
$\therefore\ A\hat{X}B = 180° - 28° - 58°$
$= 94°$

Revision Exercise 7.8

1. In Fig. 7.60, *AB* is a diameter of the circle, centre *O*, *DC* is parallel to *AB* and $B\hat{A}D = 63°$.

Calculate the following angles:

(a) $A\hat{B}D$,
(b) $C\hat{B}D$,
(c) $B\hat{O}C$.

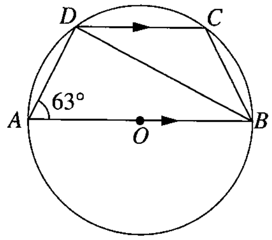


Fig. 7.60

2. In the diagram, *ADE*, *AXC*, *BXD* and *BCE* are straight lines, $A\hat{E}C = 38°$ and $E\hat{A}C = 23°$. Calculate

(a) $B\hat{D}A$,
(b) $B\hat{X}C$.

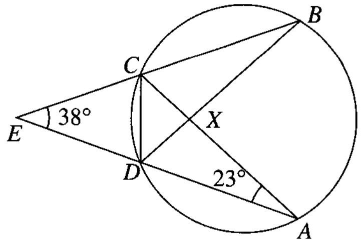


Fig. 7.61

3. In Fig. 7.62, O is the centre of the circle, $B\hat{O}C = 98°$ and $A\hat{B}O = 22°$. Calculate the following angles:

(a) $B\hat{A}C$,

(b) $A\hat{C}P$.

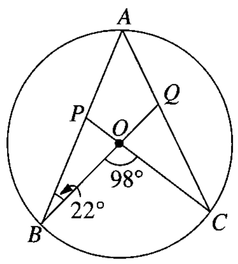


Fig. 7.62

4. In the diagram, $P\hat{A}B = 90°$, $C\hat{B}D = 68°$ and $C\hat{D}Q = 115°$. Calculate

(a) $A\hat{C}B$,

(b) $A\hat{C}D$.

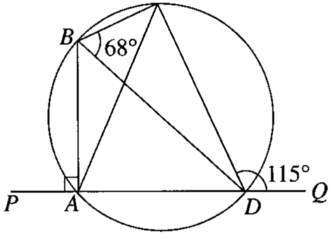


Fig. 7.63

5. In Fig. 7.64, BC is a diameter of the circle whose centre is O. TA is a tangent to the circle at A and BCT is a straight line. If $A\hat{C}O = 58°$, calculate the following angles:

(a) $A\hat{O}C$

(b) $A\hat{B}C$

(c) $C\hat{A}T$

(d) $A\hat{T}C$

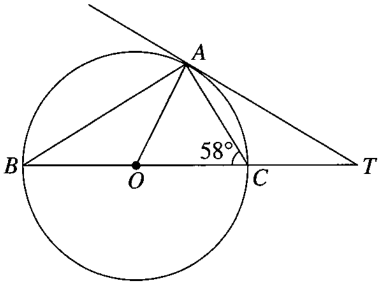


Fig. 7.64

6. In Fig. 7.65, TA and TB are tangents to the circle from an external point T. If CB is parallel to AT and $B\hat{A}T = 62°$, calculate the following angles:

(a) $A\hat{T}B$

(b) $B\hat{C}A$

(c) $B\hat{A}C$

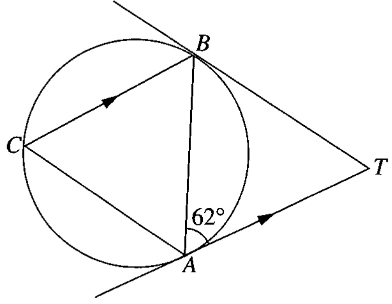


Fig. 7.65

7. In Fig. 7.66, O is the centre of the circle and TA is a tangent to the circle. If $A\hat{D}C = 66°$ and $A\hat{C}B = 40°$, calculate the following angles:

(a) $A\hat{O}C$

(b) $B\hat{A}C$

(c) $A\hat{C}O$

(d) $T\hat{A}B$

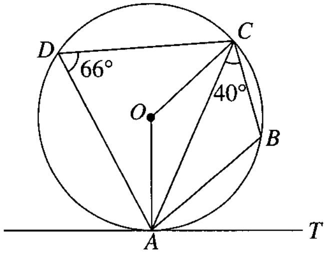


Fig. 7.66

8. In Fig. 7.67, AC is a diameter of the circle whose centre is O. AB and DC are produced to meet at T. If $A\hat{T}D = 34°$ and $B\hat{A}C = 32°$, calculate the following angles:

(a) $B\hat{D}C$

(b) $B\hat{C}D$

(c) $D\hat{B}C$

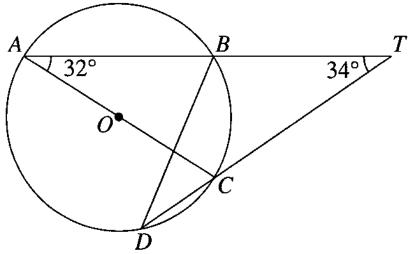


Fig. 7.67

9. In Fig. 7.68, O is the centre of the circle and TA is the tangent at A. If $TA = 5$ cm and $TB = 3.2$ cm, calculate the radius of the circle.

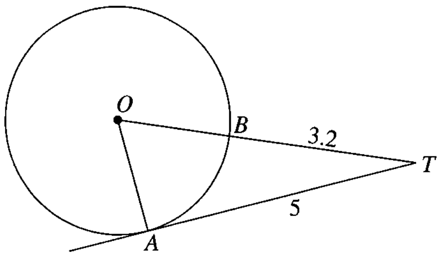


Fig. 7.68

10. In Fig. 7.69, TA and TB are the tangents to the circle with centre O and radius 5 cm. If $TA = 17$ cm, calculate

(a) the area of the quadrilateral $ATBO$,

(b) the length of the minor arc APB.

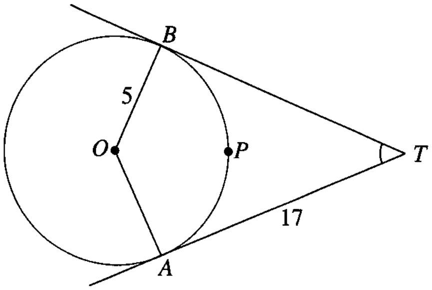


Fig. 7.69

11. A chord of length 18 cm is drawn in a circle of radius 16 cm. Calculate the perpendicular distance from the centre of the circle to the chord.

12. The perpendicular distance from the centre of a circle to a chord drawn in the circle is 7.5 cm. Calculate the radius of the circle if the chord has a length of 12 cm.

13. In Fig. 7.70, O is the centre of the circle through A, B, C and D. TA is the tangent at A and AC intersects BD at X. If $A\hat{C}B = 62°$ and $D\hat{A}C = 32°$, calculate the following angles:

(a) $B\hat{A}O$,
(b) $A\hat{O}D$,
(c) $B\hat{X}C$,
(d) $T\hat{A}D$.

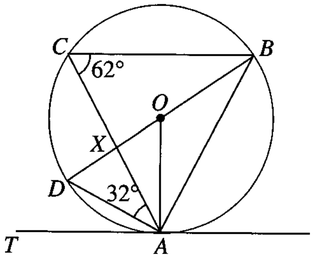


Fig. 7.70

14. A chord 8 cm long is drawn in a circle whose diameter is 10 cm. How far is the chord from the centre of the circle?

15. In Fig. 7.71, AD is a diameter of the circle whose centre is O. If $\triangle AOB$ is equilateral, and $O\hat{B}C = 50°$, calculate the following angles:

(a) $B\hat{C}D$,
(b) $O\hat{D}C$,
(c) $C\hat{B}D$,
(d) $C\hat{O}D$.

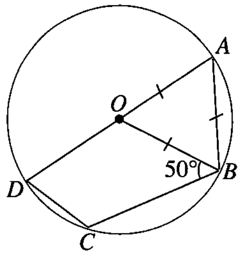


Fig. 7.71

16. In Fig. 7.72, ST is the tangent to the circle at S. $S\hat{L}N = 45°$, $N\hat{S}M = 60°$ and $L\hat{N}M = 25°$.

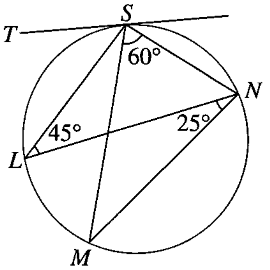


Fig. 7.72

Calculate the following angles.

(a) $L\hat{S}T$ (b) $L\hat{S}M$

17. In the diagram, AOB is a diameter with O as the centre. Given that $T\hat{A}D = 132°$ and $BC = CD$, find $A\hat{D}C$.

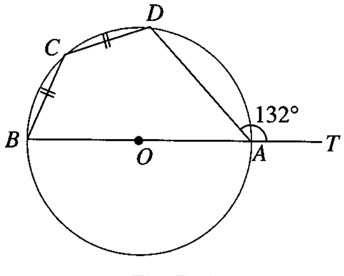


Fig. 7.73

18. In the diagram, O is the centre of the circle. Given that $A\hat{O}B = 110°$ and $B\hat{C}D = 84°$, find $O\hat{A}D$.

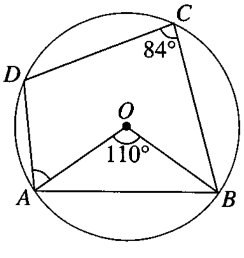


Fig. 7.74

19. In the diagram, O is the centre of the bigger circle ABC and CBP is a straight line. Given that $A\hat{B}C = 132°$, calculate

(a) $A\hat{O}C$,
(b) $A\hat{P}C$.

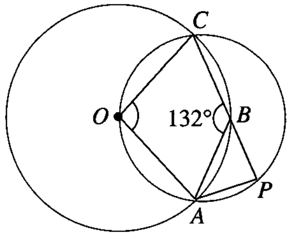


Fig. 7.75

20. In Fig. 7.76, TA is a tangent to the circle $ABCD$. If $A\hat{B}D = 70°$, $C\hat{A}D = 50°$ and $B\hat{D}C = 20°$, calculate $T\hat{A}B$.

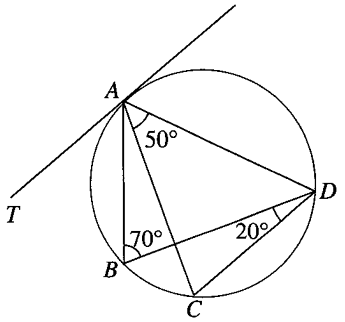


Fig. 7.76

21. In the diagram, AB is a diameter of the circle and PT is the tangent to the circle at P. If $A\hat{P}T = 118°$, calculate $A\hat{B}P$.

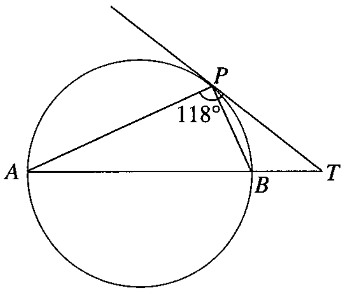


Fig. 7.77

22. In Fig. 7.78, O is the centre of the circle and PAQ is the tangent to the circle at A. Given that $B\hat{A}Q = (3x + 4y)°$, $A\hat{O}B = (7x + 6y)°$ and $A\hat{B}O = 40°$, find the value of x and of y and hence state the value of $A\hat{D}B$.

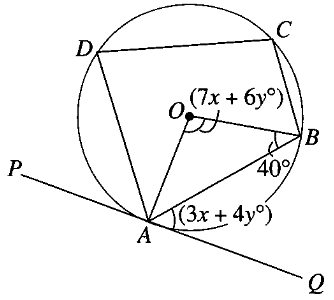


Fig. 7.78

23. In the diagram, TA and TB are tangents to the circle at A and B respectively. The lines PQ and AB intersect at X. Given that $A\hat{T}B = 48°$, $P\hat{B}T = 42°$ and $B\hat{A}Q = 44°$, calculate

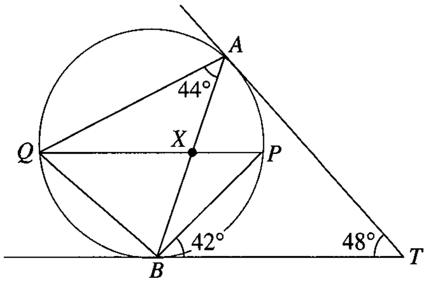


Fig. 7.79

(a) $P\hat{B}A$, **(b)** $A\hat{P}B$, **(c)** $A\hat{X}P$.

24. TA and TB are tangents to the circle at A and B respectively. Given that $A\hat{T}B = 50°$, $T\hat{B}Q = 30°$ and $A\hat{P}Q = 65°$, calculate

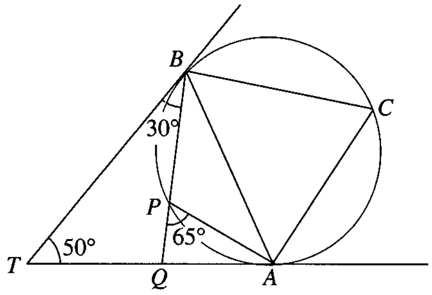


Fig. 7.80

(a) $A\hat{B}P$, **(b)** $A\hat{C}B$, **(c)** $B\hat{C}P$.

25. PAQ is the tangent to the circle at A. $B\hat{A}Q = 4x°$, $A\hat{C}E = 32°$, $A\hat{B}D = 5x°$ and $B\hat{A}D = 81°$. Find

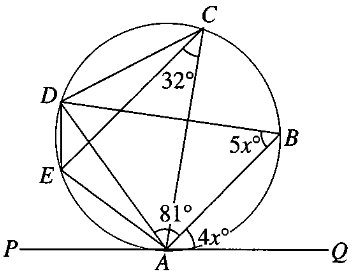


Fig. 7.81

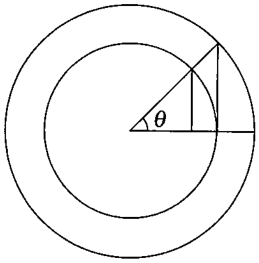
(a) the value of x,

(b) $P\hat{A}D$,

(c) $D\hat{A}E$.

7.9 Trigonometry

The trigonometric ratios of an angle remain constant whatever the length of the radius may be.



Example 1

In Fig. 7.82, $\hat{ABC} = 90°$, $BC = 15$ cm and $\hat{BAC} = 54°$.

Calculate the following:

(a) AB
(b) AC
(c) the area of △ABC

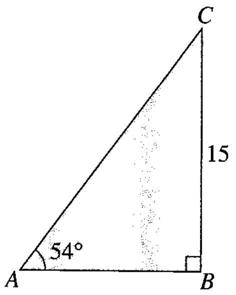


Fig. 7.82

Solution

(a) $\tan 54° = \frac{BC}{AB} = \frac{15}{AB}$

$\therefore \quad AB = \frac{15}{\tan 54°} = 10.9$ cm

(b) $\sin 54° = \frac{BC}{AC} = \frac{15}{AC}$

$\therefore \quad AC = \frac{15}{\sin 54°} = 18.5$ cm

(c) Area of $\triangle ABC = \frac{1}{2} \times 10.9 \times 15$

$= 81.75 \text{ cm}^2$

Example 2

Fig. 7.83 shows a triangle ABC, where $AB = 19$ cm, $AC = 15$ cm and $\hat{BAC} = 46°$. Calculate the following:

(a) BC
(b). the area of △ABC

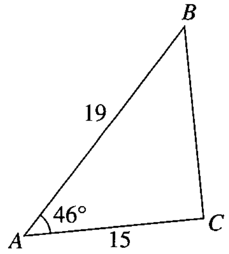


Fig. 7.83

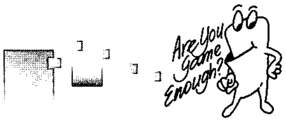
Solution

(a) $BC^2 = AC^2 + AB^2 - 2(AC)(AB) \cos \hat{BAC}$

$= 15^2 + 19^2 - 2(15)(19) \cos 46° = 190.04$

$\therefore \quad BC = \sqrt{190.04} = 13.8$ cm

(b) Area of $\triangle ABC = \frac{1}{2}(15)(19) \sin 46° = 102.5 \text{ cm}^2$



What is the angle of rotation made by the Earth in 1 hour?

Example 3

In $\triangle ABC$, $AB = 8$ cm, $AC = 15$ cm, $\hat{BAC} = 90°$ and AC is produced to D. Find

(a) the length of BC,
(b) $\sin A\hat{C}B$,
(c) $\cos B\hat{C}D$.

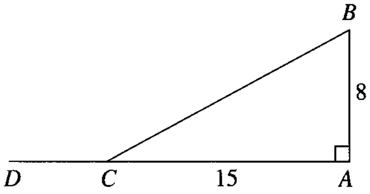


Fig. 7.84

Solution

(a) $BC^2 = 15^2 + 8^2$
$= 225 + 64$
$= 289$

$\therefore \quad BC = \pm\sqrt{289} = \pm 17$

i.e. $BC = 17$ cm (negative value is ignored here)

(b) $$\sin A\hat{C}B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{8}{17}$$

(c) $$\cos B\hat{C}D = -\frac{15}{17}$$

Example 4

From the top of a building 46 m high, the angles of depression of two points on the ground both due east of the tower are 58° and 32°. Calculate the distance between the two points.

Solution

Let AT be the building and B and C the two points on the ground.

$T\hat{C}A = 32°$, $T\hat{B}A = 58°$ (alternate ∡s)

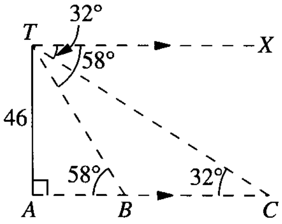


Fig. 7.85

In $\triangle ABT$, $\tan 58° = \frac{AT}{AB} = \frac{46}{AB}$

$\therefore \quad AB = \frac{46}{\tan 58°} = 28.74$ m

In $\triangle ACT$, $\tan 32° = \frac{AT}{AC} = \frac{46}{AC}$

$\therefore \quad AC = \frac{46}{\tan 32°} = 73.62$ m

$\therefore$ distance between B and C = 73.62 – 28.74 = 44.88 m

Example 5

In Fig. 7.86(a), A, B and C represent three points on a map. Calculate

(a) the bearing of C from A,
(b) the bearing of C from B.

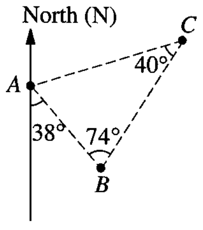


Fig. 7.86(a)

Solution

(a) $B\hat{A}C = 180° - 74° - 40°$ (∡ sum of a △)
$= 66°$

∴ $N\hat{A}C = 180° - 66° - 38°$ (∡s on a straight line)
$= 76°$

∴ the bearing of C from A is 076°.

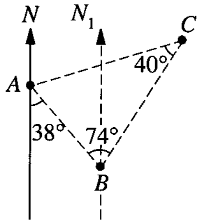


Fig. 7.86(b)

(b) Construct a line parallel to AN and passing through B.

$A\hat{B}N = 38°$ (alt. ∡s),

$N_1\hat{B}C = 74° - 38° = 36°$

∴ the bearing of C from B is 036°.

Example 6

A, B and C are three points on level ground. B is due south of A and the bearing of C from A is 085°. A vertical mast AT of height 50 m stands at A. The angle of elevation of T from B is 44° and the angle of elevation of T from C is 28°. Calculate the following, giving your answer correct to 2 decimal places.

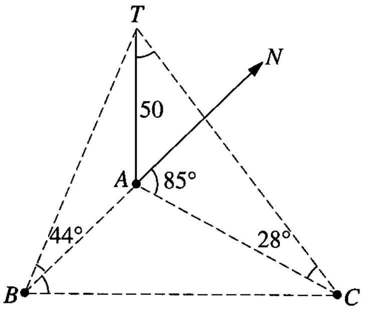


Fig. 7.87(a)

(a) AB *(b) AC*
(c) BC *(d) the bearing of C from B*

A man walks from B to C, find the greatest angle of elevation of T from any point along BC.

Solution

(a) In $\triangle ABT$, $\tan 44° = \frac{AT}{AB} = \frac{50}{AB}$

∴ $AB = \frac{50}{\tan 44°} = 51.777 = 51.78$ m

CHECK THIS OUT ☆ ☆ ☆ ☆ ☆

We all know that the base angles of an isosceles triangle are equal. The following arguments show that we have an isosceles triangle whose base angles are not equal.

In $\triangle ABC$, $a = \sqrt{3}$, $c = 1$, $B = 30°$.

Using the cosine formula,

$b^2 = a^2 + c^2 - 2ac \cos B$
$= (\sqrt{3})^2 + 1^2 - 2 \times \sqrt{3} \times 1 \times \frac{\sqrt{3}}{2}$
$= 1$

However $b > 0$, ∴ $b = 1$.

Using the sine rule,

$\sin A = \frac{a \sin B}{b}$
$= \frac{\sqrt{3} \times \sin 30°}{1} = \frac{\sqrt{3}}{2}$

Thus, $A = 60°$ *and* $C = 180° - (30° + 60°) = 90°$. *From the above discussion, we have* $b = c = 1$. *Hence,* $\triangle ABC$ *is isosceles. However the two base angles are* $B = 30°$ *and* $C = 90°$ *which are not equal.*

Thus, $\triangle ABC$ *is an isosceles triangle with unequal base angles.*

Do you believe that there is such a triangle? If not, investigate to find out any irregularities in the above arguments.

☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆

(b) In $\triangle ACT$, $\tan 28° = \frac{AT}{AC} = \frac{50}{AC}$

$\therefore \quad AC = \frac{50}{\tan 28°} = 94.036 = 94.04$ m

(c) In $\triangle ABC$, $B\hat{A}C = 180° - 85°$ (∡s on a straight line)

$= 95°$

$BC^2 = AB^2 + AC^2 - 2(AB)(AC) \cos 95°$

$= 51.777^2 + 94.036^2 - 2(51.777)(94.036) \cos 95° = 12\ 372.3$

$\therefore \quad BC = 111.23$ m

(d) In $\triangle ABC$, $\frac{AC}{\sin A\hat{B}C} = \frac{AC}{\sin B\hat{A}C}$

$$\frac{94.036}{\sin A\hat{B}C} = \frac{111.23}{\sin 95°}$$

$$\sin A\hat{B}C = \frac{94.036 \sin 95°}{111.231}$$

$$A\hat{B}C = 57.37°$$

$\therefore$ the bearing of C from B is 057.37°.

The greatest angle of elevation of T from the path of BC occurs at the point K on BC where AK is perpendicular to BC.

$\sin 57.37° = \frac{AK}{51.78}$

$AK = 51.78 \sin 57.37° = 43.61$ m

$\tan A\hat{K}T = \frac{50}{43.61}$

$\therefore \quad A\hat{K}T = 48.91°$

$\therefore$ the greatest angle of elevation of T from any point along BC is 48.91°.

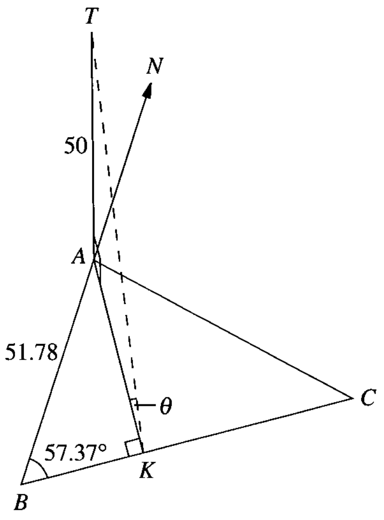


Fig. 7.87(b)

Revision Exercise 7.9

1. In Fig. 7.88, $A\hat{B}D = B\hat{D}C = 90°$, $AB = 4.2$ cm, $BD = 5.8$ cm and $B\hat{C}D = 37.6°$. Calculate the following:

   (a) $B\hat{A}D$, **(b)** CD,
   (c) BC.

   If X is the foot of the perpendicular from D to BC, calculate the length of BX.

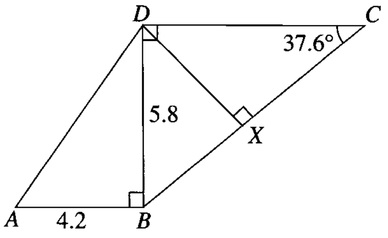


Fig. 7.88

CHECK THIS OUT

In country Y, only 5¢ and 8¢ coins are available. Jonathan has a big bag of 5¢ and 8¢ coins. He goes to a shop which also has a plentiful of 5¢ and 8¢ coins. Can Jonathan buy any product of value from 1¢ to 99¢ (To buy an item worth 1¢, Jonathan can give the shopkeeper two 8¢ coins and the shopkeeper gives him back three 5¢ coins.)

2. **(a)** In triangle ABC, $AC = 4$ cm and $BC = 5$ cm. Given that $\cos A\hat{C}B = -0.2$, calculate the length of AB.
   (b) In triangle ABC, $A\hat{B}C = 90°$, $BC = 5$ cm and $AC = \sqrt{61}$ cm. Calculate the length of AB.
   (c) In triangle ABC, $\sin \hat{A} = 0.6$, $\sin \hat{B} = 0.4$ and $AC = 10$ cm. Calculate the length of BC.

3. In $\triangle BCD$, $CD = 10$ cm, $B\hat{C}D = 90°$ and $BD = 26$ cm. CB is produced to A and CD is produced to E.

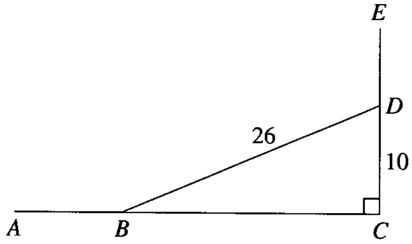


Fig. 7.89

   (a) Showing your working clearly, find the length of BC.
   (b) Express each of the following as a fraction in its simplest form:
   (i) $\sin B\hat{D}C$, **(ii)** $\cos A\hat{B}D$,
   (iii) $\tan B\hat{D}E$, **(iv)** $\sin A\hat{B}D$.

4. Three points A, B and C are on level ground, A is due south of B and C is 25 m due east of B. BT is a vertical flagpole.

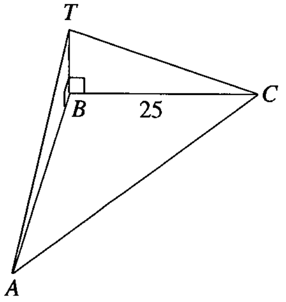


Fig. 7.90

   (a) If the angle of elevation of T from C is 18°, calculate the height of the flagpole.
   (b) If the bearing of C from A is 036°, find the angle of elevation of T from A.

5. In $\triangle ABC$, $\hat{B} = 90°$ and $\tan \hat{A} = \frac{3}{4}$, find the value of the following:
   (a) $\cos \hat{A}$, **(b)** $\sin (90° - \hat{A})$

6. In $\triangle ABC$, $\hat{B} = 90°$ and $\sin \hat{C} = \frac{3}{5}$, find the value of $\cos \hat{C} + \tan \hat{C}$.

7. A vertical flagstaff 25 m high stands on horizontal ground. If the angle of elevation of the top of the flagstaff from a point X on the ground is found to be 25°, find the distance of X from the foot of the flagstaff.

8. Find the angle of depression from the top of a tower 140 m high of an object on the ground 200 m from the foot of the tower.

9. $ABCD$ is a parallelogram where $AB = 5$ cm, $BC = 6$ cm and $A\hat{B}C = 54°$. Calculate the length of AC and of BD.

10. The area of an acute-angled triangle is 4.6 cm^2. If two of the sides are 3.5 cm and 4.2 cm, find the angle between these two sides.

11. Find the area of an equilateral triangle whose perimeter is 120 cm.

12. In $\triangle ABC$, $A\hat{B}C = 42°$, $B\hat{A}C = 41°$ and $BC = 8.6$ cm. Calculate
    (a) the length of AB,
    (b) the area of $\triangle ABC$.

13. In Fig. 7.91, r_1 and r_2 are the radii of the circles whose centres are A and B respectively and PQ is a common tangent. If r_1 is 5 cm and $r_2 = 8$ cm, calculate

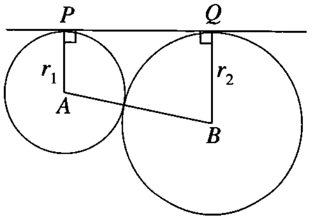


Fig. 7.91

    (a) the length of PQ, **(b)** $P\hat{A}B$.

14. The area of $\triangle ABC$ is 8 cm^2. If $AB = 6$ cm and $AC = 5$ cm, calculate the following:
    (a) $B\hat{A}C$,
    (b) the length of BC.

15. ABC is a triangle in which $B\hat{A}C = 90°$, $A\hat{B}C = 35°$ and $AC = 6$ cm. Calculate the length of BC. If BA is produced to a point D such that $CD = 11$ cm, calculate $A\hat{D}C$ and the length of BD.

16. In Fig. 7.92, $ABCD$ is a trapezium in which AB is parallel to DC and $B\hat{A}D = 90°$. If $D\hat{A}C = 64°$, $AC = 5.4$ cm and $AB = 3.6$ cm, calculate

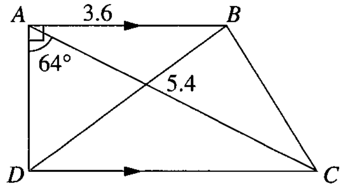


Fig. 7.92

(a) the length of CD, **(b)** $A\hat{B}D$.

17. In Fig. 7.93, AB is a common tangent to the two circles whose centres are P and Q and whose radii are 4 cm and 5 cm respectively. Calculate

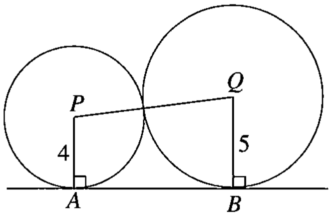


Fig. 7.93

(a) the length of AB, **(b)** $P\hat{Q}B$.

18. In a right-angled triangle, the length of the hypotenuse is 35 cm and one of its interior angles is 48°. Calculate the length of the shortest side of the triangle.

19. In $\triangle ABC$, $A\hat{B}C = 90°$. The point L on BC is such that AL bisects $B\hat{A}C$. If $AL = 4$ cm and $A\hat{L}B = 60°$, calculate the following:

(a) $A\hat{C}B$,
(b) CL,
(c) AB.

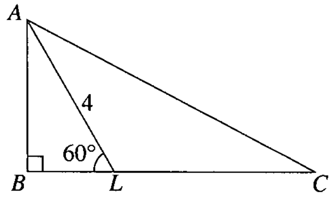


Fig. 7.94

20. Two ships A and B leave a port P at 12 00. A sails at 15 km/h on a bearing 040° and B at 24 km/h on a bearing 100°. Calculate, at 14 00,

(a) the distance between the ships,
(b) $P\hat{A}B$, to the nearest degree,
(c) the bearing of B from A.

21. In the acute-angled triangle ABC, $\hat{A} = 47°$, $AC = 12.4$ cm and $BC = 10.5$ cm.

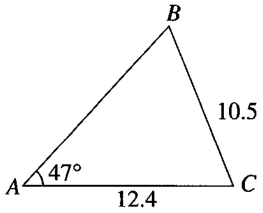


Fig. 7.95

(a) Calculate $\hat{B}$.
(b) It is also possible to draw an obtuse-angled triangle with the given dimensions. What is the size of the obtuse angle of this triangle?

22. In each of the following triangles, $OX = 20$ cm and $OY = 15$ cm.

(a) If $X\hat{O}Y = 90°$, calculate the length of XY.
(b) If $O\hat{Y}X = 90°$, calculate $O\hat{X}Y$.
(c) If $X\hat{O}Y = 112°$, calculate the length of XY.
(d) If $O\hat{X}Y = 35°$, calculate
  (i) $O\hat{Y}X$,
  (ii) the length of XY,
  (iii) the area of $\triangle OXY$.

23. A and B are two points on the coast. A is due north of B and $AB = 450$ m. A ship S is due east of A and the bearing of S from B is 038°. A second ship R is due east of B and the bearing of R from A is 128°. Calculate the distance of SR and find the bearing of R from S.

24. $ABCDE$ is a regular pentagon of side 14 cm. Calculate

(a) the length of AC,
(b) the area of the pentagon $ABCDE$.

25. In $\triangle ABC$, $AB = 24$ cm, $BC = 31$ cm and $A\hat{B}C = 57°$. Calculate the area of $\triangle ABC$ and hence or otherwise, calculate the perpendicular distance from A to BC.

26. In the figure, A, B, C and D are four points on level ground. D is due south of C and A is due west of D. If $CD = 24$ m, $AC = 43$ m, $AB = 25$ m, $B\hat{A}X = 108°$ and $A\hat{B}X = 40°$, calculate

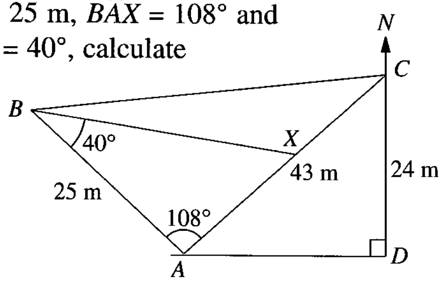


Fig. 7.96

- **(a)** the bearing of C from A,
- **(b)** the length of CX,
- **(c)** the length of BC,
- **(d)** the area of the quadrilateral $ABCD$.

27. Three points A, B and C are on level ground. The bearing of B and C from A are 195° and 305° respectively. Given that $AB = 3.4$ km and $AC = 4.5$ km, calculate

- **(a)** the distance of BC in km,
- **(b)** the bearing of C from B,
- **(c)** the area of $\triangle ABC$ in hectares.
  [1 hectare = 10 000 m^2]

28. A, B, C and D are four points on horizontal ground with D due east of A. Given that $AB = 70$ m, $BC = 80$ m, $CD = 110$ m and $AD = 190$ m, calculate

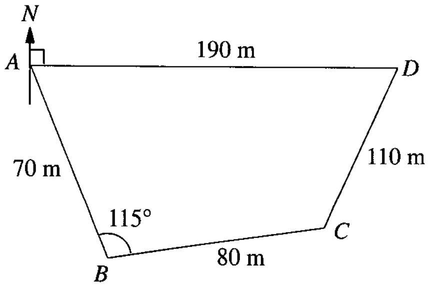


Fig. 7.97

- **(a)** the length of AC,
- **(b)** $A\hat{D}C$, **(c)** $A\hat{C}B$,
- **(d)** the bearing of B from A,
- **(e)** the area of the quadrilateral $ABCD$ in m^2.

29. In the diagram, $A\hat{C}B = A\hat{D}C = 90°$, $B\hat{A}C = 54.6°$, $AD = 4.8$ cm and $DC = 7.6$ cm. Calculate the following:

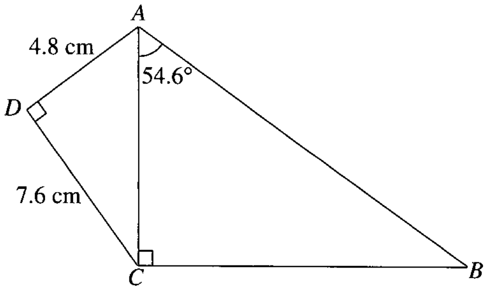


Fig. 7.98

(a) $A\hat{C}D$, **(b)** AC, **(c)** AB.

Given that E is the point on AB such that $AE = 7$ cm, calculate the area of $\triangle ACE$.

30. The diagram shows A, B, C and D, the four corners of a horizontal rectangular field $ABCD$. The corner B is 82 metres from A on a bearing of 021° and C is 173 metres from A.

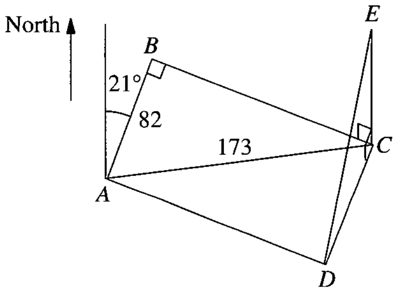


Fig. 7.99

- **(a)** Calculate
  - **(i)** the bearing of C from B,
  - **(ii)** $B\hat{A}C$,
  - **(iii)** the bearing of C from A.
- **(b)** A hot air balloon was hovering at E, which is vertically above C.

  The angle of elevation of the bottom of the balloon from D was 35°.

  Calculate
  - **(i)** the height of the bottom of the balloon above C,
  - **(ii)** the angle of elevation of the bottom of the balloon from B.
- **(c)** A bird is hovering at a height of 40 metres above the field.

  It spots its prey on the ground at an angle of depression of 63°.

Calculate the distance that the bird must fly to catch its prey. (*C*)

31. Three buoys, A, B and C, are positioned in a lake to provide a course for a yacht race. $AB = 800$ m, $A\hat{B}C = 32°$, $B\hat{A}C = 22°$ and N is the point on AB which is 200 m from A.

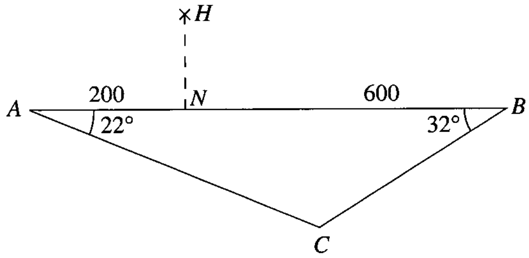


Fig. 7.100

(a) Show that the distance AC is 524 m, correct to the nearest metre.

(b) Calculate the distance NC.

(c) A helicopter, H, is hovering at a point vertically above N.

- **(i)** The angle of elevation of the helicopter from A is 12°. Calculate the height of the helicopter.
- **(ii)** P is the point on AC which is nearest to the helicopter. Calculate the angle of elevation of the helicopter from P. (*C*)

32. In the diagram, ABC represents a horizontal triangular field and AD represents a vertical tree in the corner of the field. A path runs along the edge BC of the field. $AB = 83$ m, $AC = 46$ m and angle $BAC = 67°$.

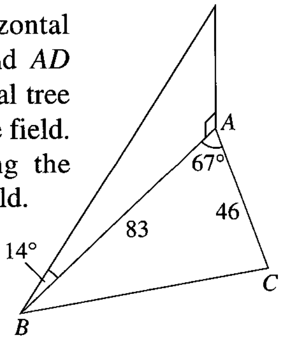


Fig. 7.101

(a) The angle of elevation of the top of the tree when viewed from B is 14°. Calculate the height of the tree.

(b) Calculate the length of the path BC.

(c) Calculate the area of the field ABC.

(d) Calculate the shortest distance from A to the path BC.

(e) Calculate the greatest angle of elevation of the top of the tree when viewed from any point on the path. (*C*)

33. A radio mast AB, of height 20 m, stands at the top of a slope which is inclined at 18° to the horizontal.

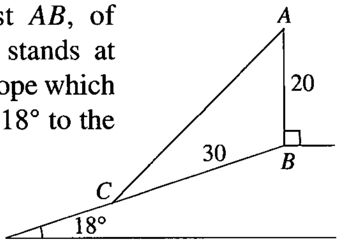


Fig. 7.102(a)

(a) The mast is supported by a wire AC attached to a point C on the slope, where $BC = 30$ m. Calculate

- **(i)** $A\hat{B}C$,
- **(ii)** the length of the wire AC.

(b) When the sun is in a certain position, the shadow cast by the mast lies down the slope, shown in the diagram by the line BD. Given that $A\hat{D}B = 42°$, calculate

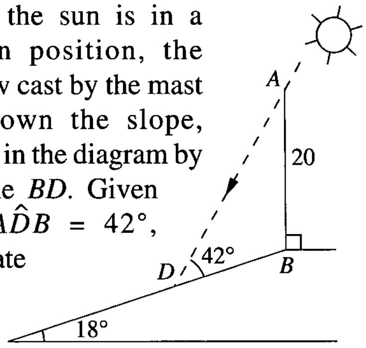


Fig. 7.102(b)

- **(i)** the angle of elevation of the sun,
- **(ii)** $D\hat{A}B$,
- **(iii)** the length of the shadow BD.

(c) The mast is supported by another wire AF. The points B, E and F lie on horizontal ground. Given that $B\hat{E}F = 90°$, $BE = 12$ m and $EF = 15$ m, calculate the length of the wire AF.

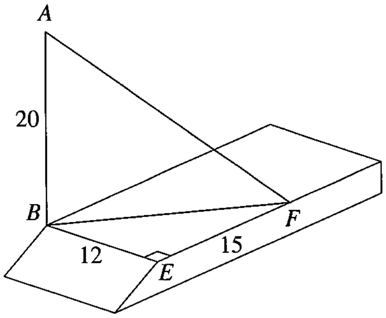


Fig. 7.102(c) (*C*)

7.10 Number Sequence and Problem Solving

Example 1

The table below refers to a certain series.

(a) Study the table and then fill in the blank spaces.

	S	*N*		*M*
Series	*Sum of series*	*Base of last term of series*	*N + 1*	*N(N + 1)*
$1^3 + 2^3$	*9*	*2*	*3*	*6*
$1^3 + 2^3 + 3^3$	*36*	*3*		
$1^3 + 2^3 + 3^3 + 4^3$		*4*	*5*	*20*
$1^3 + 2^3 + 3^3 + 4^3 + 5^3$	*225*			
$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3$				*42*

(b) Express the relationship between the numbers in column S and those in column M as a formula connecting S and M.

(c) (i) Use your answer to part (b) to find the value of S when N = 8.

(ii) Verify this by evaluating $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3$.

(d) Use your answer to part (b) to evaluate

(i) $1^3 + 2^3 + 3^3 + \dots + 3\ 375$,

(ii) $1^3 + 2^3 + 3^3 + \dots + 24^3$.

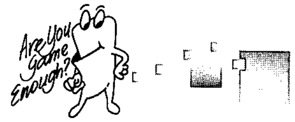
(e) Suggest a formula, in terms of n, for the sum of the series $1^3 + 2^3 + 3^3 + \dots + n^3$.

Solution

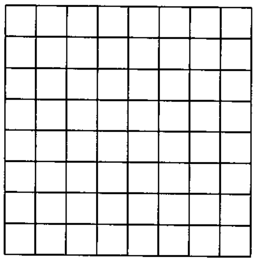
(a)

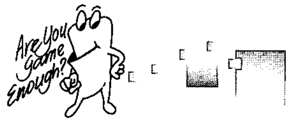
	S	*N*		*M*
Series	*Sum of series*	*Base of last term of series*	*N + 1*	*N(N + 1)*
$1^3 + 2^3$	9	2	3	6
$1^3 + 2^3 + 3^3$	36	3	4	12
$1^3 + 2^3 + 3^3 + 4^3$	100	4	5	20
$1^3 + 2^3 + 3^3 + 4^3 + 5^3$	225	5	6	30
$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3$	441	6	7	42

(b) $S = \left(\frac{M}{2}\right)^2 = \frac{M^2}{4}$



How many squares (of varying sizes) are there in a standard 8 × 8 chess board?





Start with any counting number. If it is even, ***divide it by 2****; if it is odd,* ***multiply it by 3 and add 1 to it.*** *Write down the result on a piece of paper. Repeat this with the number you have written down. For example, starting with 9 which is odd, multiply it by 3 and add 1 to obtain 28 which is even. Divide it by 2. Divide the even number 14 by 2. Obtaining 7, multiply it by 3 and add 1 to it and so on. If you continue this often enough, what interesting result will you observe? Repeat the procedure, starting with different counting numbers. Did you obtain the same result each time? Can you think of an explanation for this?*

(c) (i) When $N = 8$, $M = N(N + 1) = 8 \times 9 = 72$

$$\therefore \quad S = \left(\frac{72}{2}\right)^2 = 1\ 296$$

(ii) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3$
$= 1 + 8 + 27 + 64 + 125 + 216 + 343 + 512$
$= 1\ 296$

(d) (i) $3\ 375 = 15^3$

$\therefore \quad N = 15$ and $M = 15 \times 16 = 240$

$$S = \left(\frac{240}{2}\right)^2 = 14\ 400$$

$\therefore \quad 1^3 + 2^3 + 3^3 + \ldots + 3\ 375 = 14\ 400$

(ii) $N = 24$ and $M = 24 \times 25 = 600$

$$\therefore \quad S = \left(\frac{600}{2}\right)^2 = 90\ 000$$

i.e. $1^3 + 2^3 + 3^3 \ldots + 24^3 = 90\ 000$

(e) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ or $\frac{n^2(n+1)^2}{4}$

Example 2

All the members of a family of chemical compounds contain Carbon atoms, C, and Hydrogen atoms, H. Some of the members of the family are represented in the diagrams below.

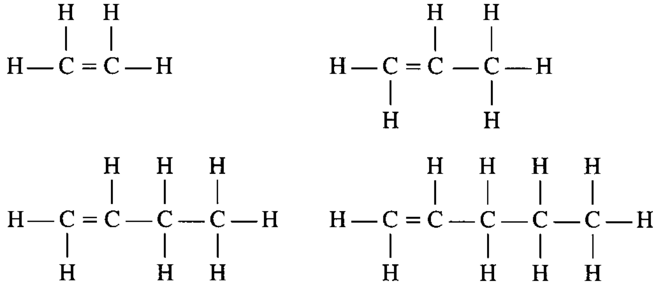


Fig. 7.103(a)

(a) Draw the next pattern in the series.

(b) If a member of the family of compounds contains x Carbon atoms and y Hydrogen atoms, then the chemical formula of this member is written as C_xH_y.

  (i) Write down the chemical formulas of the first five members of the family represented in the diagrams above and your diagram.

  (ii) A member contains 10 Carbon atoms. Write down the number of Hydrogen atoms it contains.

  (iii) Another member contains 64 Hydrogen atoms. How many Carbon atoms does it contain?

  (iv) Give a formula that connects x and y.

Solution

(a) The next member of the family is represented in the diagram below.

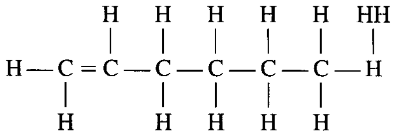


Fig. 7.103(b)

(b) (i) The first five members of the family are C_2H_4, C_3H_6, C_4H_8, C_5H_{10} and C_6H_{12}.
(ii) The number of Hydrogen atoms = 2(10) = 20.
(iii) The number of Carbon atoms = $\frac{1}{2}(64) = 32$.
(iv) $y = 2x$.

Revision Exercise 7.10

1. **(a)** Write down the next two terms in the sequence

120, 110, 101, 93, ...

(b) Write down an expression, in terms of n, for the nth term in the sequence

0, –1, –8, –27, –64, ...

2. Copy and complete the following sequence of numbers.

(a) 0, 3, 8, 15, 24, ________, ________

(b) $1, 1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{8}, \sqrt{13}$, ________, ________

(c) $\frac{1}{6}, \frac{5}{12}, \frac{2}{3}, \frac{11}{12}$, ________, ________

3. Consider the pattern

$$\begin{array}{ll} 1^2 - 1 = 0 & 5^2 - 5 = 20 \\ 2^2 - 2 = 2 & \vdots \\ 3^2 - 3 = 6 & x^2 - x = 110 \\ 4^2 - 4 = 12 & \vdots \end{array}$$

(a) Write down the 8th line in the pattern.
(b) Find the positive integer value of x satisfying the equation $x^2 - x = 110$.

4. John used toothpicks to make a series of squares. The first four squares he constructed are as shown in Fig. 7.104.

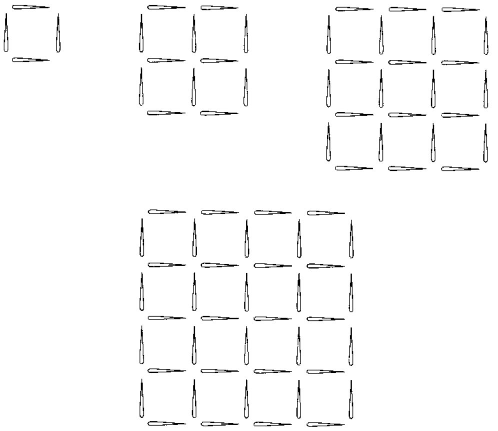


Fig. 7.104

For each square, let T represent the number of toothpicks used, S the total number of small squares formed and P the number of points at which 2 or more toothpicks meet. The values of T, S and P are tabulated as shown in the following.

Number of toothpicks used T	*Number of small squares formed S*	*Number of points at which 2 or more toothpicks meet P*
4	1	4
12	4	9
24	9	16
40	16	25
l	m	n

Study the number patterns in the table above and answer the following questions.

(a) Write down the value of l, m and n in the fifth line of the table.

(b) Form and write down, a formula connecting T, S and P.

(c) Use your answer to part **(b)** to find the value of P when $T = 364$ and $S = 169$.

(d) Give a simple reason why the number 112 could neither appear in the S column nor in the P column.

(e) Give a simple reason why the number 4 442 could not appear in the T column.

✶**5.** The table below refers to a certain series.

	S	N			M
Series	*Sum of series*	*The first factor of last term of series*	$N + 1$	$N + 2$	$N(N + 1)(N + 2)$
1×2	2	1	2	3	6
$1 \times 2 + 2 \times 3$	8	2			
$1 \times 2 + 2 \times 3 + 3 \times 4$		3	4	5	60
$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5$	40				
$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6$					210

(a) Study the table and then fill in the blank spaces.

(b) Express the relationship between the numbers in column S and those in column M as a formula connecting S and M.

(c) **(i)** Use your answer to part **(b)** to find the value of S when $N = 9$.

(ii) Verify this by evaluating

$$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6 + 6 \times 7 + 7 \times 8 + 8 \times 9 + 9 \times 10.$$

(d) Use your answer to part **(b)** to evaluate

(i) $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + 28 \times 29$,

(ii) $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + 420$.

(e) Suggest a formula, in terms of n, for the sum of the series

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n \times (n + 1).$$

(f) Give a simple reason why the number 6 663 could not appear in the column M.

6. The total number of diagonals (d) that can be drawn in polygons with a given number of sides (n) and the number of diagonals (v) that can be drawn from a vertex V are being investigated.

Fig. 7.105 and the table show the total number of diagonals that can be drawn and the number of diagonals that can be drawn from a vertex V in a triangle, a quadrilateral, a pentagon and a hexagon.

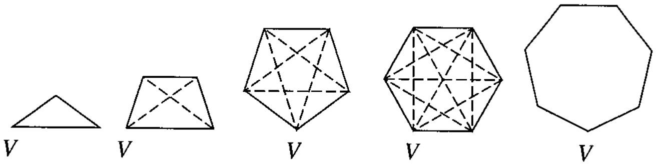


Fig. 7.105

Number of sides (n)	3	4	5	6	7	8
Number of diagonals drawn from V (v)	0	1	2	3	p	q
Total number of diagonals (d)	0	2	5	9	r	s

(a) Without drawing all the possible diagonals, or considering the number patterns, find the values of p and q in the table and explain how you find them.

(b) By drawing all the possible diagonals, or by considering the number patterns, find the values of r and s.

(c) There is a simple relationship between the number of diagonals drawn from a vertex V of a polygon with n sides.
Express this relationship as a formula connecting n and v.

(d) By studying the three rows of numbers in the table, find an equation that connects n, v and d and hence, express d in terms of n using the result in (c).

(e) Using the result in (d), find the total number of diagonals in a polygon with 30 sides.

7. Toothpicks are used to make a sequence of triangular patterns.

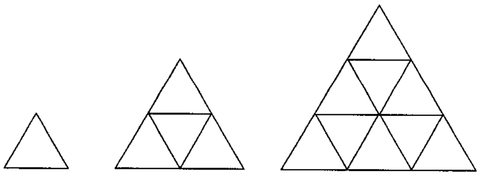


Fig. 7.106

Triangular pattern	1	2	3	4	5
Number of matchsticks used (M)	3	9	18	a	e
Number of rhombus formed with 4 toothpicks (R)	0	3	9	b	f
Number of triangles formed with 3 toothpicks (T)	1	4	9	c	g
Number of points at which 2 or more toothpicks meet (P)	3	6	10	d	h

(a) By drawing the next triangular pattern, find the values of a, b, c and d in the fourth column of the table.

(b) Without further drawing, write down the values of e, f, g and h in the fifth column of the table.

(c) Given that there are T triangles formed with 3 toothpicks in the nth triangular pattern, express T in terms of n.

(d) Form, and write down, an equation connecting the letters M, T and P.

(e) In the 11th triangular pattern, the number of points at which 2 or more toothpicks meet is 78. Find the number of triangles formed with 3 toothpicks and the number of rhombus formed with 4 toothpicks in the 12th triangular pattern.

(f) Give a simple reason why 3 335 could not possibly be a value for M.

(g) Give a simple reason why 623 could not possibly be a value for T and write down a possible value of T that is larger than 623.

8. Study the number triangle below.

1
3 5
7 9 11
13 15 17 19
21 23 a b c
d e f g 39 41

Row number	1	2	3	4	5	6
Number of terms (N)	1	2	3	4	5	6
Median of the terms (M)			9			
The first term (T)	1	3	7	13	21	
Sum of the terms (S)	1	8				

(a) By studying the triangular pattern of numbers, write down the values of a, b, c, d, e, f and g.

(b) Copy and complete the above table.

(c) Express M in terms of N and hence, find the median of the numbers in row 25.

(d) Express S in terms of N and hence, find the row in which the sum of the numbers is 46 656.

(e) There is a simple relationship between N, M and T.
Express this relationship as a formula connecting N, M and T and hence, find in terms of N, an expression for T.

(f) Using the result in (e), find
- **(i)** the first term in row 20,
- **(ii)** the row in which the first term is 1 561.

9. By considering the number patterns, complete lines (a), (b) and (c) in the table below.

	$3^4 - 16 = 1 \times 5 \times (3^2 + 4)$
	$4^4 - 16 = 2 \times 6 \times (4^2 + 4)$
	$5^4 - 16 = 3 \times 7 \times (5^2 + 4)$
(a)	$6^4 - 16 =$
(b)	$= 6 \times 10 \times (8^2 + 4)$
(c)	$n^4 - 16 =$

(d) Use your answer to part (c) and the information given below to find the three prime factors of 50 609.

$[15^2 = 225,\ 15^4 = 50\ 625.]$ (C)

10. A series of diagrams of shaded and unshaded small triangles is shown below.


Diagram 1


Diagram 2

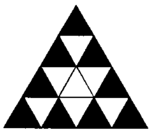


Diagram 3

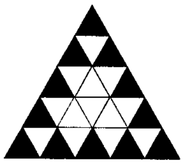


Diagram 4

The shaded triangles are those which have at least one side on the edge of the big triangle. All of the other small triangles are unshaded.

Copy the following table, which shows numbers of small triangles.

Diagram	1	2	3	4	5		n
Number of shaded triangles	3	6	9				x
Total number of triangles	4	9	16	25			y
Number of unshaded triangles	1	3	7				z

(a) Complete the column for Diagram 4 in your table.

(b) By considering the number patterns in your table,
  (i) complete the column for Diagram 5,
  (ii) find, in terms of n, expressions for x, y, and z.

(c) Find the number of unshaded small triangles in Diagram 100. *(C)*

11. Each diagram in the sequence below consists of a number of dots.

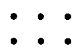


Diagram 1

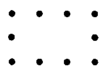


Diagram 2

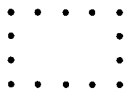


Diagram 3

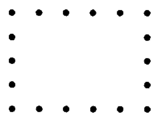


Diagram 4

(a) Draw diagram number 5 of the sequence.

(b) Copy and complete the table below.

Diagram number	1	2	3	4	5
Number of dots	6	10			

(c) By considering the number patterns, without drawing further diagrams, write down the number of dots there will be
  (i) in diagram 10,
  (ii) in diagram 500.

(d) Write down the number of the diagram that has 70 dots.

(e) The number of dots in diagram n is denoted by x. Write an equation that expresses x in terms of n. *(C)*

7.11 Locus and Constructions

Example

(a) Construct $\triangle PQR$ with $PQ = 6$ cm, $PR = 8$ cm and $RQ = 10$ cm.

(b) On the same diagram,
  (i) draw the locus of a point within $\triangle PQR$, which is 3 cm from QR,
  (ii) construct the locus of a point which is 5 cm from Q,
  (iii) construct the locus of a point equidistant from PR and RQ.

(c) *A point X lies inside the triangle PQR. The position of X is such that it is less than 3 cm from QR, more than 5 cm from Q and its distance from PR is less than its distance from RQ. Indicate clearly by shading, the region in which X must lie.*

Solution

(a) $\triangle PQR$ is constructed as shown in the diagram.

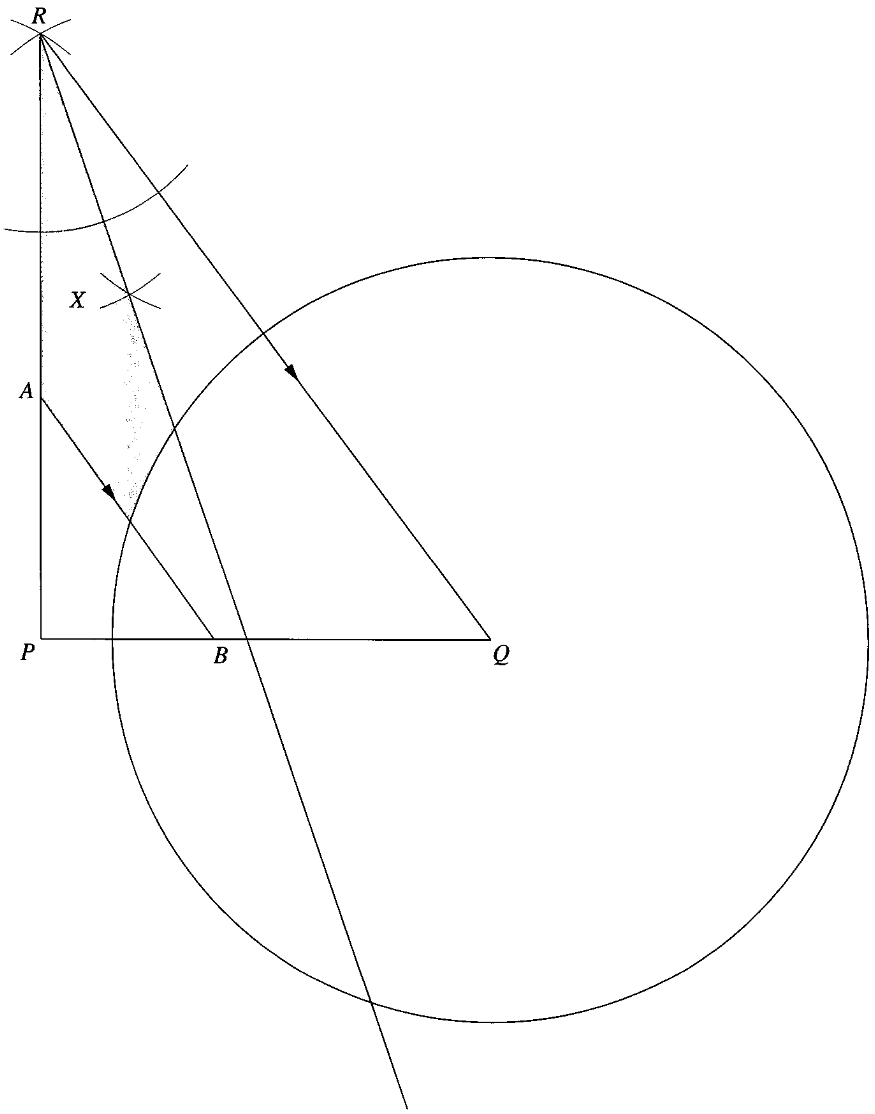


Fig. 7.107

(b) (i) The locus of a point within $\triangle PQR$ which is 3 cm from QR is the line AB parallel to QR and is 3 cm from QR.
  (ii) The locus of a point which is 5 cm from Q is the circle centre with Q and radius 5 cm.
  (iii) The locus of a point equidistant from PR and RQ is the angle bisector of PRQ.

(c) X must lie in the shaded region, as shown in the diagram.

Example 2

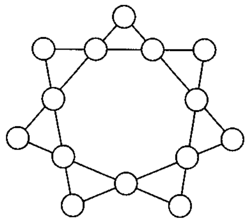
(a) Construct the quadrilateral $ABCD$ such that $AB = 7$ cm, $AD = 4$ cm, $DC = 5$ cm, $BC = 4$ cm and $B\hat{A}D = 45°$. Measure and write down the length of BD.

(b) On the same diagram, construct the locus of a point

- *(i) within the quadrilateral $ABCD$, which is equidistant from BC and CD,*
- *(ii) equidistant from the points A and B,*
- *(iii) which is 3 cm from A.*

(c) A point P lies inside the quadrilateral $ABCD$. The position of P is such that it is more than 3 cm from A, its distance from B is less than its distance from A and its distance from BC is greater than its distance from CD. Indicate clearly by shading, the region in which P must lie.

CHECK THIS OUT



Fill in the numbers from 1 to 14 in the circles so that the sum of the numbers on each side is the same.

Solution

(a) The quadrilateral $ABCD$ is constructed as shown in Fig. 7.108. The length of $BD = 5$ cm.

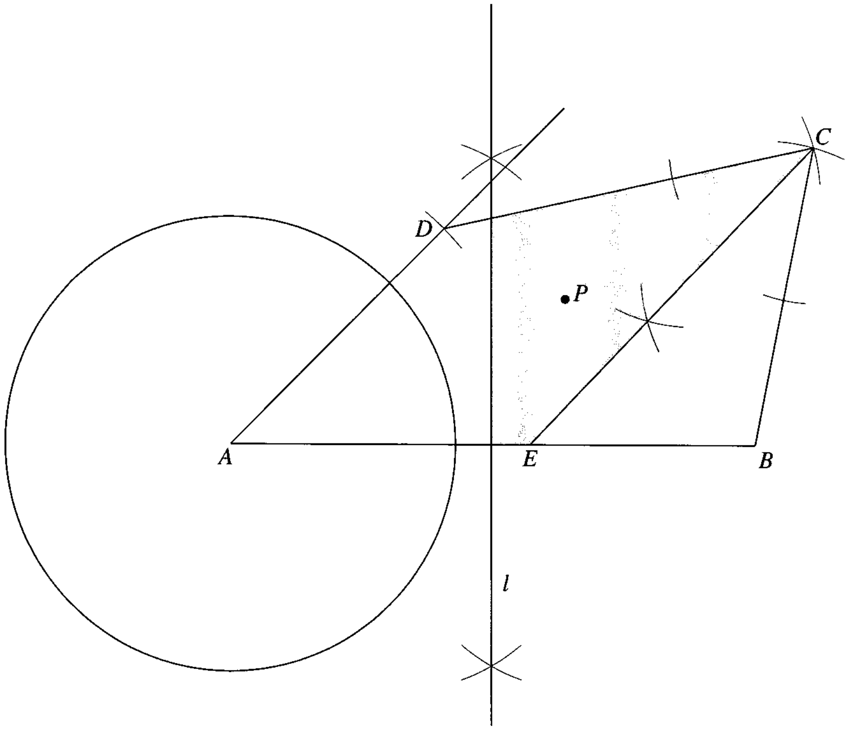


Fig. 7.108

(b) (i) The angle bisector of $B\hat{C}D$ (the line CE in the diagram) is the locus of a point within the quadrilateral $ABCD$ which is equidistant from BC and CD.

(ii) The locus of a point equidistant from A and B is the perpendicular bisector (the line l in the diagram) of AB.

(iii) The locus of a point which is 3 cm from A is a circle with centre A and radius 3 cm.

(c) P must lie in the shaded region, as shown in the diagram.

Example 3

Construct the parallelogram PQRS in which PQ = 10 cm, PS = 5 cm and $Q\hat{P}S = 50°$. Measure and write down the length of PR.

On the same diagram, construct

(a) the locus of a point A which moves so that it is equidistant from P and R,
(b) the locus of a point B which moves so that $Q\hat{B}S = 90°$.

The position of a point C, which lies inside the parallelogram PQRS, is such that $PC \geqslant RC$ and $Q\hat{C}S \geqslant 90°$.

Indicate clearly by shading, the region in which the point C must lie.

Solution

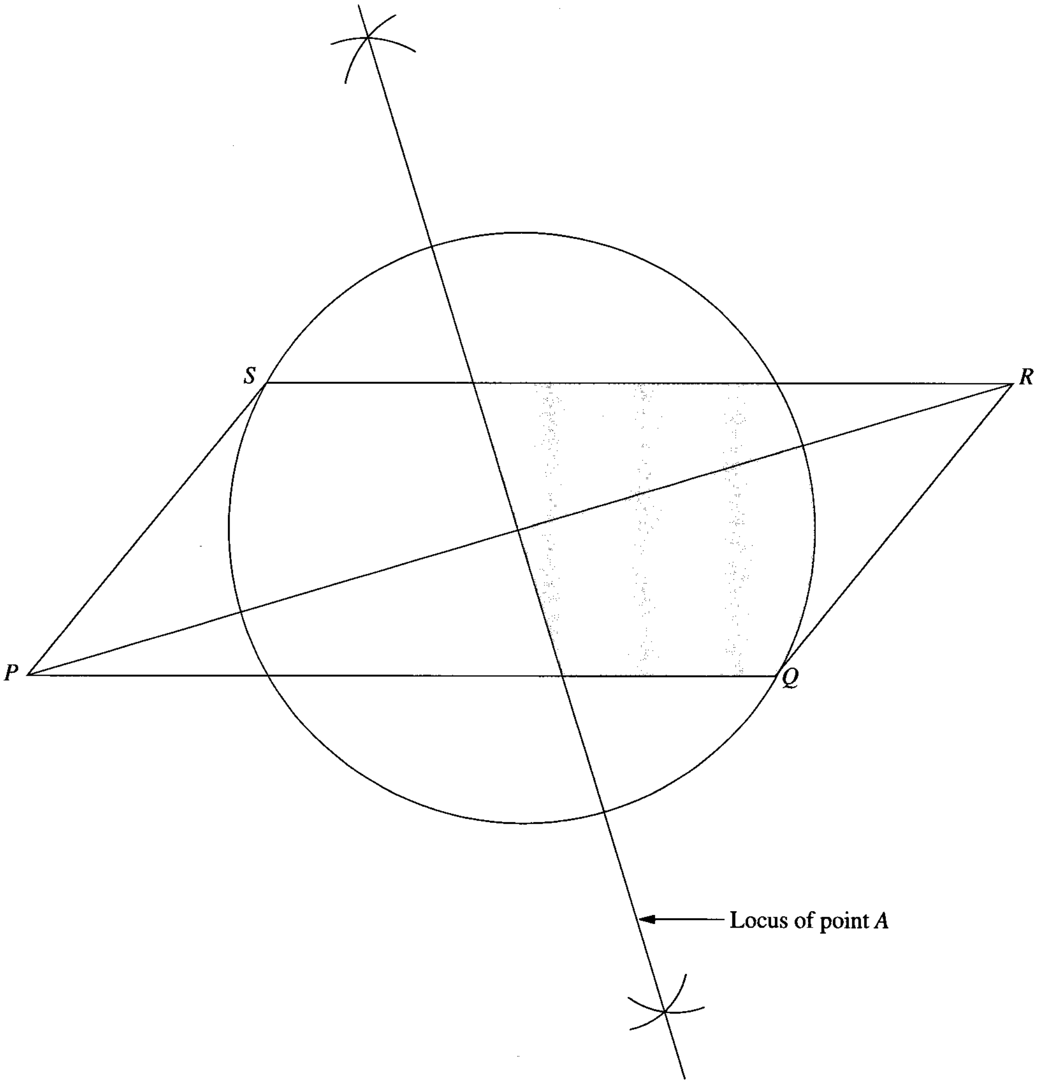


Fig. 7.109

The parallelogram *PQRS* is constructed as shown in the diagram above.

The length of $PR \approx 13.8$ cm.

(a) The locus of point A is the perpendicular bisector of PR.

(b) The locus of B is a circle with SQ as the diameter.

C lies in the shaded region.

Revision Exercise 7.11

1. A company's logo is in the form of $\triangle PQR$, where $PQ = 12$ cm, $QR = 10$ cm and $RP = 7$ cm.

(a) Construct $\triangle PQR$. Measure the largest angle in the triangle.

(b) A region Y inside the logo which is less than 5 cm from R and nearer to Q than to P is yellow in colour. On the same diagram, draw

  (i) the locus which represents all the points inside the logo which are 5 cm from R,

  (ii) the locus which represents all the points inside the logo which are equidistant from P and Q.

(c) Indicate clearly on your diagram, by shading, the region Y.

2. Three office buildings A, B and C are located at the vertices of $\triangle ABC$, where $AB = 700$ m, $BC = 600$ m and $CA = 400$ m.

(a) Using a scale of 1 cm to represent 100 m, construct a plan of $\triangle ABC$. Measure the smallest angle in the triangle.

(b) A shopping centre S is located inside $\triangle ABC$. It is 400 m from AC and equidistant from AB and BC. On the same diagram, draw

  (i) the locus which represents all the points inside the triangle which are 400 m from AC,

  (ii) the locus which represents all the points inside the triangle which are equidistant from AB and BC.

(c) Mark clearly on your diagram the position of the point S.

3. A ferry F is 600 m due east of a fishing boat B. The bearing of a lighthouse L from B is 080° and the bearing of L from F is 340°.

(a) Using a scale of 2 cm to represent 100 m, construct a plan of $\triangle FLB$. Measure and find the distance between the fishing boat and the lighthouse.

(b) A passenger on the ferry observes a sailing yacht Y in $\triangle FLB$. He estimates the possible position of the yacht. According to him, the yacht is not more than 350 m from the lighthouse, at least 200 m from LF and nearer to the ferry than the fishing boat. On the same diagram, draw the locus which represents all the points inside $\triangle FLB$

  (i) which are 350 m from L,

  (ii) which are 200 m from LF,

  (iii) which are equidistant from B and F.

(c) Indicate clearly on your diagram by shading, the region which contains the possible positions of Y.

4. **(a)** Construct $\triangle PQR$ with base $PQ = 8$ cm, $P\hat{Q}R = 42°$ and $QR = 6$ cm. Measure and write down the length of RP.

(b) On the same diagram, construct the point A, such that $Q\hat{R}A = 90°$ and A is equidistant from P and Q. Measure PA and state this length correct to the nearest millimetre.

5. Construct the parallelogram $PQRS$ in which $PQ = 8$ cm, $PS = 4$ cm and $Q\hat{P}S = 70°$. Measure and write down the length of PR.

On the same diagram, construct the locus of

- **(a)** a point A which moves so that it is equidistant from P and R,
- **(b)** a point B which moves so that $Q\hat{B}S = 90°$.

The position of a point C, which lies inside the parallelogram $PQRS$, is such that $PC \leqslant RC$ and $Q\hat{C}S \geqslant 90°$. Indicate clearly, by shading, the region in which the point C must lie.

6. Construct a parallelogram $OXYZ$ in which $OX = 9.5$ cm, $OZ = 5.5$ cm and $X\hat{O}Z = 120°$. Measure and write down the length of OY.

On the same diagram, construct the locus of

- **(a)** a point P which moves so that it is equidistant from OX and XY,
- **(b)** a point Q which moves so that $O\hat{Q}Y = 90°$.

The position of a point R, which lies inside the parallelogram is such that its distance from OX is less than or equal to its distance from XY and $O\hat{R}Y \leqslant 90°$. Indicate clearly, by shading, the region in which the point R must lie.

7. **(a)** Construct $\triangle PQR$ in which $PQ = 10$ cm, $QR = 8$ cm and $RP = 7$ cm. Construct the locus of a point which is equidistant from PQ and PR.

(b) Construct the locus of a point X, inside $\triangle PQR$, such that the area of $\triangle PQX$ is 10 cm^2.

(c) Y is the point inside $\triangle PQR$ such that it is equidistant from PQ and PR and the area of $\triangle PQY$ is 10 cm^2. Mark the position of Y and measure and write down the length RY.

8. Draw $\triangle ABC$ in which $AB = 9$ cm, $B\hat{A}C = 72°$ and $A\hat{B}C = 52°$. Construct the locus of a point which is equidistant from A and C. Construct the locus of point X, inside $\triangle ABC$, such that the area of $\triangle ABX$ is $11\frac{1}{4}$ cm^2.

Y is the point inside $\triangle ABC$ such that $AY = CY$ and the area of $\triangle ABY$ is $11\frac{1}{4}$ cm^2. Mark the position of Y, measure and write down the length CY.

✶9. **(a)** Construct the quadrilateral $ABCD$ in which $AB = 7$ cm, $BC = 5$ cm, $D\hat{A}C = 90°$, $A\hat{B}C = 105°$ and $B\hat{C}D = 60°$. Measure and write down the length of CD.

(b) On the same diagram,

- **(i)** draw the locus of a point within the quadrilateral $ABCD$ which is 4 cm from AD,
- **(ii)** construct the locus of a point equidistant from B and C,
- **(iii)** construct the locus of a point equidistant from BC and CD.

(c) A point P lies inside the quadrilateral $ABCD$. The position of P is such that it is more than 4 cm from AD, its distance from B is less than its distance from C and its distance from CD is more than its distance from BC. Indicate clearly by shading, the region in which P must lie.

✶10. **(a)** Draw accurately the quadrilateral $PQRS$ in which $PQ = 8$ cm, $QR = 5$ cm, $S\hat{P}Q = 62°$, $P\hat{Q}R = 124°$ and $Q\hat{R}S = 75°$. Measure and write down the length of PS.

(b) On the same diagram,

- **(i)** draw the locus of a point which is 6 cm from R,
- **(ii)** construct the locus of a point equidistant from R and S,
- **(iii)** construct the locus of a point equidistant from PQ and QR.

(c) A point X lies inside the quadrilateral $PQRS$. The position of P is such that it is 6 cm from R, its distance from R is more than its distance from S and its distance from QR is less than its distance from PQ. Indicate clearly, by shading, the region in which X must lie.

11. **Answer the whole of this question on a sheet of plain paper.**

(a) Draw accurately the triangle ABC with base $AB = 7$ cm, $C\hat{A}B = 82°$ and

AC = 5 cm. Measure and write down the length of BC.

(b) On the same diagram,

- **(i)** construct the locus points which are 2.4 cm from A,
- **(ii)** draw the locus of points (on the same side of BC from A) within the triangle which are 2.4 cm from BC,
- **(iii)** construct the circle of radius 2.4 cm which passes through A and touches BC, and whose centre is inside triangle ABC,
- **(iv)** construct the locus of points equidistant from AC and BC.

(c) A point P lies inside the triangle ABC. The point P is such that it is less than 2.4 cm from BC but more than 2.4 cm from A. Its distance from BC is more than its distance from AC. The point P also lies inside the circle passing through A. Indicate clearly, by shading, the region in which P must lie.

12. An amusement park is in the shape of a quadrilateral $PQRS$ where PQ = 115 m, QR = 80 m, QS = 135 m, $Q\hat{P}S$ = 80° and $P\hat{Q}R$ = 65°.

(a) Using a scale of 1 cm to represent 10 m, construct a plan of the quadrilateral $PQRS$. Measure and write down $Q\hat{R}S$.

(b) A ticket booth X is to be erected outside $PQRS$ so that $PX = QX$ and area of $\triangle PQS = \triangle PQX$. Draw two loci to locate the exact position of X.

(c) A security post Y is to be located inside $PQRS$ so that it is always equidistant from PQ, PS and QR. Draw the locus of points which

- **(i)** represents all points equidistant from PQ and PS,
- **(ii)** represents all points equidistant from PQ and QR.

Hence, mark clearly on your diagram the position of the point Y.

7.12 Vectors

Example 1

Given that $\boldsymbol{a} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ *and* $\boldsymbol{b} = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$,

(a) express $5\boldsymbol{a} - \frac{1}{4}\boldsymbol{b}$ *as a column vector,*

(b) find $|\boldsymbol{b}|$ *and* $\left|5\boldsymbol{a} - \frac{1}{4}\boldsymbol{b}\right|$.

Solution

(a) $$5\mathbf{a} - \frac{1}{4}\mathbf{b} = 5\begin{pmatrix} 5 \\ 3 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} -4 \\ 8 \end{pmatrix}$$
$$= \begin{pmatrix} 25 \\ 15 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 26 \\ 13 \end{pmatrix}$$

(b) $$|\mathbf{b}| = \sqrt{(-4)^2 + 8^2}$$
$$= \sqrt{16 + 64}$$
$$= \sqrt{80}$$
$= 8.94$ units (correct to 2 decimal places)

$$\left|5\mathbf{a} - \frac{1}{4}\mathbf{b}\right| = \sqrt{(26)^2 + (13)^2}$$
$$= \sqrt{845}$$
$= 29.07$ units (correct to 2 decimal places)

Example 2

It is given that P is the point (5, 6), Q is the point (13, 10) and X is the point on $\overrightarrow{PQ}$ such that $\overrightarrow{PX} = \frac{1}{3}\overrightarrow{XQ}$.

(a) Express as column vectors the following:

(i) $\overrightarrow{PQ}$,   (ii) $\overrightarrow{PX}$,

(iii) the position vector of X relative to the origin O.

(b) If O, P and Q are three of the vertices of a parallelogram, find the coordinates of the two possible positions of the fourth vertex.

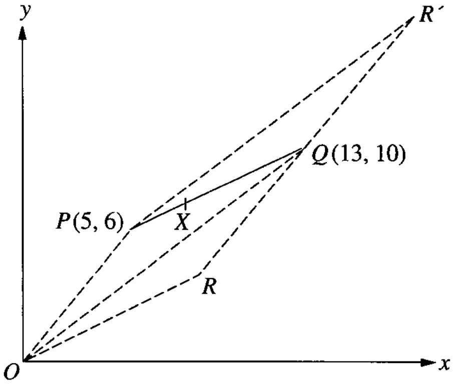


Fig. 7.110

Solution

(a) We have $\overrightarrow{OP} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 13 \\ 10 \end{pmatrix}$.

(i) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \begin{pmatrix} 13 \\ 10 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

(ii) Given that $\overrightarrow{PX} = \frac{1}{3}\overrightarrow{XQ}$, we have $\overrightarrow{PX} = \frac{1}{4}\overrightarrow{PQ} = \frac{1}{4}\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(iii) $\overrightarrow{OX} = \overrightarrow{OP} + \overrightarrow{PX}$

$$\overrightarrow{OX} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

∴ the position vector of X relative to the origin O is $\overrightarrow{OX} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$.

(b) In Fig. 7.110, R and R' are the two possible positions of the fourth vertex.

In parallelogram $OPQR$, $\overrightarrow{OR} = \overrightarrow{PQ} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$.

In parallelogram $OPR'Q$, $\overrightarrow{OR'} = \overrightarrow{OP} + \overrightarrow{OQ} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 13 \\ 10 \end{pmatrix} = \begin{pmatrix} 18 \\ 16 \end{pmatrix}$

∴ the coordinates of the two possible positions of the fourth vertex are (8, 4) and (18, 16).

Example 3

In Fig. 7.111(a), $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.

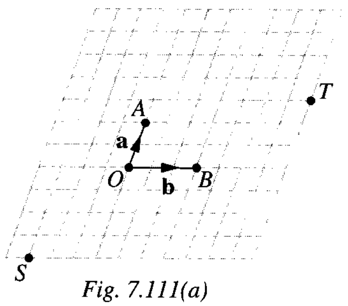


Fig. 7.111(a)

(a) Mark clearly on the diagram,

- (i) the point P, such that $\overrightarrow{OP} = 3\boldsymbol{a} + 2\boldsymbol{b}$,
- (ii) the point Q, such that $\overrightarrow{OQ} = 2\boldsymbol{b} - \boldsymbol{a}$,
- (iii) the point R, such that $\overrightarrow{OR} = -(\boldsymbol{b} - 3\boldsymbol{a})$.

(b) Write down $\overrightarrow{OS}$ and $\overrightarrow{OT}$ in terms of **a** and **b**.

Solution

(a)

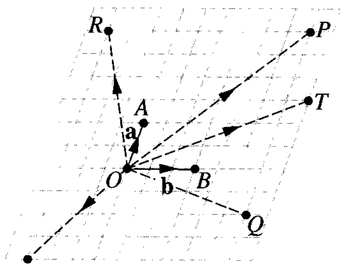


Fig. 7.111(b)

(b) $\overrightarrow{OS} = -2\mathbf{a} - \mathbf{b}$

$= -(2\mathbf{a} + \mathbf{b})$

$\overrightarrow{OT} = 1\frac{1}{2}\mathbf{a} + 2\frac{1}{3}\mathbf{b}$

$= \frac{3}{2}\mathbf{a} + \frac{7}{3}\mathbf{b}$

$= \frac{1}{6}(9\mathbf{a} + 14\mathbf{b})$

Example 4

In the diagram, BC = 3BD and CA = 4FA. E is the mid-point of DA. $\overrightarrow{BD} = \boldsymbol{p}$ and $\overrightarrow{FA} = \boldsymbol{q}$.

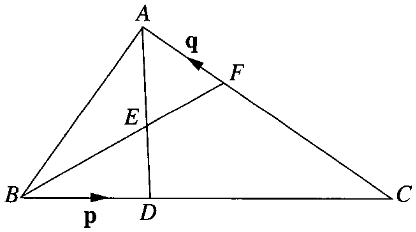


Fig. 7.112

(a) Express, as simply as possible, in terms of **p** and/or **q**,

- (i) $\overrightarrow{DC}$,
- (ii) $\overrightarrow{DA}$,
- (iii) $\overrightarrow{DE}$.

(b) Show that $\overrightarrow{BE} = 2(\boldsymbol{p} + \boldsymbol{q})$.

(c) Express $\overrightarrow{BF}$ as simply as possible, in terms of **p** and **q**.

(d) Calculate the value of

- (i) $\frac{BE}{BF}$,
- (ii) $\frac{\text{area } ABE}{\text{area } ABF}$,
- (iii) $\frac{\text{area } ABE}{\text{area } ABC}$.

Solution

(a) (i) $DC = 2BD$ $(BC = 3BD)$

$\therefore \quad \overrightarrow{DC} = 2\mathbf{p}$

(ii) $CA = 4FA$

$\therefore \quad \overrightarrow{CA} = 4\mathbf{q}$

$$\overrightarrow{DA} = \overrightarrow{DC} + \overrightarrow{CA}$$
$$= 2\mathbf{p} + 4\mathbf{q}$$
$$= 2(\mathbf{p} + 2\mathbf{q})$$

(iii) $DE = \frac{1}{2}DA$ (E is the mid-point of DA)

$$\therefore \quad \overrightarrow{DE} = \frac{1}{2} \times 2(\mathbf{p} + 2\mathbf{q})$$
$$= \mathbf{p} + 2\mathbf{q}$$

(b)
$$\overrightarrow{BE} = \overrightarrow{BD} + \overrightarrow{DE}$$
$$= \mathbf{p} + \mathbf{p} + 2\mathbf{q}$$
$$= 2\mathbf{p} + 2\mathbf{q}$$
$$= 2(\mathbf{p} + \mathbf{q})$$

(c)
$$\overrightarrow{BF} = \overrightarrow{BC} + \overrightarrow{CF}$$
$$= 3\mathbf{p} + 3\mathbf{q}$$
$$= 3(\mathbf{p} + \mathbf{q})$$

(d) (i) From (b) and (c), we have $\overrightarrow{BE} = \frac{2}{3}\overrightarrow{BF}$.

$$\therefore \quad \frac{BE}{BF} = \frac{2}{3}$$

(ii)
$$\frac{\text{area } ABE}{\text{area } ABF} = \frac{BE}{BF}$$
$$= \frac{2}{3}$$

(iii)
$$\frac{\text{area } ABF}{\text{area } ABC} = \frac{1}{4}$$
$$= \frac{3}{12}$$
$$\therefore \quad \frac{\text{area } ABE}{\text{area } ABC} = \frac{2}{12}$$
$$= \frac{1}{6}$$

★Revision Exercise 7.12

1. It is given that P is the point (–2, –1), Q is the point (4, 2) and R is the point (2, 6). Express as column vectors the position vector of

- **(a)** the point M, which is the mid-point of QR,
- **(b)** the point N on PM such that $4\overrightarrow{PN} = \overrightarrow{NM}$.

2. In the quadrilateral $OPQR$, the points M, N, X and Y are the mid-points of OP, PQ, QR and RO respectively. Given that $\overrightarrow{OP} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ and $\overrightarrow{OR} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, find the vectors $\overrightarrow{PQ}$, $\overrightarrow{RQ}$, $\overrightarrow{MN}$ and $\overrightarrow{XY}$. State the geometrical facts concerning MN and XY.

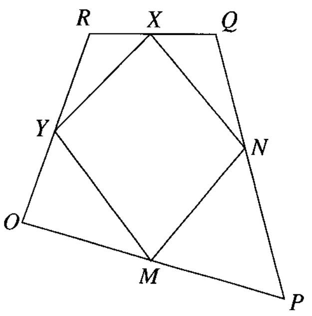


Fig. 7.113

3. The vertices of the quadrilateral $ABCD$ has coordinates $A(1, 2)$, $B(2, 4)$, $C(4, 6)$ and $D(6, 7)$.

   (a) Express $\overrightarrow{BC}$ and $\overrightarrow{AD}$ as column vectors. State the geometrical relationship between $\overrightarrow{BC}$ and $\overrightarrow{AD}$ and write down the numeral value of the ratio $\frac{BC}{AD}$.

   (b) Express $\overrightarrow{AB}$ and $\overrightarrow{CD}$ as column vectors and show that $|\overrightarrow{AB}| = |\overrightarrow{CD}|$.

4. It is given that P is the point $(0, 2)$, Q is the point $(8, 0)$ and $\overrightarrow{QR} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$. Find the following:

   (a) $|\overrightarrow{PQ}|$,

   (b) the coordinates of the point R,

   (c) $|\overrightarrow{PR}|$.

5. Given that $\mathbf{s} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$,

   (a) express $2\mathbf{s} - \frac{1}{5}\mathbf{t}$ as a column vector,

   (b) find the following:

   (i) $|\mathbf{t}|$ **(ii)** $\left|2\mathbf{s} - \frac{1}{5}\mathbf{t}\right|$

6. Given that $\overrightarrow{OP} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} m \\ 0 \end{pmatrix}$, find the following:

   (a) $|\overrightarrow{OP}|$

   (b) the value of m if $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$

7. $\overrightarrow{PQ} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix} 10 \\ y \end{pmatrix}$ and $\overrightarrow{PS} = \begin{pmatrix} x \\ 8 \end{pmatrix}$.

   (a) Given that R lies on PQ produced, write down the value of y.

   (b) Given that PS is perpendicular to PQ, calculate the value of x.

8. Given that A is the point $(1, 2)$, $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$ and M is the mid-point of BC, find

   (a) $\overrightarrow{BC}$,

   (b) $\overrightarrow{AM}$,

   (c) the coordinates of the point such that $ABPC$ is a parallelogram.

9. It is given that $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} p \\ q \end{pmatrix}$.

   (a) Find $|\mathbf{b}|$.

   (b) Express $3\mathbf{a} + 2\mathbf{b}$ as a column vector.

   (c) Given that $2\mathbf{a} - \mathbf{b} = 2\mathbf{c}$, find the value of p and the value of q.

10. It is given that $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, $\overrightarrow{QR} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} k \\ 12.5 \end{pmatrix}$.

    (a) Express as a column vector $2\overrightarrow{PQ} + 3\overrightarrow{QR}$.

    (b) Given that RS is parallel to $\overrightarrow{PQ}$, find the value of k.

    (c) Find $|\overrightarrow{PR}|$, giving your answer correct to the nearest whole number.

11. It is given that $\overrightarrow{PQ} = \begin{pmatrix} -10 \\ 24 \end{pmatrix}$.

    (a) Calculate $|\overrightarrow{PQ}|$.

    (b) Given that P is the point $(8, 19)$, find the coordinates of the point Q.

    (c) It is given that RS is parallel to PQ and it is one quarter as long as PQ. Express $\overrightarrow{RS}$ as a column vector.

12. On Fig. 7.114, A is the point (–3, 4) and O is the origin.

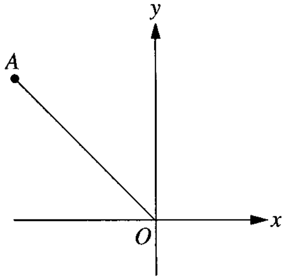


Fig. 7.114

(a) The point B lies on OA produced. Given that $\overrightarrow{OB} = 3\,\overrightarrow{OA}$, express $\overrightarrow{OB}$ as a column vector.

(b) C is the point (2, 16). Express $\overrightarrow{AC}$ as a column vector and find $|\overrightarrow{AC}|$.

(c) D is the reflection of A in the line $y = x$. Express $\overrightarrow{OD}$ as a column vector.

(d) OA is rotated in an anticlockwise direction about O so that A is mapped onto E, where E is a point on the negative x-axis.

Express $\overrightarrow{OE}$ as a column vector.

13. $PQRS$ is a parallelogram. The point T, on SR, is such that $TR = \frac{2}{3}SR$. The lines PT and QR are produced to meet at U.

$\overrightarrow{PQ} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $\overrightarrow{QR} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

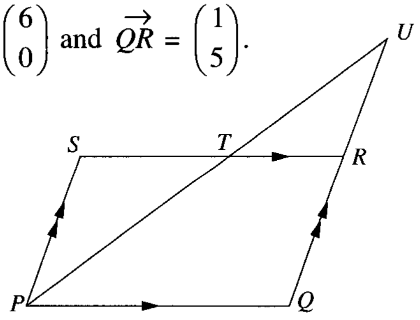


Fig. 7.115

(a) Find $|\overrightarrow{QR}|$, giving your answer correct to the nearest whole number.

(b) Express each of the following as a column vector:

(i) $\overrightarrow{SP}$ **(ii)** $\overrightarrow{ST}$

(iii) $\overrightarrow{RT}$ **(iv)** $\overrightarrow{UT}$

14. The quadrilateral $PQRS$ is such that $\overrightarrow{PQ} = 3\mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$ and $\overrightarrow{RS} = -2\mathbf{a}$.

(a) What is the special name given to the quadrilateral $PQRS$?

(b) Express $\overrightarrow{SP}$ in terms of **a** and **b**, giving your answer in the simplest form.

15. In the diagram, P is (0, 2), Q is (3, 3) and O is the origin.

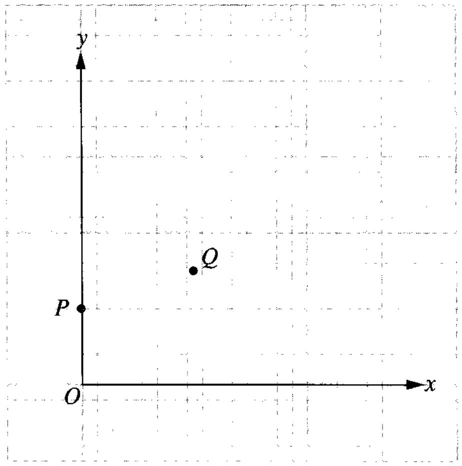


Fig. 7.116

(a) Express $\overrightarrow{PQ}$ as a column vector.

(b) Calculate the coordinates of the point R, where $\overrightarrow{QR} = 2\,\overrightarrow{PQ}$.

(c) It is given that $\overrightarrow{PS} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$.

(i) Calculate the length of PS.

(ii) Write down the gradient of the line PS.

(iii) Write down the equation of the line PS.

(iv) The point T lies on PS and TQ is parallel to the y-axis. Calculate the coordinates of T.

16. In Fig. 7.117, $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AD} = \mathbf{q}$ and X is the point on DB such that $DX = \frac{1}{4}DB$.

(a) Given that DC is parallel to AB and that it is three-quarters as long as AB, express $\overrightarrow{CD}$ in terms of **p** and/or **q**.

(b) Express the following, as simply as possible, in terms of **p** and/or **q**.

(i) $\overrightarrow{AC}$ (ii) $\overrightarrow{DB}$ (iii) $\overrightarrow{AX}$

(c) Given further that $\overrightarrow{AX} = h\mathbf{p} + k\mathbf{q}$, and that Y is the point such that $\overrightarrow{AY} = h\mathbf{p}$, mark and label the point Y on the diagram.

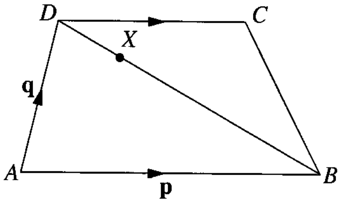


Fig. 7.117

17. $PQRS$ is a quadrilateral in which $\overrightarrow{PQ} = 3\mathbf{a}$, $\overrightarrow{PS} = 3\mathbf{b}$ and $\overrightarrow{QR} = 9\mathbf{b}$.

(a) Sketch the quadrilateral $PQRS$.

(b) Consider the pair of sides PS and QR. Write down two important facts about this pair of sides.

(c) Express $\overrightarrow{SR}$, as simply as possible, in terms of **a** and/or **b**.

(d) The sides QP and RS, when produced, meet at A. Express $\overrightarrow{AP}$, as simply as possible, in terms of **a** and/or **b**.

★18. In Fig. 7.118, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$ and M is the mid-point of QR.

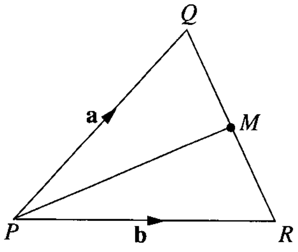


Fig. 7.118

(a) Express the following as simply as possible, in terms of **a** and/or **b**.

(i) $\overrightarrow{QR}$ (ii) $\overrightarrow{QM}$ (iii) $\overrightarrow{PM}$

(b) Given further that $\overrightarrow{PM} = \lambda\mathbf{a} + \mu\mathbf{b}$, and N is the point such that $\overrightarrow{PN} = \mu\mathbf{b}$, mark and label the point N on the diagram.

(c) The point S lies on PM produced. Given that $\overrightarrow{PS} = 2\,\overrightarrow{PM}$, express the following as simply as possible, in terms of **a** and/or **b**.

(i) $\overrightarrow{PS}$ (ii) $\overrightarrow{QS}$ (iii) $\overrightarrow{RS}$

State what sort of quadrilateral $PRSQ$ is.

19. In Fig. 7.119, $\overrightarrow{OS} = \mathbf{s}$ and $\overrightarrow{OT} = \mathbf{t}$.

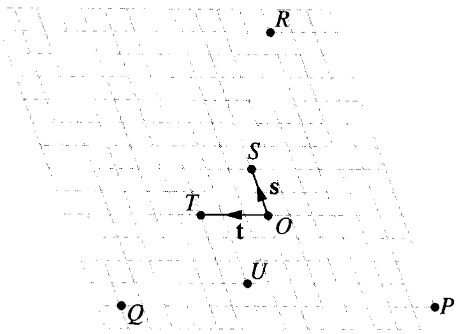


Fig. 7.119

(a) Mark clearly on the diagram

(i) the point A, such that $\overrightarrow{OA} = 2\mathbf{s} + 3\mathbf{t}$,

(ii) the point B, such that $\overrightarrow{OB} = -2(\mathbf{t} - \mathbf{s})$,

(iii) the point C, such that $\overrightarrow{OC} = 3\mathbf{t} - \mathbf{s}$.

(b) Write down $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ and $\overrightarrow{OU}$ in terms of **s** and **t**.

20. In the diagram, $PQRS$ is a parallelogram, M is the mid-point of QS, N is the mid-point of QM and L is the point on QR such that $QL = \frac{2}{3}QR$.

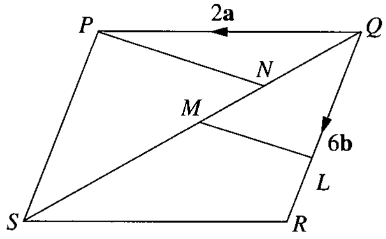


Fig. 7.120

(a) Given that $\overrightarrow{QP} = 2\mathbf{a}$ and $\overrightarrow{QR} = 6\mathbf{b}$, express, as simply as possible, in terms of **a** and/or **b**,

(i) $\overrightarrow{QS}$, **(ii)** $\overrightarrow{QM}$,

(iii) $\overrightarrow{PN}$, **(iv)** $\overrightarrow{ML}$.

(b) What do your answers to **(a)(iii)** and **(a)(iv)** tell you about *PN* and *ML*?

(c) What is the special name given to the quadrilateral *PMLN*?

(d) Write down the value of each of the following:

(i) $\dfrac{\text{Area of } \triangle PNQ}{\text{Area of } \triangle PSN}$ **(ii)** $\dfrac{\text{Area of } \triangle PSN}{\text{Area of } \triangle QML}$

7.13 Probability

What is the probability that all the 12 children in a family are daughters?

Example 1

A bag contains 52 cards numbered from 1 to 52 inclusive. A card is chosen at random from the bag. Write down the probability that the number on the chosen card will

(a) have a single digit, *(b) be a perfect square,*
(c) have at least one figure 4,
(d) ***not*** *be divisible by either 3 or 4.*

Solution

(a) There are 9 single digit numbers from 1 to 52 inclusive.

$\therefore$ P(number on the card chosen will contain a single digit) = $\dfrac{9}{52}$.

(b) The numbers from 1 to 52 inclusive which are perfect squares are 1, 4, 9, 16, 25, 36, 49.

$\therefore$ P(number on the card chosen will be a perfect square) = $\dfrac{7}{52}$.

(c) The numbers from 1 to 52 inclusive which contains at least one figure 4 are 4, 14, 24, 34, 40, 41,42, 43, 44, 45, 46, 47, 48, 49.

$\therefore$ P(number on the card chosen will contain at least one figure 4)

$$= \frac{14}{52}$$
$$= \frac{7}{26}.$$

(d) No. of numbers from 1 to 52 that are
divisible by 3 is 17 ($3 \times 17 = 51 < 52$),
divisible by 4 is 13 ($4 \times 13 = 52$),
divisible by 12 is 4 ($12 \times 4 = 48 < 52$).
Both the 17 numbers and the 13 numbers that are divisible by 3 and 4 respectively include the 4 numbers that are divisible by 12, namely 12, 24, 36 and 48. By simply adding 17 and 13 will result in double counting of the numbers that are divisible by 12.

$\therefore$ there are 17 + 13 − 4 = 26 numbers which are divisible by either 3 or 4.

P(number on the card chosen will *not* be divisible by either 3 or 4)

$= 1 - \frac{26}{52}$

$= \frac{1}{2}$

Example 2

X = {G, E, R, M, A, N} and Y = {E, N, G, L, I, S, H}.

(a) If one element is selected at random from X, write down the probability that it is a vowel.

(b) If one element is selected at random from each set, construct a possibility diagram to show all possible outcomes. How many possible outcomes are there altogether?

(c) Using the diagram, find the probability that both elements are
(i) vowels, (ii) the same.

Solution

(a) There are two vowels, in the set X.

$\therefore$ P(selecting a vowel from X) $= \frac{2}{6}$

$= \frac{1}{3}$

(b) (i) From the possibility diagram, there are 6 × 7 = 42 possible outcomes.

(ii) There are 4 outcomes, *EI*, *EE*, *AI* and *AE* (shown circled) involving 2 vowels.

$\therefore$ P(both elements are vowels)

$= \frac{4}{42}$

$= \frac{2}{21}$

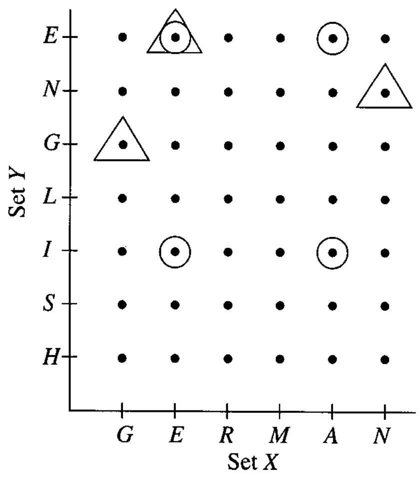
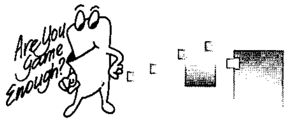


Fig. 7.121

(iii) There are 3 outcomes, *GG*, *EE* and *NN* (shown enclosed in triangles) in which two elements are the same.

P(both elements are the same) $= \frac{3}{42}$

$= \frac{1}{14}$



If you bet \$1 'small' on a 4D number, you may win the first prize of \$3 000, the second prize of \$2 000 or the third prize of \$1 000 each with the probability of $\frac{1}{10\,000}$. However, you stand to lose your bet of \$1 with the probability of $\frac{9\,997}{10\,000}$. The table below displays your possible gains.

Gain (in dollars)	*Probability*
3 000	$\frac{1}{10\,000}$
2 000	$\frac{1}{10\,000}$
1 000	$\frac{1}{10\,000}$
−1	$\frac{9\,997}{10\,000}$

Note: *−1 indicates a loss of your bet of \$1.*

Multiplying each gain amount by the corresponding probability and summing up, a value which we call the expected gain is obtained.

In this case, the expected gain $= \frac{1}{10\,000}(3\,000 + 2\,000 + 1\,000) + (-1) \times \frac{9\,997}{10\,000} = -0.40$ *(correct to two decimal places)*

Example 3

On a mini-market shelf there are 15 boxes containing white table-tennis balls and 6 boxes containing orange table-tennis balls. A boy picks any two boxes at random. Use a tree diagram to determine the probability of picking

(a) two boxes containing white table-tennis balls,
(b) two boxes containing orange table-tennis balls,
(c) a box containing white table-tennis balls followed by one containing orange table-tennis balls,
(d) two boxes which are different.

The negative value of the expected gain can be interpreted as follows:

you expect to lose 40 cents for every $1 you bet in the long run, i.e. if you bet often enough.

Find out the expected gain if you bet $1 'big' on a 4D number and interpret the value obtained.

Solution

Let W denote the event of picking a box containing white table-tennis balls and O denote the event of picking a box containing orange table-tennis balls. The tree diagram is shown below.

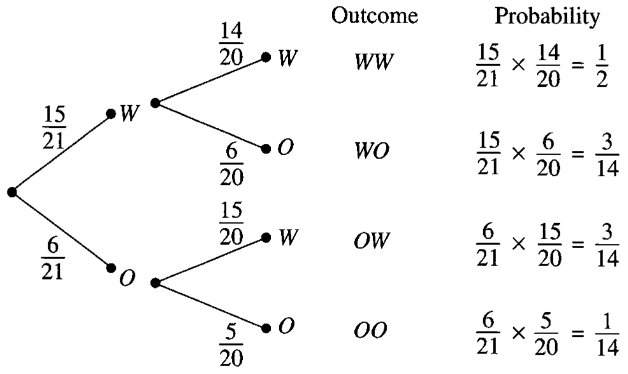


Fig. 7.122

(a) P(two boxes of white table-tennis balls) = P(WW)

$$= \frac{1}{2}$$

(b) P(two boxes of orange table-tennis balls) = P(OO)

$$= \frac{1}{14}$$

(c) P(one box of white table-tennis balls followed by one of orange table-tennis balls) = P(WO)

$$= \frac{3}{14}$$

(d) P(two boxes with table-tennis balls of different colours)
= P(WO or OW)
= P(WO) + P(OW)

$$= \frac{3}{14} + \frac{3}{14}$$

$$= \frac{3}{7}$$

Example 4

A bag of fruits contained 9 oranges and 6 apples. John took a fruit, selected at random, from the bag and ate it. Mary then took a fruit, selected at random, from the bag. Expressing each answer as a fraction in its simplest form, find the probability that

(a) John took an orange,
(b) John took an orange and Mary took an apple,
(c) both John and Mary took an apple,
(d) Mary took an apple.

Solution

(a) $P(\text{John took an orange}) = \frac{9}{15}$

$= \frac{3}{5}$

(b) $P(\text{John took an orange and Mary took an apple}) = \frac{9}{15} \times \frac{6}{14}$

$= \frac{9}{35}$

(c) $P(\text{both John and Mary took an apple}) = \frac{6}{15} \times \frac{5}{14}$

$= \frac{1}{7}$

(d) $P(\text{Mary took an apple}) = \frac{9}{15} \times \frac{6}{14} + \frac{6}{15} \times \frac{5}{14}$

$= \frac{2}{5}.$

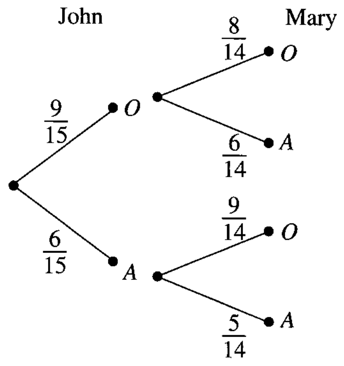


Fig. 7.123

Revision Exercise 7.13

1. Each of the letters of the word '*EXCELLENCE*' is printed on a separate card and the cards are put into a box. Two cards are drawn at random from the box, one at a time, without replacement. Find the probability that
   - **(a)** the two cards drawn both contain vowels,
   - **(b)** the first card drawn contains a vowel,
   - **(c)** the second card drawn contains a vowel,
   - **(d)** one of the two cards drawn contains a vowel,
   - **(e)** the letters on the two cards drawn are the same.

2. Two bags, X and Y, contain red and black marbles only. Bag X contains 4 red marbles and 2 black marbles. Bag Y contains 3 red marbles and 1 black marble.
   A bag is to be chosen by flipping a coin. If the coin shows heads, bag X is to be chosen, otherwise bag Y is chosen. A marble is then to be selected at random from the chosen bag.
   - **(a)** Draw a tree diagram to illustrate this experiment.
   - **(b)** Calculate the probability of selecting a red marble.

3. A bag contains 8 beads, 5 of which are green and 3 are blue. A bead is selected at random and then replaced. A second bead is then selected at random. Draw a tree diagram to show all the possible outcomes. Hence, find the probability that

- **(a)** both beads are green,
- **(b)** both beads are blue,
- **(c)** the two beads are of different colours.

4. An urn contains 2 red discs, 3 white discs and 5 black discs. Two discs are drawn at random from the urn, one at a time, without replacement. Calculate the probability that

- **(a)** two black discs are drawn,
- **(b)** two white discs are drawn,
- **(c)** one red disc and one black disc are drawn,
- **(d)** two discs of the same colour are drawn.

5. A small box contains chocolates of which 3 contain hazelnuts, 7 contain almonds and 2 contain sultanas.

- **(a)** One chocolate is picked at random from the box. Find the probability of picking
  - **(i)** one containing almonds,
  - **(ii)** one which does not contain sultanas.
- **(b)** Two chocolates are picked at random, one at a time, without replacement. Calculate the probability of picking
  - **(i)** both containing almonds,
  - **(ii)** one containing almonds and one containing hazelnuts,
  - **(iii)** both without sultanas.

6. In an election with only two candidates, x voters voted for candidate A and 36 voters voted for candidate B. If a voter is to be chosen at random, write down the expression for the probability that a voter who voted for candidate A will be chosen. Given that the probability of a voter who voted for candidate A will be chosen is $\frac{2}{5}$, find the value of x.

7. In a class of 36 boys, 18 choose soccer, 14 choose basketball and 4 choose badminton as the game they would participate during the school terms.

- **(a)** If any one boy is chosen from the class, find the probability that
  - **(i)** he does not play soccer,
  - **(ii)** he does not play basketball,
  - **(iii)** he plays badminton,
  - **(iv)** he does not play badminton.
- **(b)** If two boys are chosen at random from the class, find the probability that
  - **(i)** both boys play basketball,
  - **(ii)** the two boys play different games.

8. A survey is carried out on a group of 300 elderly people, 100 of whom are men. Find the probability that

- **(a)** one person, chosen at random, will be a woman,
- **(b)** any two people, chosen at random from the original group, will
  - **(i)** both be men,
  - **(ii)** not both be men.

9. The questions on an examination paper are numbered from 1 to 50 inclusive. A question is chosen at random.

Write down, giving your answer as a fraction, the probability that the number of the chosen question will

- **(a)** contain more than a single digit,
- **(b)** be a perfect square,
- **(c)** contain at least one figure 3,
- **(d)** not be divisible by either 2 or 3.

10. On a supermarket shelf, there are 15 bottles of tomato ketchup and 6 bottles of chilli sauce. A customer picks any two bottles at random. Use a tree diagram to find the probability of

- **(a)** picking 2 bottles of chilli sauce,
- **(b)** picking 2 bottles of tomato ketchup,
- **(c)** picking 1 bottle of tomato ketchup followed by 1 bottle of chilli sauce,
- **(d)** picking 1 bottle of tomato ketchup and 1 bottle of chilli sauce,
- **(e)** not picking 2 bottles of tomato ketchup.

11. A box contains 33 table-tennis balls, 12 of which are white and 21 of which are orange. One ball is taken at random from the box and not replaced. A second ball is then picked. Use a tree diagram to find the probability that

- **(a)** both balls are orange,
- **(b)** both balls are of the same colour,
- **(c)** the balls are of different colours.

12. A farmer has 28 white sheep and 8 black sheep. Any two sheep are taken for shearing at a time. Use a tree diagram to find the probability that

(a) both sheep are white,
(b) both sheep are not white,
(c) both sheep are of the same colour,
(d) the two sheep are of different colours.

13. (a) In Fig. 7.124, the centre of the larger circle is O, its radius is 9 cm and $P\hat{O}Q = 70°$. The radius of the smaller circle is 2 cm.

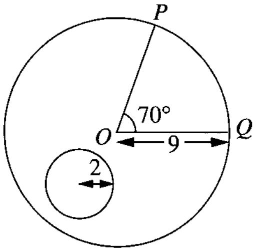


Fig. 7.124

A point is selected at random inside the larger circle.

Expressing each answer as a fraction in its lowest terms, calculate the probability that the point lies

(i) inside the sector POQ,
(ii) inside the smaller circle.

(b) In a raffle, 100 tickets are sold of which 50 are red, 30 are blue and the rest are green. After the tickets are thoroughly mixed, one is drawn for the first prize and another is drawn for the second prize.

Expressing each answer as a fraction in its lowest terms, find the probability that

(i) the first ticket drawn is green,
(ii) the first ticket drawn is red and the second ticket drawn is blue.

14. A bag contains 4 counters, one marked with the letter A, one with the letter B and two with the letter L. The counters are drawn at random from the bag, one at a time, without replacement.

In each of the following cases, calculate the probability that

(a) the first two counters drawn out will each have the letter L marked on them,
(b) the second counter to be drawn out will be that with the letter B marked on it,
(c) the order in which the counters are drawn will spell out the word $BALL$.

15. A boy has 40 records in a rack, 24 of which are 'singles' and the remainder 'long players'. In Fig. 7.125, S and L are the sets of 'singles' and 'long players' and the set P, indicated by the dotted outline, is the set of 'pop records'. The numbers and letters represent the number of records in each subset. (For example, there are 20 singles which are 'pop records'.)

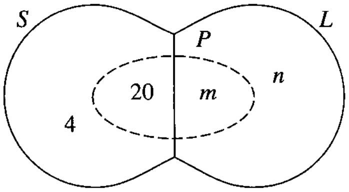


Fig. 7.125

(a) If a 'single' is drawn at random from the rack, find the probability that it is not a 'pop record'.
(b) If, instead, two records are taken at random from the rack, find the probability that they are both 'pop singles'.
(c) On another occasion, a record is chosen at random from the rack. The probability that this record is either a 'long playing pop record' or any 'single' is $\frac{17}{20}$. Find the value of m and the value of n.

16. John has fifteen coins in his pocket. He has eight 20¢ coins, four 50¢ coins and three $1 coins. He takes two coins at random from his pocket, one after the other without replacement.

(a) Complete the probability tree diagram shown below.

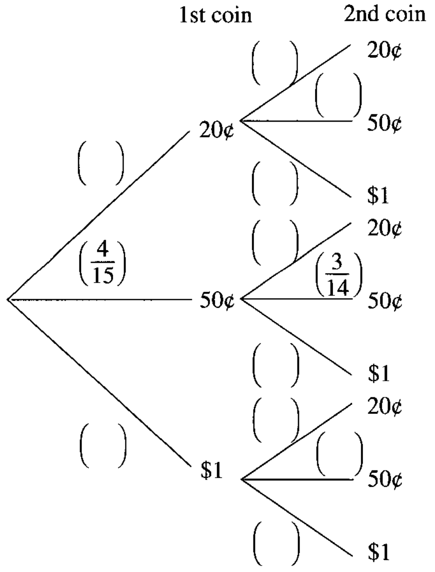


Fig. 7.126

(b) Find the probability that

(i) the first coin is a 20¢ coin and the second is a $1 coin,

(ii) the two coins have different denominations,

(iii) the second coin he takes from his pocket is a 50¢ coin,

(iv) the total value of the two coins is more than one dollar.

17. Alvin has 3 white and 5 blue shirts, and 9 pairs of shorts of which 4 are white and 5 are yellow, in his cupboard. Every morning he selects a shirt and a pair of shorts at random.

(a) Complete the probability diagram.

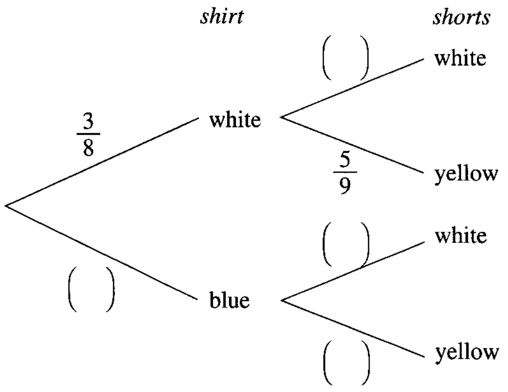


Fig. 7.127

(b) Calculate the probability that he will select a shirt and a pair of shorts of the same colour.

(c) Calculate the probability that Alvin will wear a blue shirt and yellow pair of shorts.

18. When Jack leaves for work in the morning, he takes his umbrella with a probability of $\frac{1}{5}$ if it is fine, and with a probability of $\frac{5}{6}$ if it is wet.

(a) Given that the probability that a day is fine is $\frac{3}{4}$, complete the tree diagram in the answer space.

(b) Find the probability that in a particular morning

(i) it is fine and he does not take his umbrella,

(ii) he goes to work with his umbrella.

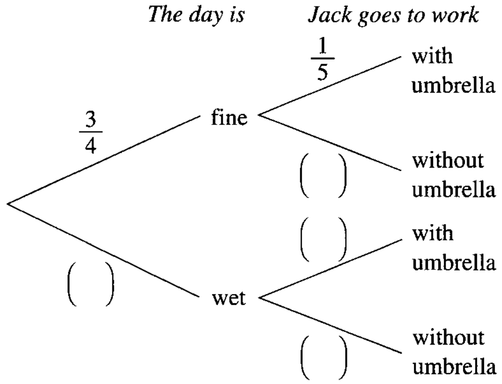


Fig. 7.128

19. A bag contains 1 red ball, 2 black balls and 3 white balls.

(a) A ball is drawn at random from the bag. What is the probability that the ball drawn is

(i) red, **(ii)** white.

(b) The first ball drawn is replaced and a ball is again drawn at random. What is the probability that the two balls drawn out are of different colours?

(c) If, instead, the first ball drawn is not replaced and a second ball is drawn at random, what is the probability in this case of drawing two balls of the same colour?

20. There are 24 white marbles, x red marbles and y blue marbles in a box. One marble is drawn at random. Given the probability that a red marble is drawn is $\frac{1}{5}$ and that a blue marble is drawn is $\frac{2}{5}$, calculate the value of x and of y.

With these values of x and y, calculate the probability that two marbles drawn in succession without replacement are

(a) of the same colour,

(b) a white marble followed by a red marble.

21. The number on a six-sided die is 1, 3, 5, 7, 9, 11 while the number on the other die is 2, 3, 5, 7, 11, 13. The two dice are thrown together and the product of the resulting numbers was calculated. Some of the products are shown in the given possibility diagram.

×	1	3	5	7	9	11
2		6		14		22
3			15		27	
5		15		35		
7	7		35	49		77
11		33		77	99	
13	13			91		143

Fig. 7.129

(a) Copy and complete the possibility diagram.

(b) Using the diagram or otherwise, find the probability that the product of the two numbers is

(i) odd,

(ii) a prime number,

(iii) a multiple of 3,

(iv) less than 15,

(v) a perfect square.

7.14 Statistics

Example 1

720 pupils were asked which of four subjects they preferred. The pie chart illustrates the results.

(a) How many pupils preferred Science?

(b) Given that 204 pupils preferred Mathematics, calculate the angle of the sector representing this.

(c) Given that 35% preferred English, find the angle of the sector representing this.

(d) Draw a bar graph to illustrate the data.

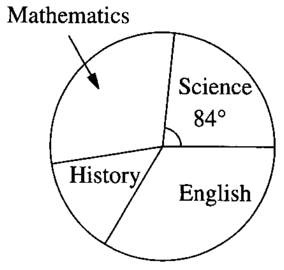


Fig. 7.130

Solution

(a) No. of pupils who preferred Science = $\frac{84}{360} \times 720 = 168$

(b) The angle of Mathematics sector = $\frac{204}{720} \times 360 = 102°$

(c) The angle of the English sector = 35% of 360 = $\frac{35}{100} \times 360 = 126°$

(d) No. of pupils who preferred English = $\frac{126}{360} \times 720 = 252$

preferred History = $720 - 168 - 204 - 252 = 96$

The table below shows how the pupils voted.

Subject	*No. of pupils*
English	252
History	96
Mathematics	204
Science	168

The bar graph shows the number of pupils voting for various subjects.

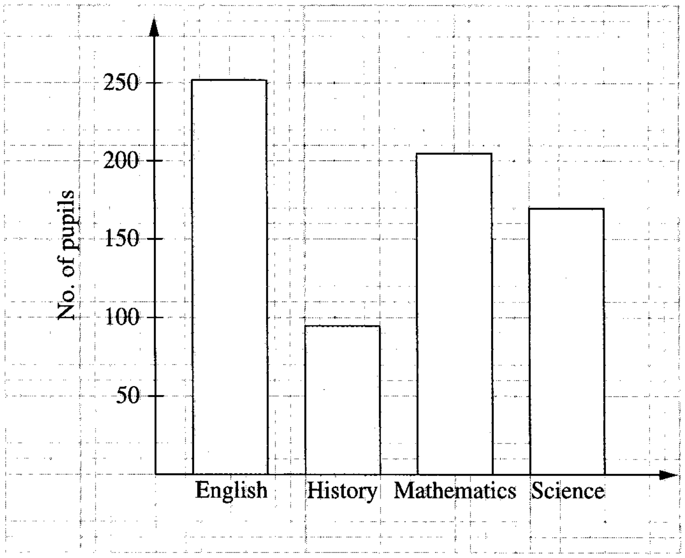


Fig. 7.131

Example 2

The bar graph illustrates the number of non-fiction books in a school library.

(a) Calculate

- *(i) the total number of non-fiction books in the library,*
- *(ii) the percentage of Commerce books.*

(b) Illustrate the information on a clearly-labelled pie chart.

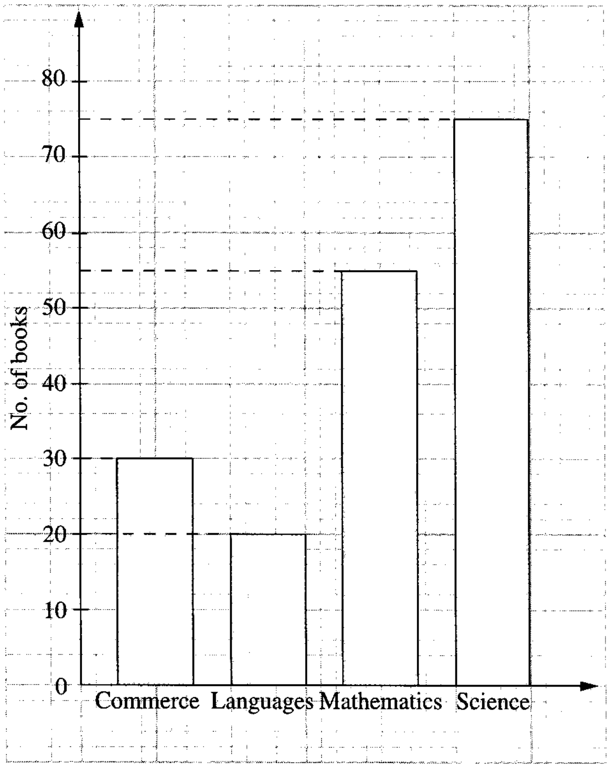


Fig. 7.132

Solution

(a) (i) The total number of non-fiction books = 30 + 20 + 55 + 75 = 180

(ii) The percentage of Commerce books = $\frac{30}{180} \times 100\% = 16\frac{2}{3}\%$

(b) Angle of sectors for

Commerce: $\frac{30}{180} \times 360° = 60°$ Languages: $\frac{20}{180} \times 360° = 40°$

Mathematics: $\frac{55}{180} \times 360° = 110°$ Science: $\frac{75}{180} \times 360° = 150°$

The required pie chart is drawn as shown in Fig. 7.133.

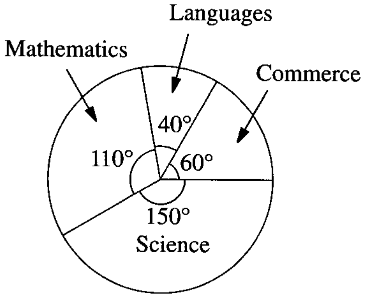


Fig. 7.133

Example 3

The scores obtained by 30 players in a certain game in which the highest possible score was 9 are shown in the table below.

Score	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*> 7*
Frequency	2	3	3	6	5	5	4	2

Draw a histogram to represent the data.

Solution

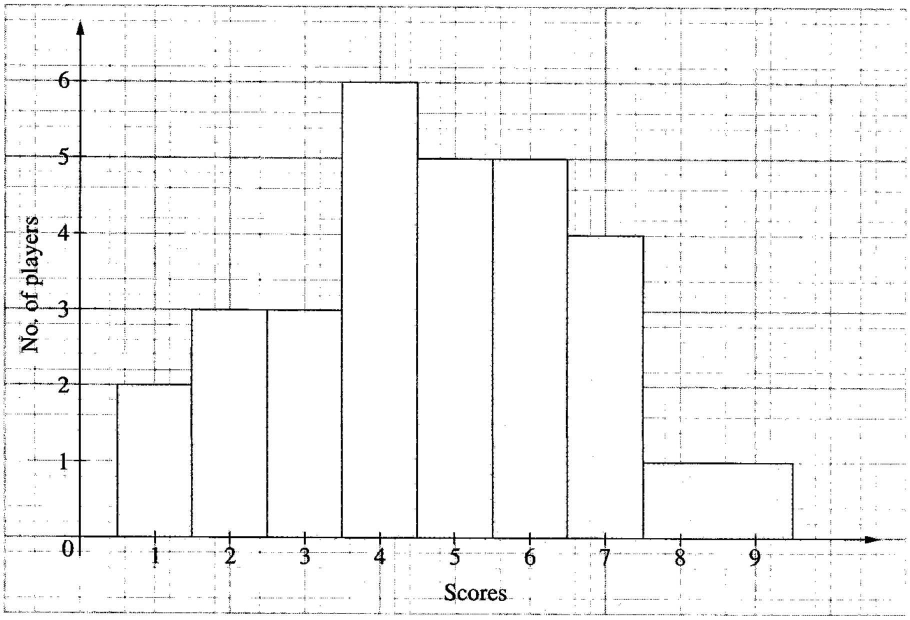


Fig. 7.134

Example 4

For the distribution 6, 8, 15, 11, 4, 6, 7, 11, 21, 6, find the following:

(a) Mode *(b) Median* *(c) Mean*

Solution

(a) The mode is 6.

(b) Arranging the numbers in order of size, we have

4, 6, 6, 6, 7, 8, 11, 11, 15, 21.

The median is the average of 7 and 8.

$\therefore$ the median is $\frac{7 + 8}{2} = 7.5$

(c) $$\text{The mean} = \frac{4 + 6 + 6 + 6 + 7 + 8 + 11 + 11 + 15 + 21}{10}$$
$$= \frac{95}{10}$$
$$= 9.5$$

In 1996, Singaporeans spent a whopping \$5 billion on legal betting alone. \$2.8 billion was spent on four-digit lottery, \$1.26 billion on horse-racing, \$310 million on Toto, \$72 million on Singapore Sweep and another \$500 million on Jackpot machines in private clubs. The government collected a total of \$1.144 billion taxes from betting, sweepstakes and private lotteries in 1996. Singaporeans was ranked Top gamblers in South-East Asia on per capital basis. Do you think we should celebrate for this great achievement?

Example 5

In an experiment, the lengths of 60 bean sprouts, cultivated in a laboratory for six days, were measured. The results are tabulated below.

Length (x) in mm	$110 < x \leqslant 130$	$130 < x \leqslant 140$	$140 < x \leqslant 145$	$145 < x \leqslant 160$
No. of bean sprouts	*14*	*21*	*10*	*15*

(a) Calculate an estimate of the mean length of the bean sprouts.
(b) Draw a histogram to represent this distribution. Label your axes carefully.
(c) A bean sprout is chosen at random and is then replaced.
Find the probability that its length lies in the range $130 < x \leqslant 140$.
(d) Two bean sprouts are chosen at random.
Find the probability that the mass of one is in the range $130 < x \leqslant 140$ and the mass of the other is in the range $145 < x \leqslant 160$.

Solution

(a)

Length (x mm)	120	135	142.5	152.5
Frequency	14	21	10	15

$$\text{The estimate of the mean length} = \frac{120 \times 14 + 135 \times 21 + 142.5 \times 10 + 152.5 \times 15}{60}$$
$$= 137.125 \text{ mm}$$
$$= 137 \text{ mm (correct to the nearest mm)}$$

(b)

Length (x mm)	$110 < x \leqslant 130$	$130 < x \leqslant 140$	$140 < x \leqslant 145$	$145 < x \leqslant 160$
Frequency Density	$\frac{14}{20} = 0.7$	$\frac{21}{10} = 2.1$	$\frac{10}{5} = 2$	$\frac{15}{15} = 1$

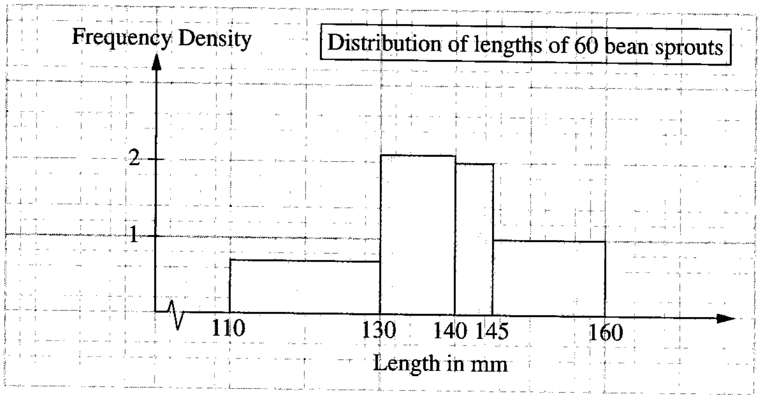


Fig. 7.135

(c) Required probability $= \frac{21}{60} = \frac{7}{20}$.

(d) Required probability $= \frac{21}{60} \times \frac{15}{59} \times 2 = \frac{21}{118}$.

Example 6

A class of 50 pupils sat for a Physical Science examination. The marks obtained by the pupils were tabulated and a cumulative frequency curve was drawn as shown in Fig. 7.136.

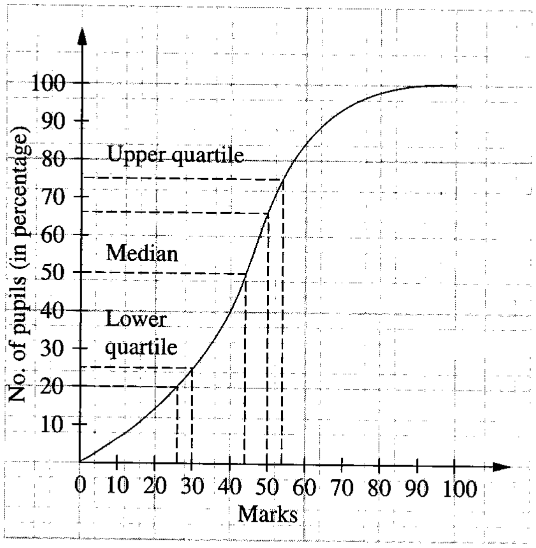


Fig. 7.136

Study the graph and answer the following.

(a) Find the following:

(i) the lower quartile, (ii) the median,
(iii) the upper quartile, (iv) the interquartile range.

A former Prime Minister of Britain, Mr Disraeli, once said: 'There are three kinds of lies — lies, damned lies and statistics'. It is very true that statistical information can be distorted to mislead people to support a particular case.

(b) If 75% of the pupils passed the test, what was the passing mark?

(c) If not more than 20% failed, what was the passing mark?

(d) How many pupils passed the test if the passing mark is 50?

Solution

(a) From the graph,
lower quartile = 30, median = 44,
upper quartile = 54, interquartile range = 54 – 30 = 24.

(b) If 75% of the boys passed the test, then the passing mark = the lower quartile = 30.

(c) The passing mark = the 20th percentile = 26.

(d) 66th percentile = 50, so 34% of the pupils passed the test.

$\therefore$ number of pupils who passed = $\frac{34}{100} \times 50 = 17$.

Example 7

The mass x g of each 90 oranges of a certain variety was recorded. The data obtained was illustrated as shown in the table.

Mass x (g)	*No. of oranges*
$60 < x \leqslant 80$	4
$80 < x \leqslant 90$	9
$90 < x \leqslant 100$	28
$100 < x \leqslant 110$	37
$110 < x \leqslant 120$	8
$120 < x \leqslant 130$	4

(a) Copy and complete the following cumulative frequency table.

Mass x (g)	60	80	90	100	110	120	130
No. of oranges of this mass or less	0	4					90

(b) Using a horizontal scale of 1 cm to represent a mass of 10 g and a vertical scale of 1 cm to represent 10 oranges, draw a smooth cumulative frequency curve for this distribution.

(c) Use your graph to estimate the following for this distribution.
(i) the median, *(ii) the interquartile range.*

(d) Use your graph to find, as accurately as possible, the probability that an orange chosen at random from the sample has a mass of
(i) 98 g or less, *(ii) more than 98 g but not more than 116 g.*

(e) Find the probability that one orange, chosen at random, has a mass of more than 110 g.

(f) Two oranges are chosen at random from these 90 oranges. Find the probability that neither has a mass of more than 90 g.

Solution

(a)

Mass x (g)	60	80	90	100	110	120	130
No. of oranges of this mass or less	0	4	13	41	78	86	90

(b)

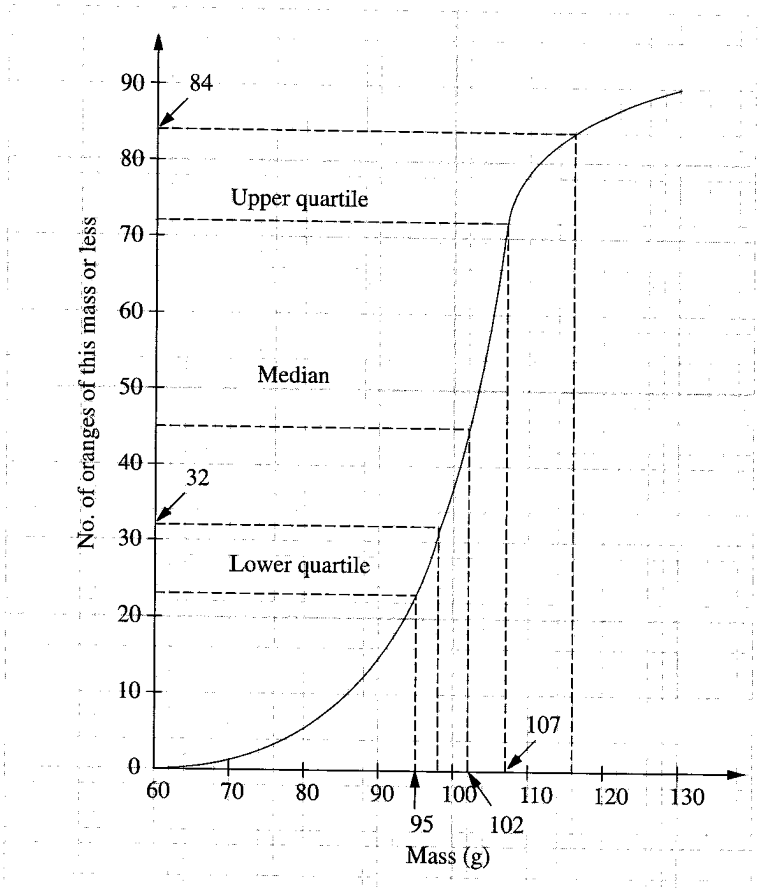


Fig. 7.137

(c) From the graph, the median = 102 g and the interquartile range = 107 – 95 = 12 g.

(d) (i) From the graph, the estimated number of oranges having a mass of 98 g or less is 32.

$\therefore$ P(an orange chosen at random has mass 98 g or less) = $\frac{32}{90} = \frac{16}{45}$

(ii) No. of oranges having a mass of not more than 116 g = 84

$\therefore$ no. of oranges having a mass of more than 98 g but not more than 116 g = 84 – 32
= 52

$\therefore$ the required probability = $\frac{52}{90} = \frac{26}{45}$

(e) No. of oranges having a mass of more than 110 g = 8 + 4 = 12

$\therefore$ the required probability = $\frac{12}{90} = \frac{2}{15}$

(f) No. of oranges having a mass of 90 g or less = 4 + 9 = 13

$\therefore$ the required probability = $\frac{13}{90} \times \frac{12}{89} = \frac{26}{1\ 335}$

Revision Exercise 7.14

1. The pie chart illustrates the value of various goods sold by a certain shop.
   - **(a)** Calculate the value of x.
   - **(b)** Given that the total value of the sales was \$21 600, find the sales value of the following:
     - **(i)** food
     - **(ii)** stationery

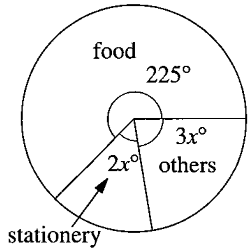


Fig. 7.138

2. The bar graph shown in the figure illustrates the production of edible oils in a certain country.

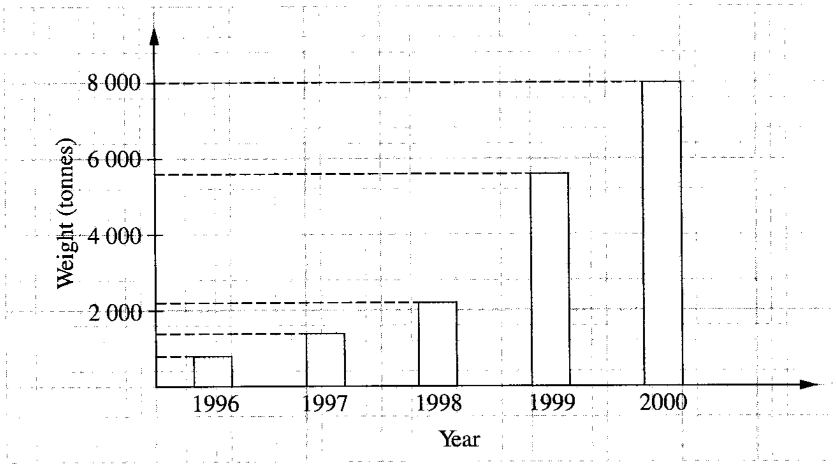


Fig. 7.139

   - **(a)** Calculate the total production of edible oils from 1996 to 2000.
   - **(b)** Illustrate the information on a clearly-labelled pie chart.

3. The pie chart represents the production of steel in four districts in a certain country.
   - **(a)** Express the production of steel in District C as the percentage of the total production in the four districts.
   - **(b)** Given that the production of steel in District C was 800 000 tonnes, calculate the production of steel in District A.
   - **(c)** Given that the production of steel in District B is three times that in District D, calculate the angle of the sector which represents the production of steel in District B.

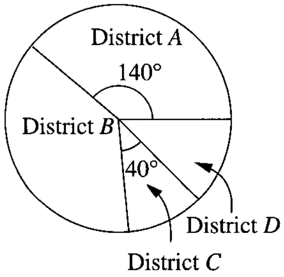


Fig. 7.140

4. The pie chart illustrates the cost of producing a TV programme.
   - **(a)** Find what percentage of the total cost is made up of
     - **(i)** production costs,
     - **(ii)** musicians' fees.
   - **(b)** Given that the total cost of producing the programme is \$72 000, find
     - **(i)** the production costs,
     - **(ii)** the musicians' fees.
   - **(c)** Given that the actors' fees amount to \$14 000, find x.

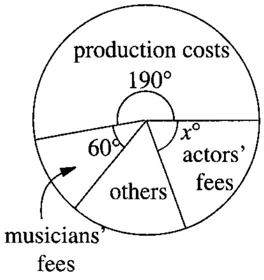


Fig. 7.141

5. The bar graph illustrates the means of travelling to school by a group of secondary one pupils.

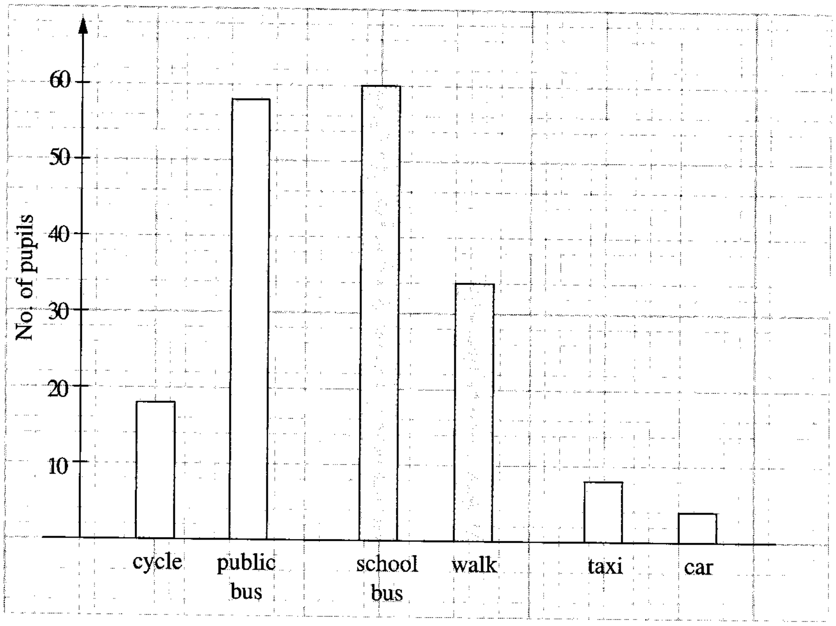


Fig. 7.142

(a) Calculate
   (i) the total number of secondary one pupils,
   (ii) the percentage of pupils who do not travel to school by bus.

(b) Which is the most popular means of transport?

6. The pie chart shows the number of road deaths in a certain country during one year for boys aged up to and including 19 years. Given that the total number of road deaths was 120, find the number of road deaths for each category. Hence, illustrate the information on a clearly-labelled bar graph.

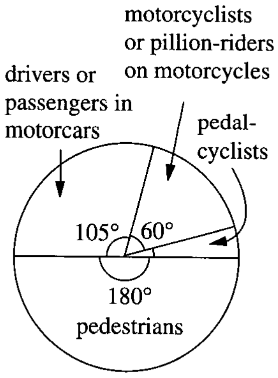


Fig. 7.143

7. The marks scored by 14 pupils in a test marked out of 60 are as follows:

46, 53, 42, 38, 50, 44, 46,
45, 46, 39, 48, 45, 47, 41

Find, for this set of marks, the following:

(a) Mode (b) Median (c) Mean

8. The following is a set of twelve numbers.

6, 7, 14, 7, 4, 9, 4, 4, 3, 4, 5, 17

(a) Find the following:
   (i) Mode (ii) Median (iii) Mean

(b) When the number a is added to the above set, the new mean is 9. Calculate the value of a.

9. The table below shows the distribution of the heights of a number of children.

Height (x) in cm	$60 < x \leqslant 70$	$70 < x \leqslant 80$	$80 < x \leqslant 100$
Number of children	14	16	10

(a) Calculate an estimate of the mean height of the children.
(b) Draw a histogram to represent this distribution. Label your axes carefully.
(c) A child is chosen at random and then replaced. Find the probability that his height lies in the range $70 < x \leqslant 80$.
(d) Two children are chosen at random. Find the probability that the mass height of one is in the range $70 < x \leqslant 80$ and the height of the other is in the range $80 < x \leqslant 100$.

10. Diameters of 100 tree trunks were measured and were divided into four groups.

I: those with diameter less than 50 cm,
II: those with diameter greater than or equal to 50 cm, but less than 60 cm,
III: those with diameter greater than or equal to 60 cm, but less than 70 cm,
IV: those with diameter greater than or equal to 70 cm, but less than 100 cm.

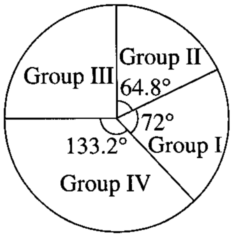


Fig. 7.144

Each of the 100 tree trunks is in one of the four groups, as shown on the given pie chart.

(a) Complete the table in the answer space

	Group I	*Group II*	*Group III*	*Group IV*
Diameter (d cm)	$0 < d \leqslant 50$	$50 < d \leqslant 60$	$60 < d \leqslant 70$	$70 < d \leqslant 100$
Number of tree trunks		18		

(b) Calculate an estimate of the mean diameter of the tree trunks.
(c) Draw a histogram to represent this distribution. Label your axes carefully.
(d) Two tree trunks are chosen at random. Find the probability that the diameters of both tree trunks lie in the range $50 < d \leqslant 60$.

11. The table below shows the number of hours a group of 300 people spent on watching a particular TV channel during a given week.

No. of hours (x)	10	11	12	13	14	15	16	17
No. of people (f)	4	16	34	104	76	52	10	4

(a) Draw the frequency polygon of this distribution, taking values of x from 10 to 17 on the horizontal axis and using a scale of 2 cm to represent 1 hour. On the vertical axis, take values of f from 0 to 110 and use a scale of 2 cm to represent 20 people.
(b) For this distribution, find the following:
(i) Mode **(ii)** Median **(iii)** Mean

12. The following marks were obtained by 50 pupils in an examination.

Marks	0	1	2	3	4	5	6	7
No. of pupils	2	3	4	6	7	10	11	7

(a) Draw a histogram to represent the data given in the table.

(b) For this distribution, find the following:
(i) Mode **(ii)** Median **(iii)** Mean

13. **Answer the whole of this question on a sheet of graph paper.**
In an athletics event, the distance x, in kilometres, run by each of a group of 80 children was recorded. The data obtained was then expressed in two ways as shown in the tables below.

Distance x, (km)	$0 < x \leqslant 1$	$1 < x \leqslant 2$	$2 < x \leqslant 3$	$3 < x \leqslant 4$	$4 < x \leqslant 5$	$5 < x \leqslant 6$	$6 < x \leqslant 7$	$7 < x \leqslant 8$
Number of children	2	5	11	18	17	A	8	5

Distance x, (km)	1	2	3	4	5	6	7	8
Number of children running this distance or less	2	7	B	36	53	67	75	80

(a) Find the value of A and of B.

(b) Using a vertical scale of 2 cm to represent 10 children and a horizontal scale of 2 cm to represent 1 km, draw a smooth cumulative frequency curve to represent the results in the second table.

(c) Showing your method clearly, use your graph to estimate
(i) the median, **(ii)** the interquartile range.

(d) Find the probability that one child, chosen at random, has run more than 4 km.

(e) A child is chosen at random from those who have run more than 4 km. Find the probability that the child has run more than 7 km.

(f) Two children are chosen at random from the group of 80 children. Find the probability that neither has run more than 1 km. *(C)*

14. Fig. 7.145 shows the cumulative frequency curve for the distance travelled to work by a number of commuters. Use the curve to estimate, as accurately as possible,

(a) the median distance,

(b) the interquartile range,

(c) the probability that a commuter selected at random travels 34 km or less to work.

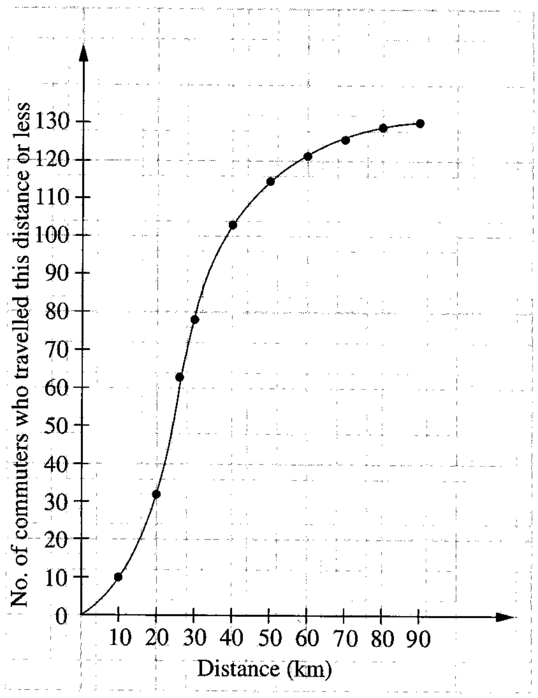


Fig. 7.145

15. Fig. 7.146 shows the cumulative frequency curve for the number of goals scored in a certain season by 95 Football League clubs.

(a) Using the given curve, find, for this distribution,

(i) the median,

(ii) the interquartile range.

(b) Given that at the end of the season, the 12 clubs which have scored the most goals are invited to play in a tournament, estimate the minimum number of goals scored which will enable a club to qualify.

(c) Estimate the probability that a club selected at random scored 66 goals or less.

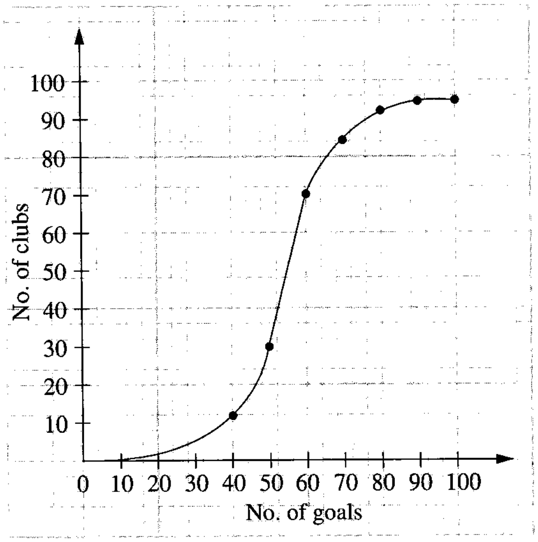


Fig. 7.146

★16. Answer the whole of this question on a sheet of graph paper.

The length x mm of each of 100 leaves of a plant was measured. The data obtained is illustrated in the table below.

Length x (mm)	$40 < x \leqslant 50$	$50 < x \leqslant 60$	$60 < x \leqslant 70$	$70 < x \leqslant 80$	$80 < x \leqslant 90$	$90 < x \leqslant 100$	$100 < x \leqslant 110$	$110 < x \leqslant 120$
No. of leaves	6	8	14	21	26	14	7	4

(a) Copy and complete the following cumulative frequency table.

Length x	40	50	60	70	80	90	100	110	120
No. of leaves of this length or less	0	6							100

(b) Using a vertical scale of 2 cm to represent a length of 20 mm and a vertical scale of 2 cm to represent 10 leaves, draw a smooth cumulative frequency curve for this distribution.

(c) Showing your method clearly, use your graph to estimate, for this distribution,

(i) the median,

(ii) the interquartile range.

(d) Find the probability that one leaf, chosen at random, has a length of more than 80 mm.

(e) A leaf is chosen at random from those having lengths more than 80 mm. Find the probability that its length is greater than 100 mm.

(f) Two leaves are chosen at random from these 100 leaves. Find the probability that neither has a length greater than 50 mm.

★17. Answer the whole of this question on a sheet of graph paper.

The height of 56 plants, grown under experimental conditions, are given in the following table.

Height (cm)	$x \leqslant 10$	$10 < x \leqslant 20$	$20 < x \leqslant 30$	$30 < x \leqslant 40$	$40 < x \leqslant 50$	$50 < x \leqslant 60$	$60 < x \leqslant 70$
No. of plants	1	2	4	6	13	22	8

(a) Copy and complete the following cumulative frequency table.

Height (cm)	0	10	20	30	40	50	60	70
No. of plants of this height or less	0	1						56

(b) Using a horizontal scale of 2 cm to represent a height of 10 cm and a vertical scale of 2 cm to represent 10 plants, draw a smooth cumulative frequency curve for these results.

(c) Showing your method clearly, use your graph to estimate, for this distribution,

(i) the median, **(ii)** the interquartile range.

(d) If one plant is selected at random, find as a fraction in its simplest form, the probability that its height is

(i) greater than 50 cm,

(ii) either not greater than 10 cm or greater than 60 cm.

(e) The probability that a plant produces a flower is $\frac{6}{7}$ and, whether it does or not, the probability that it suffered from disease is $\frac{1}{4}$. These probabilities are independent of the height of the plant.

(i) Calculate, as a fraction in its simplest form, the probability that a single plant selected at random produces a flower and does not suffer from disease.

(ii) Given that $\frac{7}{12}$ of the flowers produced are red, calculate, as a fraction in its simplest form, the probability that a single plant selected at random is of height greater than 50 cm and produces a red flower.

★**18. Answer the whole of this question on a sheet of graph paper.**

Each of the 560 pupils in a girls' school was asked how far she travelled from home. The results are given in the table below.

Distance x km	$x \leqslant 1$	$1 < x \leqslant 2$	$2 < x \leqslant 3$	$3 < x \leqslant 4$	$4 < x \leqslant 5$	$5 < x \leqslant 6$	$6 < x \leqslant 7$
Number of girls	8	34	118	244	106	40	10

(a) Copy and complete the following cumulative frequency table.

Distance in km	0	1	2	3	4	5	6	7
Number of girls travelling this distance or less	0	8						560

(b) Using a horizontal scale of 2 cm to represent 1 km and a vertical scale of 2 cm to represent 100 girls, draw a smooth cumulative frequency curve for these results.

(c) Use your graph to estimate the number of girls who travel 4.5 km or more.

(d) Showing your method clearly, use your graph to estimate

(i) the median,

(ii) the interquartile range of this distribution.

(e) One girl is selected at random from the 560.

(i) Find the probability that the distance she travels is less than or equal to 3 km.

(ii) If the probability that she travels more than y kilometres is $\frac{5}{56}$, find y.

(f) Two girls are selected at random from the 560. Find the probability that they each travel a distance less than or equal to 1 km.

★19. Answer the whole of this question on a sheet of graph paper.

The following table gives the frequency distribution of marks obtained by 80 candidates in examinations in Mathematics and English.

Mark	$0 \leqslant x \leqslant 20$	$20 < x \leqslant 40$	$40 < x \leqslant 60$	$60 < x \leqslant 80$	$80 < x \leqslant 100$
Mathematics	8	12	18	25	17
English	2	10	33	31	4

(a) Copy and complete the table below showing the cumulative frequency distribution in each subject.

	Mark	20	40	60	80	100
Number of children with this mark or less	*Mathematics*	8	20			80
	English					

(b) Using a scale of 2 cm to represent 20 marks on the horizontal axis and 2 cm to represent 20 candidates on the vertical axis, draw separate cumulative frequency diagrams for each of the subjects Mathematics and English.

Showing your method clearly, use your graph to estimate

(i) the median mark in Mathematics,

(ii) the interquartile range in English,

(iii) the number of candidates who will obtain a distinction in English, if the minimum mark for a distribution is 76,

(iv) how many *more* candidates will fail to achieve a credit in Mathematics than in English if the minimum mark for a credit is 60 in each subject.

20. Answer the whole of this question on a sheet of graph paper.

160 electric light bulbs of brand A were tested to find the life span of each bulb (that is the time it lasted before it failed). The results are given in the table below.

Life span (t hours)	*Number of bulbs*
$t \leqslant 500$	2
$500 < t \leqslant 1\,000$	4
$1\,000 < t \leqslant 1\,500$	13
$1\,500 < t \leqslant 2\,000$	68
$2\,000 < t \leqslant 2\,500$	51
$2\,500 < t \leqslant 3\,000$	18
$3\,000 < t \leqslant 3\,500$	3
$3\,500 < t \leqslant 4\,000$	1

(a) Copy and complete the following cumulative frequency table.

Life span in hours	$\leqslant 500$	$\leqslant 1\,000$	$\leqslant 1\,500$	$\leqslant 2\,000$	$\leqslant 2\,500$	$\leqslant 3\,000$	$\leqslant 3\,500$	$\leqslant 4\,000$
Number of bulbs	2	6						160

(b) Using a horizontal scale of 2 cm to represent 500 hours and a vertical scale of 2 cm to represent 20 bulbs, draw a smooth cumulative frequency curve for these results.

(c) Showing your method clearly, use your graph to estimate

(i) the median, **(ii)** the 10th percentile of the distribution.

160 brand B bulbs were also tested and a report on the test gave the following information.

4 bulbs had a life span $\leqslant$ 500 hours.
None lasted beyond 3 200 hours.
The median life span was 2 300 hours.
The upper quartile of the distribution was 2 600 hours.
The interquartile range of the distribution was 600 hours.

(d) Use this information to draw, on the same axes, a smooth cumulative frequency curve for the brand B bulbs.

(e) Use your graphs to estimate the number of bulbs with a life span 2 750 hours or less
  (i) for brand A,
  (ii) for brand B.

(f) Both brands are the same price.
Which do you think is the better buy? Give a reason for your choice. *(C)*

7.15 Geometrical Transformations

Example 1

The coordinates of $\triangle ABC$ are $A(2, 2)$, $B(2, 5)$ and $C(4, 5)$. Draw and label $\triangle ABC$. $\triangle ABC$ is transformed into $\triangle A_1B_1C_1$ by a stretch with the y-axis invariant and a stretch factor of 2. Draw and label the image of $\triangle ABC$ under the stretch.

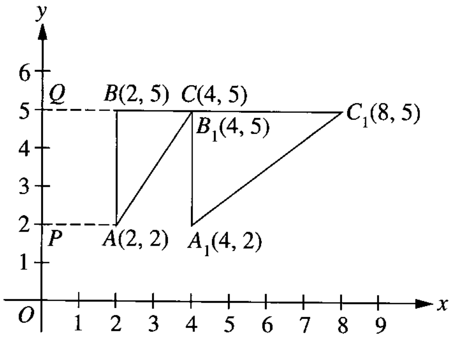


Fig. 7.147

Solution

Let P and Q be the perpendiculars from A and B to the y-axis respectively.

$$\text{Now} \quad \frac{QB_1}{QB} = \frac{PA_1}{PA} = \frac{QC_1}{QC} = 2$$

$$\text{i.e.} \quad \frac{QB_1}{2} = \frac{PA_1}{2} = \frac{QC_1}{4} = 2$$

$\therefore$ $QB_1 = 4$, $PA_1 = 4$ and $QC_1 = 8$

$\therefore$ the coordinates of B_1 is (4, 5), C_1 is (8, 5) and A_1 is (4, 2).

Example 2

The triangle formed by the points A(1, 1), B(1, 4) and C(3, 1) is reflected in the y-axis and then the image PQR is reflected in the line x = 3 to give a second image LMN. Use a sheet of graph paper to plot all the image points and describe the single transformation which is equivalent to the combination of the two reflections.

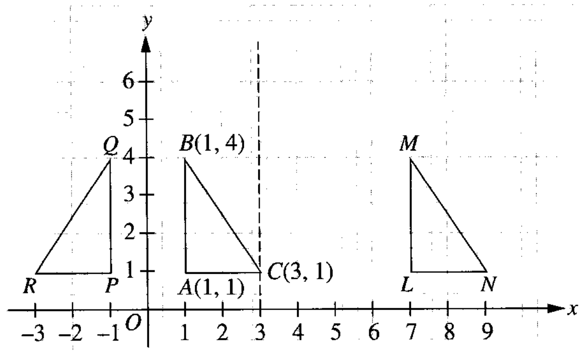


Fig. 7.148

Solution

The single transformation that will map ABC onto LMN is a translation represented by $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$.

Example 3

Fig. 7.149 shows a big equilateral triangle of side 6 cm which is divided into 9 congruent equilateral triangles. Describe a single transformation that will map

(a) $\triangle ABC$ onto $\triangle EBC$,
(b) $\triangle DEH$ onto $\triangle EFI$,
(c) $\triangle ABC$ onto $\triangle AGJ$,
(d) $\triangle GDH$ onto $\triangle FIE$,
(e) $\triangle BCE$ onto $\triangle IHE$.

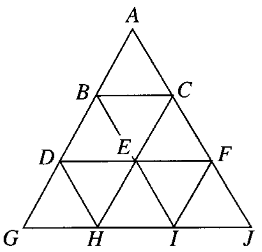


Fig. 7.149

Solution

(a) $\triangle ABC$ can be mapped onto $\triangle EBC$ by a reflection in the line BC.

(b) $\triangle DEH$ can be mapped onto $\triangle EFI$ by a translation of 2 cm parallel to DE.

(c) $\triangle ABC$ can be mapped onto $\triangle AGJ$ by an enlargement with centre A and scale factor 3.

(d) $\triangle GDH$ can be mapped onto $\triangle FIE$ by a 180° rotation with centre at the mid-point of EH.
Alternatively, $\triangle GDH$ can also be mapped onto $\triangle FIE$ by an enlargement with centre at mid-point of EH and scale factor -1.

(e) $\triangle BCE$ can be mapped onto $\triangle IHE$ by an enlargement of scale factor -1 and centre of enlargement at E.
Alternatively, $\triangle BCE$ can also be mapped onto $\triangle IHE$ by a 180° rotation about the point E.

CHECK THIS OUT

A man has two cars, a big one and a small one. The petrol tanks of these two cars can hold a total of 70 litres of petrol. When the tank of each is empty, a full tank of petrol for the small car costs \$45 while the full tank of petrol for the big car costs \$68. Suppose the man uses regular grade and premium grade petrol for the small and the big cars, respectively, and the difference of the two grades of petrol is 20 cents per litre, what is the capacity of the petrol tank of each car?

Example 4

(a) *Draw and label the triangle ABC whose vertices are A(1, 2), B(4, 4) and C(6, 2).*

(b) *The translation T maps $\triangle ABC$ onto $\triangle A_1B_1C_1$. Given the point $C_1(1, 1)$, write down the column vector representing T. Draw and label $\triangle A_1B_1C_1$.*

(c) *R is a reflection in the x-axis which maps $\triangle ABC$ onto $\triangle A_2B_2C_2$. Draw and label $\triangle A_2B_2C_2$.*

(d) *The transformation S is a shear with the y-axis invariant and it maps $\triangle ABC$ onto $\triangle A_3B_3C_3$. Given that the coordinates of C_3 is (6, 8), find the shear factor and the coordinates of A_3 and B_3.*

Solution

(a)

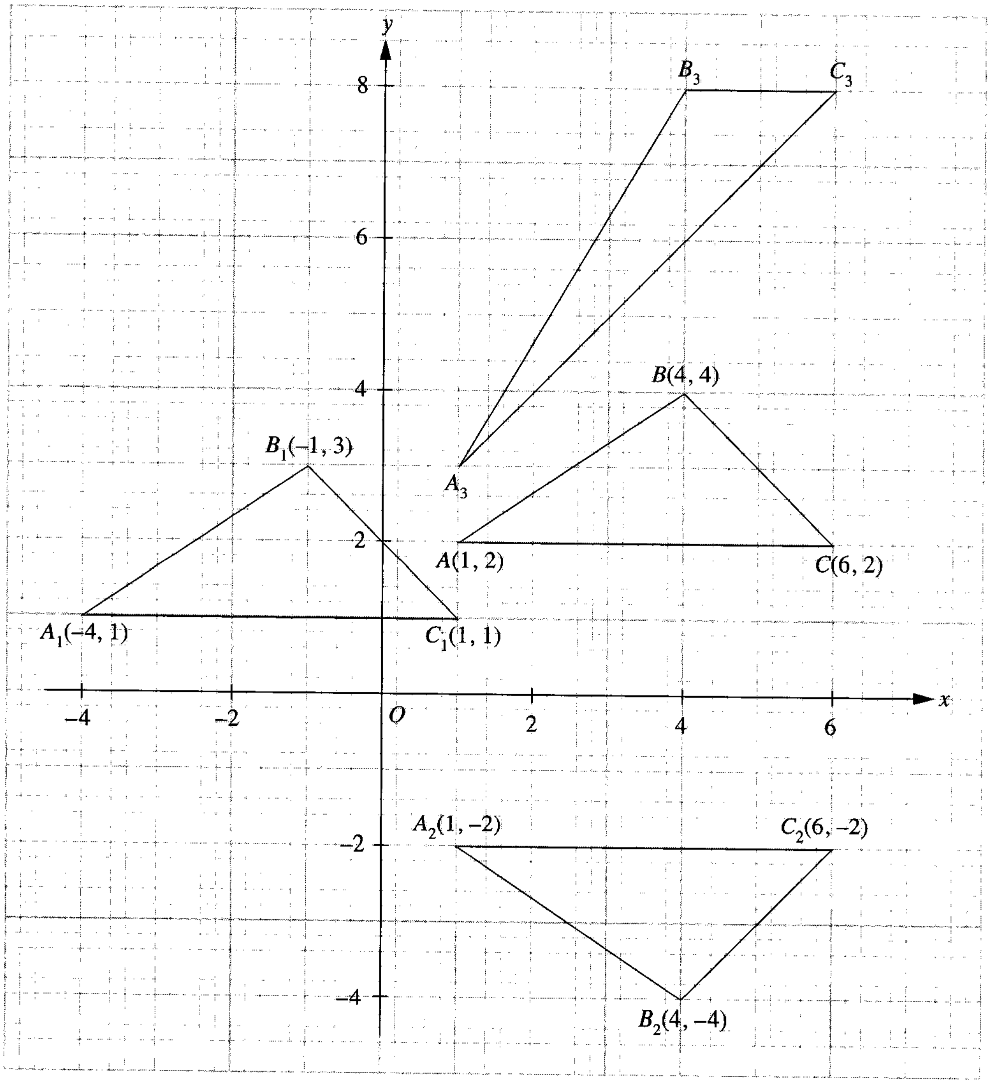


Fig. 7.150

(b) Let the vector representing T be $\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

(d) Shear factor = $\dfrac{\text{distance of } CC_3}{\text{distance of } C \text{ from } y\text{-axis}} = \dfrac{8-2}{6} = 1$

Coordinates of B_3 is given by $\dfrac{\text{distance of } BB_3}{\text{distance of } B \text{ from } y\text{-axis}} = 1$

$BB_3 = 4$ units

$\therefore$ B_3 is the point (4, 8).

Similarly, A_3 is found to be the point (1, 3).

Example 5

The vertices of $\triangle ABC$ are $A(2, 1)$, $B(2, 2)$ and $C(4, 1)$.

(a) *Using a scale of 1 cm to represent 2 units on each axis, draw x- and y-axes for $-2 \leqslant x \leqslant 8$ and $-4 \leqslant y \leqslant 7$. Draw and label $\triangle ABC$.*

(b) *The transformation X maps $\triangle ABC$ onto $\triangle A_1B_1C_1$ with coordinates $A_1(-1, 4)$, $B_1(-2, 4)$ and $C_1(-1, 6)$. Draw and label $\triangle A_1B_1C_1$. Describe the transformation X fully.*

(c) *A reflection in the line l maps the point C onto $C_2(1, -2)$. Draw and label the line l and draw $\triangle A_2B_2C_2$ on your graph.*

(d) *Under a shear with the line x = 1 as the invariant line and shear factor 2, $\triangle ABC$ is mapped onto $\triangle A_3B_3C_3$, draw and label $\triangle A_3B_3C_3$.*

(e) *Under an enlargement centre at E(4, 2) and scale factor –2, $\triangle ABC$ is mapped onto $\triangle PQR$, draw and label $\triangle PQR$.*

Solution

(b) X is a 90° anticlockwise rotation about (–1, 1).

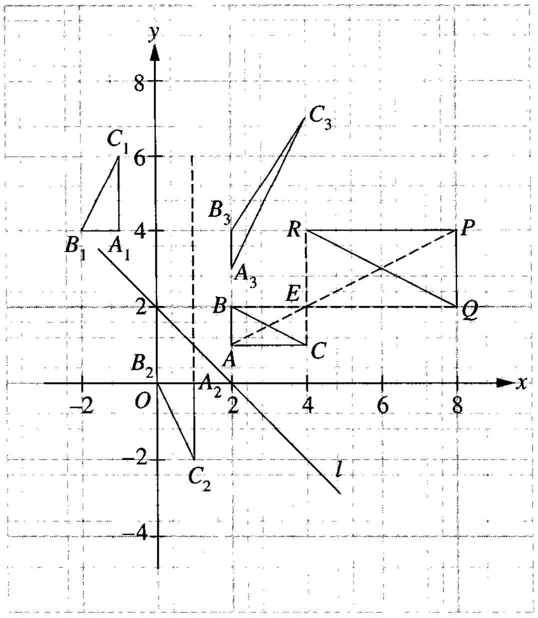


Fig. 7.151

Example 6

Using a scale of 1 cm to represent 2 units on each axis, draw x- and y-axes for $-4 \leqslant x \leqslant 10$ and $-4 \leqslant y \leqslant 9$. Draw $\triangle ABC$ with vertices $A(1, 2)$, $B(3, 2)$ and $C(3, 4)$.

(a) $\triangle ABC$ is mapped onto $\triangle A_1B_1C_1$ with vertices $A_1(4, 2)$, $B_1(6, 2)$ and $C_1(9, 4)$. Describe fully the transformation X that maps $\triangle ABC$ onto $\triangle A_1B_1C_1$.

(b) H is a shear that maps $\triangle ABC$ onto $\triangle A_2B_2C_2$ with vertices $A_2(1, 4)$, $B_2(3, 6)$ and $C_2(3, 8)$. Draw and label $\triangle A_2B_2C_2$. Describe fully the transformation H.

(c) $\triangle A_2B_2C_2$ is mapped onto $\triangle A_3B_3C_3$ by a transformation K. Given that the coordinates of $\triangle A_3B_3C_3$ are $A_3(9, 8)$, $B_3(7, 6)$ and $C_3(7, 4)$. Draw and label $\triangle A_3B_3C_3$ and describe fully the transformation K.

(d) The coordinates of $\triangle A_4B_4C_4$ are $A_4(-1, 2)$, $B_4(-3, 2)$, $C_4(-3, 4)$. Draw and label $\triangle A_4B_4C_4$. Describe the transformation that will map $\triangle ABC$ onto $\triangle A_4B_4C_4$.

(e) Under a 90° anticlockwise rotation about the point (4, 0), $\triangle ABC$ is mapped onto $\triangle A_5B_5C_5$, draw and label $\triangle A_5B_5C_5$ on your graph.

Solution

(a) CA produced and C_1A_1 produced meet at a point on the x-axis. Thus, X is a shear with shear factor $= \frac{AA_1}{2} = \frac{3}{2} = 1.5$.

(b) BA produced and B_2A_2 produced meet at a point on the line $x = -1$. Shear factor $\frac{AA_2}{AA_4} = \frac{2}{2} = 1$. Thus, H is a shear with $x = -1$ as the invariant line and shear factor 1.

(c) K is an enlargement centre at (5, 6) and scale factor −1. K can also be a 180° rotation about the point (5, 6).

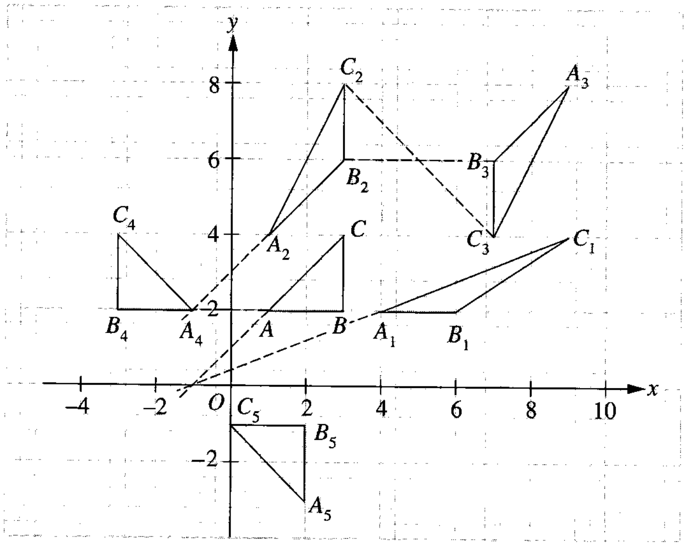


Fig. 7.152

Revision Exercise 7.15

1. The coordinates of $\triangle ABC$ are $A(1, 1)$, $B(5, 1)$ and $C(4, 3)$. Draw and label $\triangle ABC$. $\triangle ABC$ is transformed onto $\triangle A_1B_1C_1$ by a stretch with the x-axis invariant and stretch factor 2. Draw and label $\triangle A_1B_1C_1$.

2. $\triangle ABC$ has vertices $A(1, 2)$, $B(1, 6)$ and $C(4, 4)$. It is mapped onto $\triangle PQR$ by a translation T represented by the column matrix $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ followed by a clockwise rotation of 90° about the origin. Draw and label $\triangle ABC$ and $\triangle PQR$.

3. $\triangle LMN$ has vertices $L(3, 1)$, $M(5, 3)$ and $N(1, 7)$. It is mapped onto $\triangle PQR$ by a clockwise rotation of 90° about the origin followed by a reflection in the y-axis. Draw and label $\triangle LMN$ and $\triangle PQR$.

4. The transformation S is a shear with the x-axis invariant and shear factor 2.
   (a) Find the image of (4, 7) under S.
   (b) Find the image of (2, –5) under S.
   (c) Find the coordinates of the point whose image is (15, 4) under S.

5. Find the equation of the image of the line $y = 2x + 1$ under reflection in
   (a) the x-axis,
   (b) the y-axis.
   (c) the line $x = 2$.

6. Under a translation T, the point (5, 2) is mapped onto (1, 5). Given that the image of the point (x, y) is (11, –4), find the value of x and of y.

7. Triangle ABC has vertices $A(3, 4)$, $B(4, 2)$ and $C(1, 2)$.
   (a) $\triangle ABC$ can be mapped onto $\triangle ADE$ by an enlargement. It is given that D is the point $(k, 0)$.
      (i) State the coordinates of the centre of enlargement.
      (ii) Find the value of k and the coordinates of the point E.
   (b) $\triangle ABC$ is mapped onto $\triangle LMN$ by a shear with the line $y = 1$ as the invariant line and shear factor 2. Find the coordinates of L, M and N. Draw and label $\triangle LMN$.
   (c) Under a stretch with the line $x = 2$ as the invariant line and stretch factor –2, $\triangle ABC$ is mapped onto $\triangle GHJ$. Find the coordinates of G, H and J. Draw and label $\triangle GHJ$.
   (d) Describe completely the single transformation that will map $\triangle ABC$ onto $\triangle PQR$.

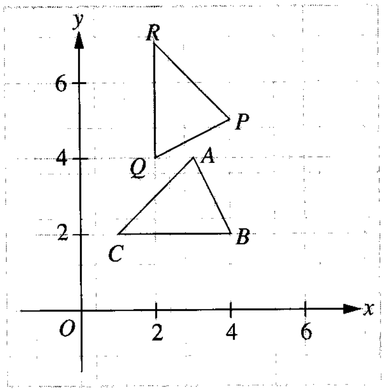


Fig. 7.153

8. The vertices of a square $ABCD$ are $A(1, 1)$, $B(1, –1)$, $C(–1, –1)$ and $D(–1, 1)$. Find the image of $ABCD$ under an enlargement centre at the origin and with the following scale factor.
   (a) 2
   (b) –1

9. Under the reflection in the line $x = 5$, the point (3, 6) is mapped onto the point (h, k). Under a reflection in the line $y = 5$, the point (h, k) is mapped onto the point (p, q). Find the values of h, k, p and q.

10. T is the translation $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and R is a 180° rotation about the origin.

    $\triangle ABC$ has vertices $A(2, 4)$, $B(3, 7)$ and $C(5, –2)$. Find the coordinates of the following.
    (a) T(A)
    (b) R(B)
    (c) TR(C)

11. Fig. 7.154 shows two squares $ABCD$ and $PQRS$. Describe completely the transformation that will map
    (a) $ABCD$ onto $PQRS$,
    (b) $ABCD$ onto $QPSR$,
    (c) $ABCD$ onto $RSPQ$.

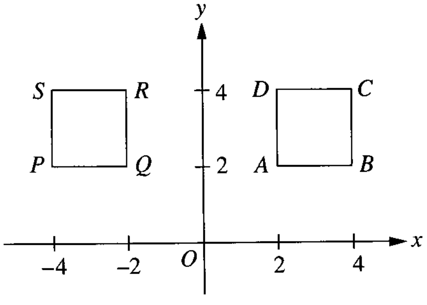


Fig. 7.154

12. Three transformations are defined in the following ways.

T: translation of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

M: reflection in the line $y = 1$.

R: anticlockwise rotation of 90° about (0, 0).

The coordinates of $\triangle ABC$ are $A(2, 2)$, $B(5, 2)$ and $C(2, 6)$.

(a) Using a sheet of graph paper, plot the points A, B and C.

(b) $\triangle ABC$ is mapped onto $\triangle A_1B_1C_1$ by the transformation represented by TM. Mark and label $\triangle A_1B_1C_1$.

(c) $\triangle ABC$ is mapped onto $\triangle A_2B_2C_2$ by the transformation represented by TR. Mark and label $\triangle A_2B_2C_2$.

(d) $\triangle A_2B_2C_2$ is mapped onto $\triangle A_3B_3C_3$ by the transformation M. Mark and label $\triangle A_3B_3C_3$.

13. Fig. 7.155 shows two parallelograms $OABC$ and $OPQR$. Describe completely the transformation that will map

(a) $OABC$ onto $OPQR$,

(b) $\triangle OPQ$ onto $\triangle ORQ$,

(c) $\triangle ORQ$ onto $\triangle OCB$.

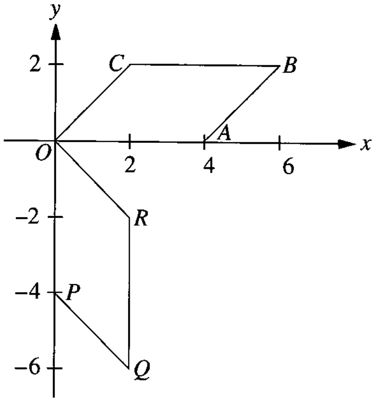


Fig. 7.155

14. The vertices of $\triangle ABC$ are $A(2, 1)$, $B(4, 6)$ and $C(5, 2)$. $\triangle ABC$ is reflected in the x-axis followed by a reflection in the line $y = x$. Plot the points A, B and C on a sheet of graph paper and also their images under the two successive reflections. Describe a single transformation which is equivalent to the two successive reflections.

15. The coordinates of A and B are (1, 3) and (1, 6). Under a rotation, the images of A and B are (4, 0) and (k, 0) where $k > 4$. Find the value of k. Given that (6, 0) is the image of the point C under the rotation, find the coordinates of C.

16. Three transformations A, B and C are defined in the following ways.

A: 90° anticlockwise rotation about the origin.

B: reflection in the line $y = x$.

C: reflection in the line $y + x = 0$.

Describe a single transformation equivalent to each of these successive transformations.

(a) AB **(b)** BA **(c)** BC **(d)** CAC

17. The figure represents an equilateral $\triangle ADF$ of side 6 cm. B, C and E are the mid-points of AD, AF and DF. Describe completely

(a) the transformation that will map $\triangle ABC$ onto $\triangle CEF$,

(b) two successive reflections which will map $\triangle ABC$ onto $\triangle EBD$,

(c) a reflection followed by a rotation that will map $\triangle ABC$ onto $\triangle FEC$,

(d) the transformation which will map $\triangle ADF$ onto $\triangle ABC$.

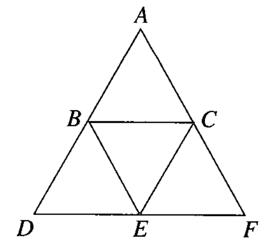


Fig. 7.156

18. On a sheet of graph paper, using a scale of 2 cm to represent 2 units on both axes, draw $\triangle ABC$ whose coordinates are $A(1, 1)$, $B(4, 2)$ and $C(3, 4)$. T is the translation represented by $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and R is the anticlockwise rotation of 90° about the origin. $\triangle A_1B_1C_1$ is the image of $\triangle ABC$ under the transformation RT. Find the image of $\triangle A_1B_1C_1$, mark and label $\triangle A_1B_1C_1$ on your graph.

$\triangle ABC$ is mapped onto $\triangle A_2B_2C_2$ by a shear with $x = 0$ as the invariant line and shear factor 2. Find the coordinates of A_2, B_2 and C_2. Draw and label $\triangle A_2B_2C_2$.

19. The triangle ABC has vertices $A(2, 2)$, $B(5, 2)$ and $C(2, 4)$.

The triangle $A_1B_1C_1$ has vertices $A_1(-2, 2)$, $B_1(1, 2)$ and $C_1(-6, 4)$.

The triangle $A_2B_2C_2$ has vertices $A_2(3, 3)$, $B_2(9, 3)$ and $C_2(3, 7)$.

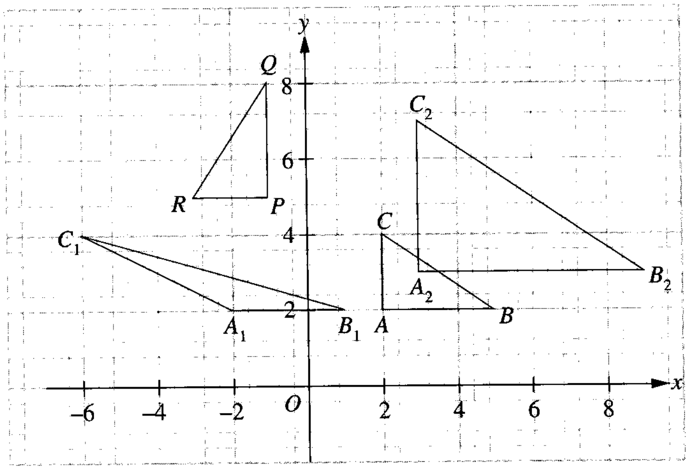


Fig. 7.157

(a) Describe completely the single transformation that will map $\triangle ABC$ onto $\triangle A_1B_1C_1$.

(b) Describe completely the single transformation that will map $\triangle ABC$ onto $\triangle A_2B_2C_2$.

(c) Under a reflection in the line l, $\triangle ABC$ is mapped onto $\triangle A_3B_3C_3$. Given that the coordinates of C_3 is $(0, 2)$.
Draw the line l and find its equation.

(d) Describe a single transformation that will map $\triangle ABC$ onto $\triangle PQR$.

20. On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on both axes, draw the $\triangle T$ whose vertices are (3, 1), (3, 3) and (5, 3). M represents a reflection in the line $x = 2$, R represents a 180° rotation about (2, 0). Draw and label the image of $\triangle T$ under the following transformations.

(a) M **(b)** R

Describe completely the transformation P such that R = PM. The coordinates of the image under a transformation Q are (4, 0), (6, 0) and (6, –2). Describe the transformation Q completely.

21. The transformation S is a shear with the y-axis invariant and shear factor 4.

(a) Find the coordinates of the image of the point (2, –3) under S.

(b) Given that the image of the point A under S is (5, –8), find the coordinates of A.

(c) S maps $\triangle PQR$ whose area is 14 cm² onto $\triangle LMN$. Find the area of $\triangle LMN$.

22. Fig. 7.158 shows a regular hexagon $ABCDEF$ whose centre is O.
Describe completely a single transformation that will map

(a) $\triangle OAB$ onto $\triangle ODE$,

(b) $\triangle FOA$ onto $\triangle EDO$,

(c) $\triangle EOF$ onto $\triangle AOB$,

(d) $\triangle DOC$ onto $\triangle COB$.

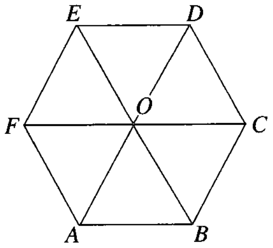


Fig. 7.158

23. **(a)** Given that the transformation H maps $\triangle ABC$ onto $\triangle PQR$, describe the transformation H completely.

(b) Given that the transformation K maps $\triangle ABC$ onto $\triangle LMN$, describe the transformation K completely.

(c) Given that the transformation U maps $\triangle PQR$ onto $\triangle XYZ$, describe the transformation U.

(d) Given that $\triangle LMN$ can be mapped onto $\triangle PQR$ by a transformation V, describe the transformation V.

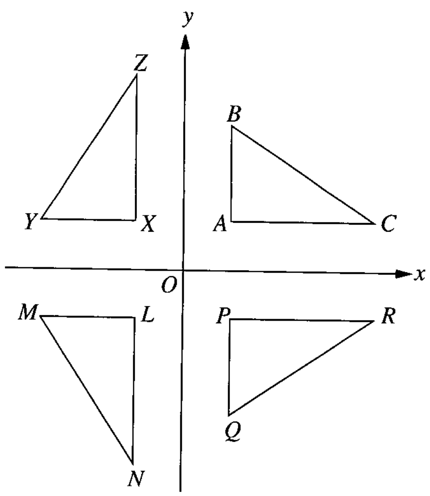


Fig. 7.159

24. The vertices of $\triangle ABC$ are $A(1, 2)$, $B(5, 3)$and $C(5, 7)$. $\triangle ABC$ is rotated through 90° about the point (2, 0) to $\triangle A_1B_1C_1$. $\triangle A_1B_1C_1$ undergoes a one-way stretch parallel to the x-axis with stretch factor 2 to $\triangle A_2B_2C_2$. $\triangle A_2B_2C_2$ is then enlarged to $\triangle A_3B_3C_3$ under an enlargement centre (5, 4) and scale factor 3. $\triangle A_3B_3C_3$ finally undergoes a shear parallel to the y-axis with shear factor 5 to $\triangle A_4B_4C_4$.

Find the area of the following:

(a) $\triangle A_1B_1C_1$ **(b)** $\triangle A_2B_2C_2$ **(c)** $\triangle A_3B_3C_3$ **(d)** $\triangle A_4B_4C_4$

25. **Answer the whole of this question on a sheet of graph paper.**

(a) Using a scale of 1 cm to 1 unit on each axis, draw x- and y-axes for $-8 \leqslant x \leqslant 8$ and $-10 \leqslant y \leqslant 10$, draw and label $\triangle ABC$ with vertices $A(5, 4)$, $B(6, 1)$ and $C(2, 2)$.

(b) $\triangle ABC$ is mapped onto $\triangle APQ$ under an enlargement. Given that Q is a point on the x-axis, draw and label $\triangle APQ$. Write down the coordinates of P and Q.

(c) The coordinates of $\triangle A_1B_1C_1$ are $A_1(5, 8)$, $B_1(6, 6)$ and $C_1(2, 3)$. Draw and label $\triangle A_1B_1C_1$. Describe a single transformation that will map $\triangle ABC$ onto $\triangle A_1B_1C_1$.

(d) Under a reflection in the line l, the image of C_1 is (–1, 0), draw the line l in your diagram and find the equation of the line l.

(e) The triangle ABC is mapped onto $\triangle LMN$ by a translation represented by the vector $\begin{pmatrix} -6 \\ -4 \end{pmatrix}$ followed by a rotation. Draw and label $\triangle LMN$ whose vertices are $L(-3, 4)$, $M(0, 5)$ and $N(-1, 1)$. Find the centre of rotation and state the angle of rotation.

26. The diagram shows the line l and the triangles ABC and $A_1B_1C_1$.

(a) Find the equation of the line l.

(b) $\triangle ABC$ is mapped onto $\triangle PQR$ under a reflection in the line l. Find the coordinates of P and Q.

(c) $\triangle A_1B_1C_1$ is the image of $\triangle ABC$ under a shear.

Find **(i)** the equation of the invariant line,
(ii) the shear factor.

(d) $\triangle ABC$ is mapped onto $\triangle LMN$ by a translation represented by $\begin{pmatrix} -6 \\ -1 \end{pmatrix}$ followed by a 90° clockwise rotation about (–3, 0). Find the coordinates L and M.

(e) Under an enlargement centre at (0, 2) and scale factor –2, $\triangle ABC$ is mapped onto $\triangle A_2B_2C_2$. Find the coordinates of A_2 and B_2.

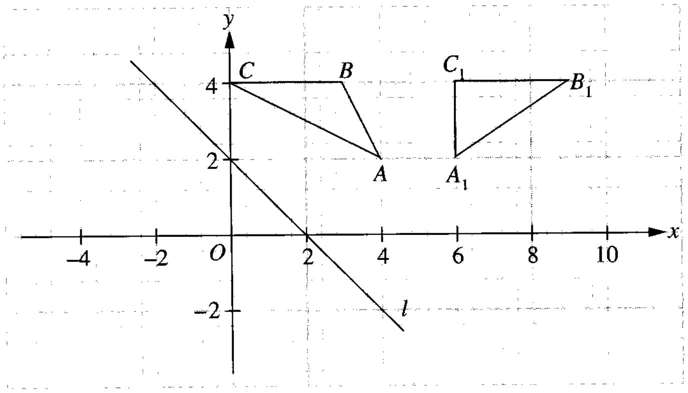


Fig. 7.160

27. Answer the whole of this question on a sheet of graph paper.

Triangle A has vertices (4, 2), (6, 2) and (4, 6). Triangle B has vertices (4, 8), (6, 11) and (4, 12). Triangle C has vertices (3, 7), (3, 9) and (–1, 7).

(a) Using a scale of 1 cm to represent 1 unit on each axis, draw axes for values of x and y in the ranges $-6 \leqslant x \leqslant 12$ and $-4 \leqslant y \leqslant 14$. Draw and label triangles A, B and C.

(b) Triangle A can be mapped onto triangle B by a single transformation H. Describe transformation H completely.

(c) Describe fully the single transformation which maps triangle A onto triangle C.

(d) An enlargement with centre at (2, 4) and scale factor $2\frac{1}{2}$ maps triangle A onto triangle D. Draw and label triangle D on your diagram.

(e) Triangle A is mapped onto triangle E by a translation represented by $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, followed by a reflection in the line $y + x = 0$.

(i) Draw and label the line $y + x = 0$.

(ii) Draw and label triangle E.

Specimen Paper A

Paper 1 **Time: 2 hours**

Answer **all** the questions. Omission of essential working will result in loss of marks. The use of **all calculating aids** is **prohibited**. The number of marks given is shown in the brackets [ ] at the end of each question or part question.

1. Simplify $\left(2\frac{1}{2} \times 2\frac{1}{2} - 2\frac{1}{4} \times 2\frac{3}{4}\right) \times 64$. [2]

2. Find the exact value of $\frac{21}{0.35} + \frac{65}{0.52}$. [2]

3. Find the value of

(a) $9^{-\frac{3}{2}} + 5^0$, [1]

(b) $4^{-11} \times 4^4 \times 4^5$. [1]

4. Ten cards numbered 1, 2, 2, 3, 3, 3, 5, 7, 8, 9 are placed in a box. Peta draws a card at random from the box. Find the probability that he draws

(a) a prime number, [1]

(b) a number divisible by 3. [1]

5. **(a)** Express 45 minutes after 11.36 p.m. in 24-hour time. [1]

(b) A tradesman charges \$84 for a job lasting $2\frac{1}{2}$ hours. How much should he charge for one lasting $4\frac{1}{2}$ hours, if charges are made at the same hourly rate? [1]

6. Given that $x = 12$, $y = -5$, $z = \frac{1}{3}$, evaluate

(a) $-2(x + z)$, [1]

(b) $\frac{9z + x}{y}$. [1]

7. A radar station transmits a signal which travels at 298 000 km per second. This signal, when reflected from an aircraft, returns to the transmitter at the same speed.

(a) Write down the speed at which the signal is transmitted, giving your answer in standard form. [1]

(b) Find the difference in time between the signals received by reflection from two aircrafts if one is $372\frac{1}{2}$ metres farther away from the station than the other. [2]

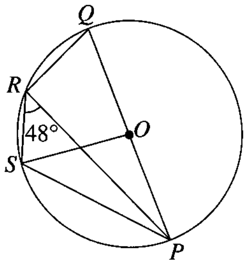
8. **(a)** Simplify $7p^7 \times 2p^{-\frac{1}{3}}$. [1]

(b) Solve the equation

$$2(3x - 8) = x + 19. \quad [1]$$

(c) Find 12% of 75 km. [1]

9. In the diagram, O is the centre of the circle and POQ is a straight line.



(a) Write down the value of $P\hat{R}Q$. [1]

(b) Calculate

(i) $Q\hat{O}S$, [1]

(ii) $Q\hat{P}S$. [1]

10. Express 4 116 in prime factors. [2]

Find the smallest whole number by which 4 116 must be multiplied, in order to give a perfect square. [1]

11. **(a)** Factorise completely $6t^2 - 18t$. [1]

(b) Factorise $x^2 + 5x - 14$. [1]

(b) Factorise $9a^2 - b^2$. [1]

12. **(a)** Given that $a = 2b$, find the numerical value of $\frac{2a + 7b}{24b - a}$. [1]

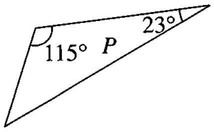
(b) In a bag of x sweets, $\frac{2x}{5}$ are yellow, $\frac{x}{10}$ are green and the rest are red.

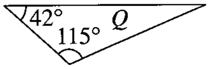
(i) Find the number of red sweets. [1]

(ii) Mary takes a sweet at random from the bag. The sweet is replaced in the bag because it is not the colour

Mary prefers. She draws a second sweet at random from the bag. What is the probability that the first sweet drawn is yellow and the second sweet drawn is green? [2]

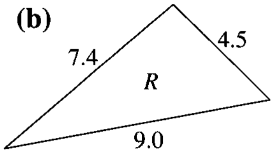
13. (a)

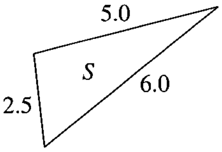




Are triangles P and Q similar?
Explain the reason for your answer. [1]

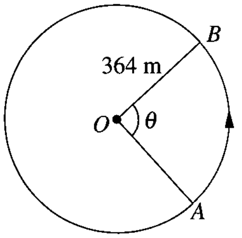
(b)



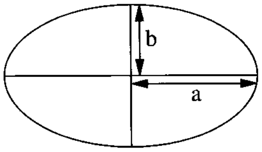


Are triangles R and S similar?
Explain the reason for your answer. [2]

14. (a) The diagram represents a circular motor-racing track, radius 364 m. A car travels at a constant speed of 286 km/h around the track from A to B. If the car takes 8 seconds to reach B, find the angle subtended by the arc AB at the centre O. $\left(\text{Take } \pi = \frac{22}{7}.\right)$ [2]



(b) The scale of a survey map is 1 : 20 000. A lake is approximately oval on the map, being 8.4 cm long and 6 cm wide. Find the area of the lake in km^2.



$\Big[$ The area of an oval $2a$ long and $2b$ wide is given by the formula $A = \pi ab$. Take $\pi = \frac{22}{7}.\Big]$ [2]

15. (a) If 15 pumps of a standard pattern can raise 12 000 litres of water in 14 hours, find an expression (but do not try to simplify it) for the number of such pumps required to raise 13 440 litres of water in 20 hours. [2]

(b) A television set is sold for \$1 845 at a profit of 23% of its cost price. Calculate this profit. [2]

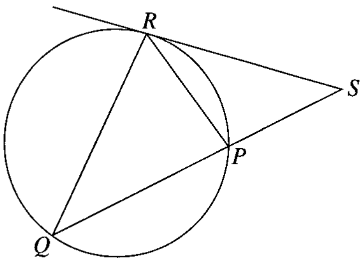
16. (a) Express as a single fraction in its lowest terms

$$\frac{3x-1}{2} - \frac{2x+5}{7} + 1. \qquad [2]$$

(b) Solve the simultaneous equations

$$4x - 3y = 17$$
$$5x + 6y = -8. \qquad [2]$$

17. SR is a tangent and SPQ is a straight line.



(a) Explain why triangles SPR and SQR are similar. [1]

(b) Given that SQ = 14 cm and SR = 7 cm, find

(i) SP, [2]

(ii) $\frac{\text{area } \triangle SPR}{\text{area } \triangle SQR}$. [1]

18. (a) Given that $s = \sqrt{\left(\frac{u-v}{t}\right)}$, express u in terms of s, v and t. [2]

(b) Calculate the exact value of $\sqrt[3]{27 \times 10^{12}}$. [1]

19. $\overrightarrow{PQ} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, $\overrightarrow{QR} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$ and $\overrightarrow{OP} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$.

(a) Express $\overrightarrow{PR}$ as a column vector. [1]

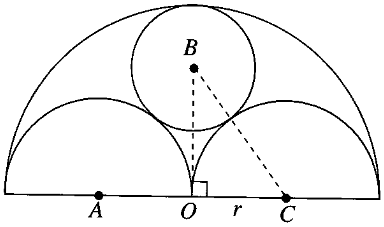
(b) Find the coordinates of the point Q. [1]

(c) If $\overrightarrow{XY} = \begin{pmatrix} -4 \\ 14 \end{pmatrix}$, what is the special name given to the quadrilateral $RXYQ$? [1]

(d) A is the point $(-3, k)$ and B is $(-2, 5)$.

If $|\overrightarrow{AB}| = |\overrightarrow{PQ}|$, find the two possible values of k. [2]

20. In the diagram, the circle centre B has radius 1 cm. The semicircle centre A and the semicircle centre C are identical and each has radius r cm. O is the centre of the larger semicircle.



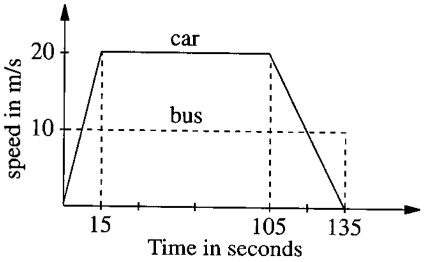
(a) Write an expression for BO in terms of r. [1]

(b) Using Pythagoras theorem, form an equation in r and show that it reduces to

$$2r^2 - 3r = 0. \quad [3]$$

(c) Solve the equation to find the radius of the larger semicircle. [1]

21. The diagram shows the speed-time graphs of a bus and a car during a period of 2 minutes and 15 seconds.



(a) Calculate

(i) the acceleration of the car, during the first seconds, [1]

(ii) the retardation of the car during the last seconds, [1]

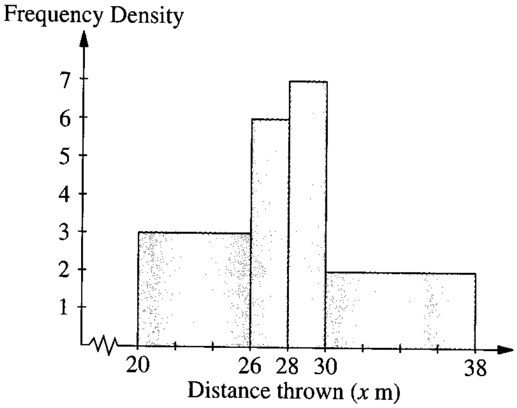
(iii) the distance travelled by the car when moving at uniform speed, [1]

(iv) the distance travelled by the car during the 135 seconds. [1]

(b) The bus and the car start from the same point at the same time and travel in the same direction.

After T seconds, where $15 < T < 135$, the bus travelled twice as far as the car. Find the value of T. [1]

22. The results of the discus event at a school's athletics competition were analysed and the following histogram illustration of these results was obtained.



(a) Copy and complete the following table.

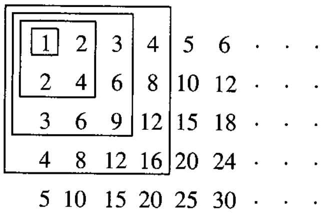
Distance thrown (x m)	*Number of students*
$20 \leqslant x < 26$	
$26 \leqslant x < 28$	12
$28 \leqslant x < 30$	
$30 \leqslant x < 38$	

[2]

(b) Work out an estimate of the mean distance thrown. [2]

(c) Draw a pie chart to illustrate the information in the table. [2]

23. Squares are placed to enclose numbers in the number array as shown in the diagram.



1	2	3	4	5	6	· · ·
2	4	6	8	10	12	· · ·
3	6	9	12	15	18	· · ·
4	8	12	16	20	24	· · ·
5	10	15	20	25	30	· · ·

The sum of the numbers in the first square, $S_1 = 1 = 1^2$.

The sum of the numbers in the second square, $S_2 = 1 + 2 + 2 + 4 = 9 = 3^2$.

(a) Find S_3, S_4 and S_5. [2]

(b) Find a formula for S_n in terms of n. [3]

(c) The sum of the numbers in the kth square is 44 100. Find k. [1]

Paper 2 Time: 2 h 30 min

Answer **all** the questions. All working must be clearly shown. The number of marks given is shown in the brackets [ ] at the end of each question or part question.

1. (a) Two points P and Q are on level ground and a vertical mast 315 metres high is erected at P. On a map of scale 1 : 20 000, the distance PQ is represented by a line of length 1.35 cm. Find

(i) the actual length of PQ in metres, [1]

(ii) the angle of elevation of the top of the mast from Q. [2]

(b) There are 1 200 pupils in a secondary school. The school receives an allowance of $93.16 per pupil each to spend on books and stationery.

(i) Calculate the total annual school allowance for books and stationery. [1]

The school spends $95 023.20 to buy books.

(ii) Express the amount spent on books as a percentage of the total annual school allowance for books and stationery. [2]

Two-fifths of the pupils are in the Upper Secondary, and books for each Upper Secondary pupil cost $94 per year.

(iii) Calculate the total cost of Upper Secondary books per year. [1]

(iv) Calculate the mean expenditure on books per year for each pupil not in the Upper Secondary. [2]

2. (a) The weights of 100 members of a sports club are given in the following table.

Weight (x kg)	*Number of members*
$45 < x \leqslant 55$	5
$55 < x \leqslant 65$	10
$65 < x \leqslant 75$	21
$75 < x \leqslant 85$	45
$85 < x \leqslant 95$	16
$95 < x \leqslant 105$	3

(i) Construct a frequency polygon to show this information. [3]

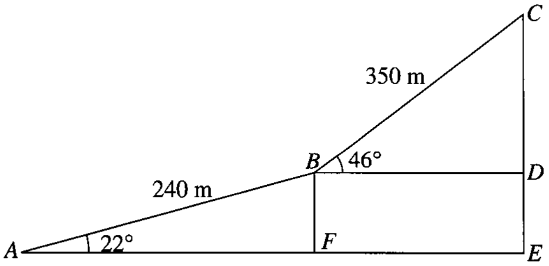
(ii) Calculate an estimate of the mean weight of these members. [3]

(b) In a pie chart drawn to represent the number of animals on a large farm, the angle representing deer is 165°. A quarter of the animals are goats and there are 287 cows. Given that there are no other animals, calculate

(i) the percentage of deer on the farm, giving your answer correct to the nearest whole number, [1]

(ii) the total number of animals on the farm. [2]

3. $BFED$ is a rectangle. Angle $BAF = 22°$, angle $CBD = 46°$, $AB = 240$ m and $BC = 350$ m. Calculate



(a) AE, [2]

(b) the area of trapezium $BFEC$, [3]

(c) angle CAB. [2]

4. A lighthouse L is 9 km due north of a point P. A boat travels at 8 km/h on a course 035°, leaving the port at 09 00.

(a) Using a scale of 1 cm to represent 1 km, construct the position A at which the boat is equidistant from P and L. [2]

At A, the boat alters course and sails on a constant bearing so that its nearest distance from the lighthouse is 2 km.

(b) On the same diagram, draw the locus which represents points 2 km from L. [1]

(c) Obtain by geometrical construction the two possible courses and measure the bearings. [3]

(d) Hence, estimate to the nearest minutes the time when the boat is nearest the lighthouse. [2]

5. Mr Lee smoked x packets of cigarettes and drank y cans of beer in one month when a packet of cigarettes cost \$4.50 and a can of beer cost \$2.85. His total expenditure was \$288. After the budget, the cost of a packet of cigarettes was increased by 70 cents and the cost of a can of beer was increased by 40 cents. He calculated that his monthly expenditure on cigarettes and beer would increase by \$43.50.

(a) Form two simultaneous equations in x and y and show that they reduce to

$$30x + 19y = 1\,920$$

and $$7x + 4y = 435.$$ [3]

(b) Solve these equations to find the total number of packets of cigarettes Mr Lee smoked and the total number of cans of beer Mr Lee drank in one year. [3]

(c) After attending a series of talks on how to stay healthy, Mr Lee decided to give up smoking entirely and cut down the drinking of beer by one-third. Calculate how much Mr Lee could save in a period of one year as a result of quitting smoking and changing drinking habit. [2]

(d) Mr Lee's only son is expected to go to university in 5 years' time and the cost of providing his son with the university education is estimated to be \$50 000. Mr Lee plans to use the money saved by giving up smoking and cutting down the drinking of beer to finance part of the cost. Calculate the amount saved in 5 years as a percentage of his son's estimated university cost. [2]

6. In an experiment, a small spherical drop of oil is allowed to fall on to the surface of water so that it produces a thin film of oil covering a large area. 9 426 such drops are needed to completely fill a small container in the form of a cone of base radius 5 cm and height 4.5 cm.

(a) Taking π to be 3.142, find, in mm^3, the volume of one drop of oil. [3]

(b) Given that the volume V of a sphere of radius r is $\frac{4}{3}\pi r^3$,

(i) express r in terms of V and π, [1]

(ii) calculate the radius of one drop, taking π to be 3.14. [2]

A single drop is found to produce a circular oil film of 1 150 cm^2.

(c) Calculate the thickness of the film in mm, expressing your answer in standard form. [3]

7. A harbour H and an oil rig P are 62 km apart with P due east of H. A supply ship leaves H for a second oil rig Q which is 44 km from P on a bearing 048° from H.

Find

(a) the bearing of Q from P, [3]

(b) the distance HQ. [2]

A seaside resort R is situated west of the line HQ. R is 45 km and 61 km from H and Q respectively.

(c) The supply ship leaves H at 11 15. It sails directly to R, where it stays for 40 minutes, then returns to H. When moving, it may be assumed that it travels at a constant speed of 15 km/h. At what time does it return to H? [2]

(d) Calculate the angle HQR. [2]

(e) What is the shortest distance from R to HQ? [1]

(f) Find the area of quadrilateral $HPQR$. [2]

8. The vertices of the triangle ABC are $A(1, 1)$, $B(4, 1)$ and $C(1, 3)$.

(a) Using a scale of 1 cm to represent 1 unit on both axes, draw the x- and y-axes for $-8 \leqslant x \leqslant 8$ and $-8 \leqslant y \leqslant 8$. Draw and label $\triangle ABC$. [1]

(b) An enlargement with centre at (0, 0) and scale factor 2 maps $\triangle ABC$ onto $\triangle LMN$.

- **(i)** Draw and label $\triangle LMN$. [1]
- **(ii)** Write down the ratio of
  - **(a)** angle ABC : angle LMN, [1]
  - **(b)** $AC : LN$, [1]
  - **(c)** area of triangle ABC : area of triangle LMN. [1]
- **(iii)** If $\triangle L'M'N'$ is an enlargement of $\triangle LMN$, centre (0, 0) and scale factor 2, write down the ratio of area of triangle ABC : area of triangle $L'M'N'$. [1]

(c) A reflection in the y-axis followed by a reflection in the x-axis maps $\triangle ABC$ onto $\triangle PQR$.

- **(i)** Draw and label $\triangle PQR$. [1]
- **(ii)** $\triangle ABC$ can also be mapped onto $\triangle PQR$ by two single transformations, a rotation and an enlargement. Describe the two single transformations clearly. [2]

(d) $\triangle PQR$ is mapped onto $\triangle XYZ$ by a shear along the x-axis with shear factor 2.

- **(i)** Draw and label $\triangle XYZ$. [2]
- **(ii)** Write down the ratio of area of $\triangle PQR$: area of $\triangle XYZ$. [1]

9. A bag contains equal number of black and white marbles. Two marbles are drawn at random from the bag, one after the other, and are not replaced. The probabilities that the second marble drawn is black when there are 2, 4, 6 and 8 marbles respectively in the bag are calculated as shown below.

2(1) = 2 marbles :

$$\text{Probability} = \frac{1}{2} \times \frac{1}{1} + \frac{1}{2} \times \frac{0}{1} = \frac{1}{2} + \frac{0}{2}$$
$$= \frac{1}{2}.$$

2(2) = 4 marbles :

$$\text{Probability} = \frac{2}{4} \times \frac{2}{3} + \frac{2}{4} \times \frac{1}{3} = \frac{2}{6} + \frac{1}{6}$$
$$= \frac{1}{2}.$$

2(3) = 6 marbles :

$$\text{Probability} = \frac{3}{6} \times \frac{3}{5} + \frac{3}{6} \times \frac{2}{5} = \frac{3}{10} + \frac{2}{10}$$
$$= \frac{1}{2}.$$

2(4) = 8 marbles :

$$\text{Probability} = \frac{4}{8} \times \frac{4}{7} + \frac{4}{8} \times \frac{3}{7} = \frac{4}{14} + \frac{3}{14}$$
$$= \frac{1}{2}.$$

(a) Write down, in the similar manner, the probability of obtaining a black marble in the second draw when there are

- **(i)** 10 marbles,
- **(ii)** 18 marbles in the bag. [2]

(b) When there are $2n$ marbles in the bag, we have

P(second marble drawn is black) =

$$\frac{a}{p} \times \frac{a}{q} + \frac{a}{p} \times \frac{b}{q} = \frac{a}{r} + \frac{b}{r} = \frac{1}{2}.$$

Express a, b, p, q and r in terms of n. [5]

(c) When the number of marbles in the bag is 20, write down the probability that a black marble is drawn after a white marble. [1]

(d) When the number of marbles in the bag is 24, find the probability that **not** both the two marbles drawn are black. [2]

(e) Find the values of c and d such that

$$\frac{c}{1998} + \frac{d}{1998} = \frac{1}{2}.$$ [2]

10. Answer either (I) or (II) of this question.

(I) **Answer the whole of this question on a sheet of graph paper.**

The volume of an **open** rectangular box, made of thin metal, is 35 cm^3.

The base of the box is a square of side x cm.

(a) Find, in terms of x, an expression for the height of the box. [1]

(b) The total external area, of the base and the four sides, is A cm^2. Show that

$$A = x^2 + \frac{140}{x}. \qquad [2]$$

(c) Complete the following table which gives values of x and the corresponding values of A.

x	2	2.5	3	3.5	4	4.5	5	5.5	6
A	74			52.3	51	51.4		55.7	

[2]

(d) Using a scale of 4 cm to 1 unit, draw a horizontal x-axis for $2 \leqslant x \leqslant 6$. Using a scale of 4 cm to 5 units, draw a vertical A-axis for $50 \leqslant A \leqslant 75$. Draw the graph of A against x. [3]

(e) Use your graph to find

- **(i)** the side of the largest base, which will give a total surface area of 55 cm^2, [1]
- **(ii)** the minimum value of A, [1]
- **(iii)** the height of the box for which the least amount of metal is used, [1]
- **(iv)** the total surface area when the base has side 2.6 cm. [1]

(II) Answer the whole of this question on a sheet of graph paper.

The following table gives the frequency of marks obtained by 60 candidates in an examination.

Mark	*Number of candidates*
$10 \leqslant x < 20$	4
$20 \leqslant x < 30$	8
$30 \leqslant x < 35$	11
$35 \leqslant x < 40$	15
$40 \leqslant x < 45$	10
$45 \leqslant x < 50$	5
$50 \leqslant x < 65$	7

(a) Copy and complete the following cumulative frequency table.

Mark	10	20	30	35	40	45	50	65
Number of candidates with this mark or less	0	4						60

[1]

(b) Using a horizontal scale of 2 cm to represent 10 marks and a vertical scale of 2 cm to represent 10 candidates, draw a smooth cumulative frequency curve for these results. [3]

(c) Showing your method clearly, use your graph to estimate

- **(i)** the median, [1]
- **(ii)** the 90th percentile of the distribution. [1]

Grade E was awarded to candidates scoring less than 30 marks and Grade A to those scoring 50 marks or more.

(d) A candidate was selected at random from the sixty. Find the probability that

- **(i)** the candidate was awarded Grade E, [1]
- **(ii)** the candidate was awarded either Grade E or Grade A. [1]

(e) Two candidates were selected at random from the sixty. Find the probability that

- **(i)** exactly one of them was awarded Grade E, [2]
- **(ii)** neither of the candidates was awarded Grade E or Grade A. [2]

Give your answers to **(d)** and **(e)** as fractions in their lowest terms.

Specimen Paper B

Paper 1 **Time: 2 hours**

Answer **all** the questions. Omission of essential working will result in loss of marks. The use of **all calculating aids** is **prohibited**. The number of marks given is shown in the brackets [ ] at the end of each question or part question.

1. **(a)** Evaluate $2^{-3} \times 27^{\frac{2}{3}}$. [1]
   (b) If $4^{2x-1} = 8$, find x. [1]

2. **(a)** Write down the next two terms in the sequence
   2, 3, 7, 16, 32, 57, ______, ______. [1]
   (b) Write down an expression, in terms of n, for the nth term of the sequence
   3, 5, 9, 17, 33, 65, … . [1]

3. Find the largest integer x such that
   $$\frac{x}{2} < 23 - 3x. \qquad [2]$$

4. An overnight train took 8 hours 45 minutes to cover a distance of 665 km and arriving at its destination at 06 20 on Monday. Find
   (a) the time and day at which the train started its journey, [1]
   (b) the average speed of the train, giving your answer in kilometres per hour. [1]

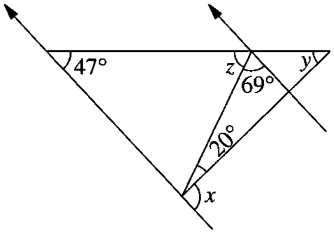
5. Factorise the following completely
   (a) $12x^2 - 3y^2$, [1]
   (b) $2x^2 - 3x + 2xy - 3y$. [1]

6. When p is decreased by 10% and then increased by 20%, it becomes q. Express the ratio of $p : q$ in its simplest form. [2]

7. Given that $y = \dfrac{a^3 + b}{b - c}$.
   (a) Find the value of y when $a = -2$, $b = 4$ and $c = -1$. [1]
   (b) Express b in terms of a, c and y. [2]

8. A map is drawn to a scale of 1 : 40 000.
   (a) Two towns are 25 cm apart on the map. Calculate the actual distance of the two towns in km. [1]
   (b) A forest reserve has an area of 8 km^2. Calculate, in cm^2, the area of the forest reserve on the map. [2]

9. Calculate the angles marked x, y and z in the figure.

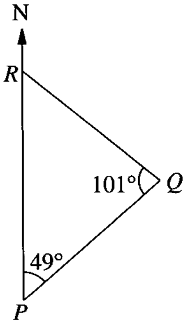


[3]

10. Solve the simultaneous equations
    $$5x + 3y = 1, \; x - 4y = 14. \qquad [2]$$

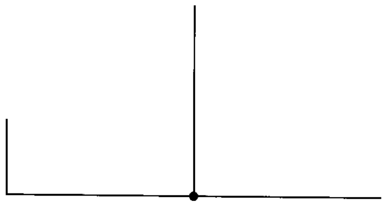
11. Express 200 cm^3 as a percentage of 0.04 m^3. [2]

12. The points P, Q and R are on level ground and $P\hat{Q}R = 101°$. R is due north of P, and Q is on a bearing 049° from P. Find
    (a) the bearing of Q from R, [1]
    (b) the bearing of R from Q. [1]

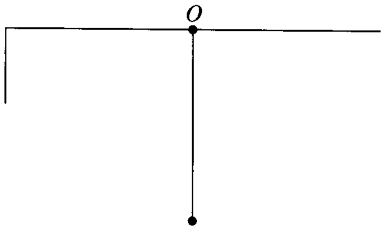


13. A man left $\frac{5}{12}$ of his money to his wife, $\frac{2}{3}$ of the remainder to his two children and the rest to be divided equally among seven charitable organisations. If each charitable organisation received $12 500, how much had the man left behind? [3]

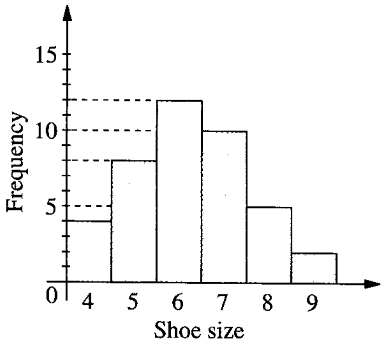
14. **(a)** Add a line segment to the figure so that it will give one axis of symmetry. [1]



(b) Add two line segments to the figure so that it will have order of rotational symmetry of 2 about the point O. [2]



15. The bar graph shows the sizes of shoes worn by a group of prefects in a secondary school.



(a) Calculate the total number of prefects in the school. [1]

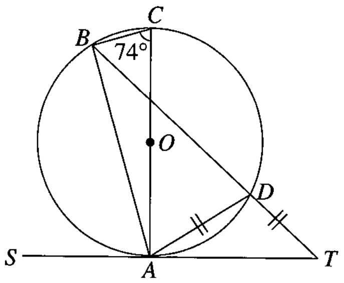
(b) If you are the owner of a shoe shop catering mainly to school children, state 3 shoe sizes that you would like to stock in the shop, giving your reasons clearly. [2]

16. The line $5x + 6y = 30$ cuts the x-axis at P and the y-axis at Q. Calculate the area of $\triangle PQR$ where R is the point $(6, 7)$. [3]

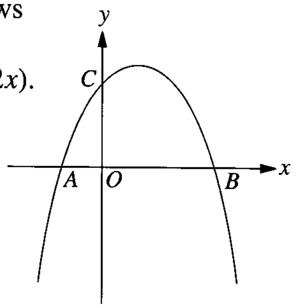
17. In the diagram, O is the centre of the circle $ABCD$. Given that $AD = TD$, SAT is the tangent to the circle at A and $A\hat{C}B = 74°$. Calculate

(a) $A\hat{B}D$, [2]

(b) $B\hat{A}C$. [1]



18. The diagram shows part of the graph $y = (2x + 3)(5 - 2x)$.



(a) Write down the coordinates of A and C. [2]

(b) Write down the equation of the axis of symmetry of the graph

$$y = (2x + 3)(5 - 2x).$$ [1]

19. 25 cards bearing the numbers 1, 2, 3, ..., 24, 25 are put inside a box A.

(a) A card is picked from the box A at random. Find the probability that the selected card is

(i) a multiple of 3 as well as a multiple of 7, [1]

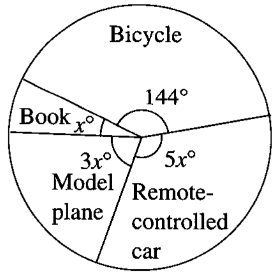
(ii) a multiple of 3 or a multiple of 7. [1]

(b) Cards which are either a multiple of 3 or a multiple of 4 are selected and put in box B. A card is then picked at random from box B. Find the probability that the selected card is

(i) a multiple of 7, [1]

(ii) a multiple of 3, 4 or 7. [2]

20. Each child in a group of 240 was asked to choose a bicycle, a remote-controlled car, a book or a model plane. The pie chart illustrates their choices.

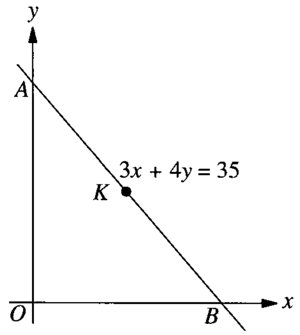


(a) Calculate the value of x. [1]

(b) How many children chose a remote-controlled car? [1]

(c) Express the number of children who chose a model plane as a fraction of the total number of pupils in its lowest terms. [1]

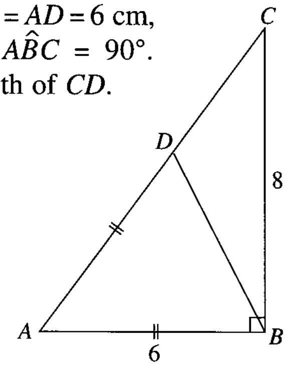
21. The equation of the line $3x + 4y = 35$ cuts the x- and y-axes at A and B.



(a) Write down the gradient of AB. [1]

(b) A point K lies on the line and is equidistant from the x- and y-axes. Find the coordinates of K. [2]

22. In the diagram, $AB = AD = 6$ cm, $BC = 8$ cm and $A\hat{B}C = 90°$. Calculate the length of CD.

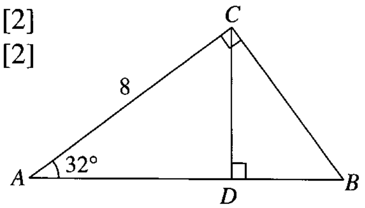


[2]

23. In the figure, $A\hat{C}B = A\hat{D}C = 90°$, $C\hat{A}B = 32°$ and $AC = 8$ cm. Using as much of the information given below as is necessary, calculate the length of

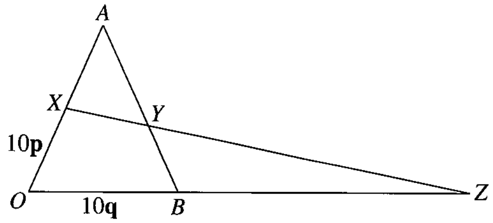
(a) BC, [2]

(b) AD. [2]



[Given: sin 32° = 0.53, cos 32° = 0.85, tan 32° = 0.62.]

24. In the diagram, $\overrightarrow{OA} = 10\mathbf{p}$, $\overrightarrow{OB} = 10\mathbf{q}$ and X is the mid-point of OA. Z is a point on OB produced such that $\frac{OB}{BZ} = \frac{1}{2}$.



Express in terms of $\mathbf{p}$ and $\mathbf{q}$, as simply as possible,

(a) $\overrightarrow{AB}$, (b) $\overrightarrow{XZ}$. [2]

The lines AB and XZ intersect at Y.

(c) Given that $\overrightarrow{AY} = s\overrightarrow{AB}$, show that $\overrightarrow{OY} = 10(1 - s)\mathbf{p} + 10s\mathbf{q}$. [1]

(d) Given that $\overrightarrow{XY} = t\overrightarrow{XZ}$, show that $\overrightarrow{OY} = 5(1 - t)\mathbf{p} + 30t\mathbf{q}$. [1]

(e) Using the results of parts **(c)** and **(d)**, find the values of s and t. [3]

(f) Find the numerical value of the ratio $\frac{XY}{YZ}$. [1]

Paper 2 Time: 2 h 30 min

Answer **all** the questions. All working must be clearly shown. The number of marks given is shown in the brackets [ ] at the end of each question or part question.

1. Rain water collected in a rectangular container, with a base measuring 5 m by 8 m, reached a height of 4.5 cm. All the water was then allowed to run into a cylindrical tank of internal diameter 2.4 m.

(a) Calculate the depth of water in the cylindrical tank. [3]

(b) Part of the water in the cylindrical tank was used to completely fill 5 hemispherical containers of internal radius 26 cm. Find the drop in the water level in the cylindrical tank. [3]

(c) The remaining water in the cylindrical tank was allowed to drain through a valve at a rate of 2.5 litres per minute. Calculate the time needed to drain all the remaining water from the cylindrical tank. Give your answer correct to the nearest minute. [3]

2. Consider the pattern

$1^3 = 1 = 1^2$

$1^3 + 2^3 = 9 = (1 + 2)^2$

$1^3 + 2^3 + 3^3 = 36 = (1 + 2 + 3)^2$

$1^3 + 2^3 + 3^3 + 4^3 = 100 = (1 + 2 + 3 + 4)^2$

$\vdots \qquad \vdots \qquad \vdots$

(a) Write down the seventh line in the pattern. [1]

(b) Find the value $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3$. [1]

(c) Given that $1^3 + 2^3 + \ldots + k^3 = 55^2$, find the value of k. [2]

(d) What is the value of $1^3 + 2^3 + 3^3 + \ldots + 50^3$? [2]

3. A playground occupies a triangular site ABC where AB = 172 m, BC = 156 m and AC = 194 m.

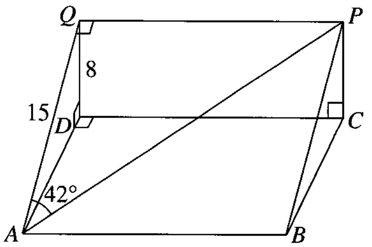
(a) Using a scale of 1 cm to represent 20 m, construct a plan of $\triangle ABC$. Measure and write down the smallest angle in $\triangle ABC$. [2]

(b) A boy walks from A in a straight line so that he is always equidistant from AB and AC. Draw on your diagram, the locus which represents the path that the boy must take. [2]

(c) A girl walks from the mid-point of AB in such a way that she will always be equidistant from A and B. Draw on the same diagram, the locus which represents the path of the girl. [2]

(d) The authority erects a vertical pole of length 30 m at the point P where $AP = BP$ and P is equidistant from AB and AC. By drawing a right-angled triangle on the diagram, find the angle of elevation of the pole from the point A. [3]

4. In the figure, $ABCD$ is a rectangle, $CPQD$ is a vertical plane and $ABPQ$ is an inclined plane where $A\hat{D}Q = A\hat{Q}P = A\hat{D}C = 90°$.



Given that $P\hat{A}Q = 42°$, AQ = 15 cm and DQ = 8 cm, calculate

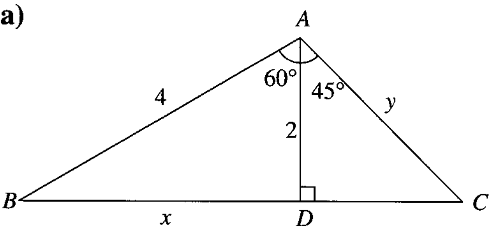
(a) the length of AP, [2]

(b) the area of rectangle $ABCD$, [3]

(c) $P\hat{B}C$, [2]

(d) $P\hat{A}C$. [3]

5. (a)



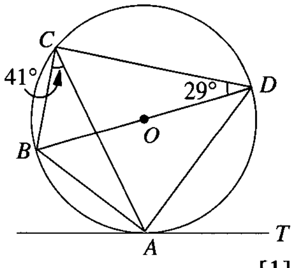
In the above figure, AD is perpendicular to the base BC of $\triangle ABC$. Given that $B\hat{A}D = 60°$, $D\hat{A}C = 45°$, AB = 4 cm, AD = 2 cm, $BD = x$ cm and $AC = y$ cm, express

(i) sin 60° and tan 30° in terms of x, [2]

(ii) cos 45° in terms of y. [1]

Hence, express $\frac{\sin 60°}{\tan 30°} + (\cos 45°)^2$ in terms of x and y, and then calculate its exact numerical value. [4]

(b) In the figure, O is the centre of the circle passing through A, B, C and D. TA is the tangent to the circle at A. If $B\hat{D}C$ = 29° and $A\hat{C}B$ = 41°, calculate



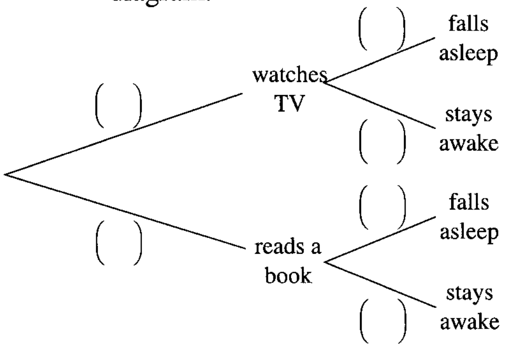
(i) $A\hat{B}C$, [1]

(ii) $D\hat{A}T$, [2]

(iii) $A\hat{C}D$. [1]

6. Every evening, Mr Cook either watches television or reads a book. The probability that he watches television is $\frac{3}{5}$. If he watches television, the probability that he will fall asleep is $\frac{1}{4}$. If he reads a book, the probability that he will fall asleep is $\frac{3}{8}$.

(a) Copy and complete the given tree diagram.



[2]

(b) Find the probability that Mr Cook will stay awake in the evening. [2]

(c) Find the probability that Mr Cook will be able to stay awake for at least one out of two evenings. [3]

7. A night train leaves Singapore for Segamat and returns to Singapore.

(a) If the train leaves Singapore at 22 30 and arrives at Segamat at 04 15 the next morning, find the time taken for the journey. [1]

(b) The fare for an adult ticket from Segamat to Singapore is RM22.40 and the child fare is RM12.60. On a certain journey, there are 250 adult and 80 child passengers. Calculate the total amount of fares collected in RM. Given that the exchange rate is S\$100 = RM228.50, calculate the total amount of fares collected for the above trip in S\$, giving your answer correct to the nearest 10 cents. [4]

(c) The distance between Singapore and Segamat is 200 km.

(i) If the train travels from Singapore to Segamat at an average speed of x km/h, write down an expression for the time taken, in hours, for the journey. [1]

(ii) On the return trip from Segamat to Singapore, the train increases its speed by 5 km/h, write down an expression for the time taken. [1]

(iii) If the difference in time between the two journeys is 1 hour 15 minutes, form an equation in x and show that it reduces to $x^2 + 5x - 800 = 0$. [3]

(iv) Solve the equation $x^2 + 5x - 800 = 0$ and hence, find the speed of the train from Segamat to Singapore. [2]

8. (a) Copy and complete the following table of values for $y = 2 + 1\frac{1}{5}x - \frac{1}{2}x^2$.

x	–3	–2	–1	0	1	2	3	4	5
y	–6.1	–2.4		2	2.7	2.4		–1.2	

[2]

(b) Using a scale of 2 cm to 1 unit on both axes, draw the graph of $y = 2 + 1\frac{1}{5}x - \frac{1}{2}x^2$ for $-3 \leqslant x \leqslant 5$. [3]

(c) Use your graph to find

(i) the value of y when $x = 1.5$, [1]

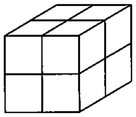
(ii) the values of x when $y = 0$, [2]

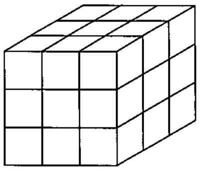
(iii) the solution of the equation

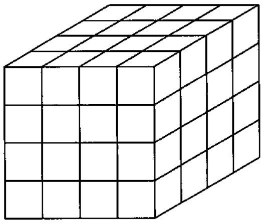
$$2 + 1\frac{1}{5}x - \frac{1}{2}x^2 = 1, \quad [2]$$

(iv) the gradient of the curve $y = 2 + 1\frac{1}{5}x - \frac{1}{2}x^2$ at the point $x = 2$, by drawing a suitable tangent to the curve. [2]

9. A boy arranges small cubes of 1 cm to form cubes with sides 2 cm, 3 cm, 4 cm, etc.







He then painted the outer surfaces of each cube. He observed that for cubes with side 2 cm ($n = 2$), there are 8 cubes with 3 faces painted. For cubes with side 3 cm ($n = 3$), there are 8 cubes with 3 faces painted, 12 cubes with 2 faces painted, 6 cubes with only 1 face painted and 1 cube not painted at all. He tabulated his finding as shown in the table below.

Size of cubes	*Number of faces painted*			
	3	*2*	*1*	*0*
$n = 2$	8	0	0	0
$n = 3$	8	12	6	1
$n = 4$	a	b	c	d
$n = 5$				
⋮				

(a) Find the values of a, b, c, and d. [3]

(b) When $n = 10$, how many cubes will have 3 faces painted? [1]

(c) When $n = 5$, how many cubes will not be painted? [2]

(d) When a cube has size n cm, find in terms of n the number of cubes with

- **(i)** 2 faces painted, [2]
- **(ii)** 1 face painted, [2]
- **(iii)** 0 face painted. [2]

10. Answer either (I) or (II) of this question.

(I) **Answer the whole of this question on a sheet of graph paper.**

The coordinates of the vertices of $\triangle T$ are (1, 2), (1, 7) and (4, 8).

(a) Using a scale of 1 cm to represent 1 unit on both axes, draw x- and y-axes for $-8 \leqslant x \leqslant 8$ and $-8 \leqslant y \leqslant 8$. Draw and label $\triangle T$. [1]

(b) Draw and label clearly the triangles P and Q on your diagram such that

- **(i)** P is the image of T under a rotation of 90° anticlockwise about the origin,
- **(ii)** Q is the image of T under a translation $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$. [2]

(c) $\triangle P$ may be mapped onto $\triangle Q$ by a rotation. State the centre and angle of rotation. [2]

(d) Draw the line $y = 2x$ on your graph. T may be mapped onto R by a reflection in the line $y = 2x$. Draw $\triangle R$ and find the sum of the area formed by $(R + T)$. [3]

(e) $\triangle T$ is mapped onto $\triangle S$ by an enlargement E with scale factor -1 and centre at the origin. Draw $\triangle S$. [2]

(f) A shear, with the x-axis invariant, maps $(-1, 5)$ onto $(4, 5)$. Find the shear factor. [2]

(II) Table 1 below gives the distribution of the scores of 60 pupils obtained in a mathematics quiz.

Score (x)	*No. of pupils*
$x \leqslant 10$	0
$10 < x \leqslant 20$	4
$20 < x \leqslant 30$	8
$30 < x \leqslant 35$	11
$35 < x \leqslant 40$	15
$40 < x \leqslant 45$	10
$45 < x \leqslant 50$	5
$50 < x \leqslant 65$	7

Table 1

Score (x)	*No. of pupils with scores* $\leqslant x$
10	0
20	4
30	
35	
40	
45	
50	
65	

Table 2

(a) Copy and complete the cumulative frequency Table 2. [1]

(b) Using a horizontal scale of 2 cm to represent a score of 10, and a vertical scale of 2 cm to represent 10 pupils, draw a cumulative frequency curve for the range $10 \leqslant x \leqslant 65$. [2]

(c) Use your graph to estimate

- **(i)** the median score, [1]
- **(ii)** the interquartile range, [1]
- **(iii)** the fraction of pupils who scored less than 37 marks. [1]

(d) One pupil is selected at random from the 60 pupils. Use your graph to estimate the probability that his score is between 24 and 56. [2]

(e) Out of the 60 pupils, 12 of them are girls.

- **(i)** Two pupils are selected at random from the 60 pupils. Find the probability that at least one will be a girl. [2]
- **(ii)** One pupil is chosen at random from the 60 pupils. Estimate the probability that she will be a girl and has a score of more than 41. [2]

Specimen Paper C

Paper 1 **Time: 2 hours**

Answer **all** the questions. Omission of essential working will result in loss of marks. The use of **all calculating aids** is **prohibited**. The number of marks given is shown in the brackets [ ] at the end of each question or part question.

1. (a) Express the number 0.003 048 in standard form, giving your answer correct to 2 significant figures. [1]

(b) Estimate the value of $\frac{56.04 \times \sqrt{3.99}}{7.96}$, giving your answer correct to 1 significant figure. [1]

2. (a) On a certain day in the Gobi Desert, the day temperature was 32°C. At night, the temperature fell to –5°C. Find the difference in the temperature on that day. [1]

(b) Find the value of x in the following sequence:

$$9, 5, 1, x, -7, -11, \ldots \qquad [1]$$

3. (a) Arrange the following numbers in descending order:

$$-3.14, -3\frac{1}{7}, 0.314, -0.314. \qquad [1]$$

(b) Evaluate $(0.25)^{-1} + \left(5\frac{1}{3}\right)^0$. [1]

4. Given that $3 \leqslant x \leqslant 5$ and $-1 \leqslant y \leqslant 3$, find the smallest value of

(a) $\frac{1}{x} + y$, **(b)** $\frac{y^2}{x} + xy$. [3]

5. If $(a + b)^2 = 73$ and $ab = 6.5$, calculate the value of $2a^2 + 2b^2$. [2]

6. Solve the simultaneous equations

$$5x - 2y = 29$$
$$x + 4y = -3. \qquad [3]$$

7. Gleneagles Hospital purchased a $1.2 million computer-operated robot called SurgiScope in 1999. It enables the brain surgeon to home in on the tumours with minimal damage to the rest of the brain. The machine will help to cut operation time by a third and save patients' time and money. The extra cost for using the robot per operation is approximately $2 000 to $3 000. What is the maximum and minimum number of operations to be performed to recover the cost of the machine? [3]

8. The government is planning to spend $4.4 billion over 7 years to upgrade and rebuild schools in Singapore under the Programme for Rebuilding and Improving Existing Schools (Prime). Each school will have bigger classrooms, from the present 64 square metres to 90 square metres to allow more computers to be put in. Calculate the percentage increase in the floor area of the classroom, giving your answer correct to 1 decimal place. [3]

9. It is given that $\overrightarrow{AB} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 20 \\ 6x \end{pmatrix}$.

(a) Evaluate $|\overrightarrow{AB}|$. [1]

(b) If AB is parallel to CD, find the value of x. [2]

10. The line $3y = kx + h$ is parallel to the line $5x + 6y = 54$ and it also passes through the point (2, 7). Find the value of h and of k. [2]

11. Factorise the following completely:

(a) $x^3 + x^2 - 6x$, [2]

(b) $36 - 9a^2$. [2]

12. The Defence Ministry (Mindef) started the construction of an ammunition storage complex deep underground in a disused quarry in Mandai in 1999. The project will help to save more than 300 ha of land for other valuable use in the land scarce Singapore. If 1 square metre of land is valued at $2 580, calculate the cost of a land of area 300 ha. [3]

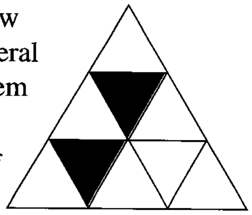
13. Given that y is directly proportional to the square of $(x + 1)$, and that $y = 9$ when $x = 1$,

(a) express y in terms of x, [2]

(b) find the value of y when $x = 3$. [1]

14. Mr Tan attended a family outing to the zoo organised by his company. He noticed that the ratio of the number of men to women was 2 : 3, and that the ratio of the number of women to children was 12 : 17. Find the ratio of the number of men to children at the outing. [3]

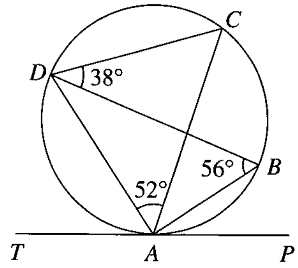
15. The diagram below shows 9 small equilateral triangles with 2 of them shaded.



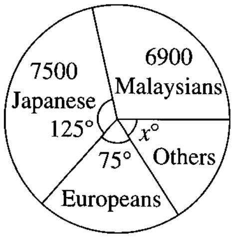
(a) Draw a line of symmetry on the diagram. [1]

(b) Shade another equilateral triangle on the diagram so that the diagram will have a rotational symmetry of order 3. [2]

16. In the diagram, TAP is the tangent to the circle at A, $A\hat{B}D = 56°$, $D\hat{A}C = 52°$ and $B\hat{D}C = 38°$. Calculate $P\hat{A}B$. [2]



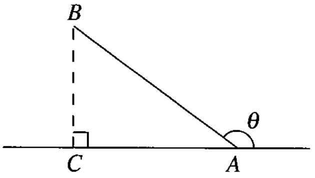
17. The pie chart shows the number of foreign visitors to an exhibition.



(a) How many Europeans visited the exhibition? [1]

(b) Find x. [2]

18. In the diagram, θ is obtuse and $\sin\theta = \frac{5}{13}$.



(a) Find the value of $\cos\theta - 2\tan\theta$. [2]

(b) If BC = 8 cm, calculate the length of AC. [2]

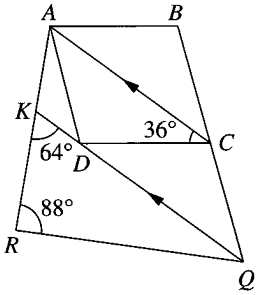
19. In November 1999 the Ba'alwi Mosque distributed part of the 45 tonnes of dates donated by the Saudi Arabia government to 20 non-Muslim charity organisations in Singapore and the general public of all races and religion on the first come first served basis. If 12 tonnes are for the public and each person is given 750 g, how many people received the dates from the mosque? [3]

20. In the diagram, $ABCD$ is a rhombus, BCQ, KDQ and AKR are straight lines. Given that $AC//KQ$, $K\hat{R}Q = 88°$, $R\hat{K}Q = 64°$ and $A\hat{C}D = 36°$, calculate

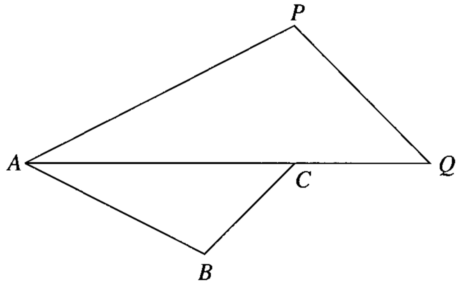
(a) $A\hat{B}C$, [1]

(b) $K\hat{A}D$, [1]

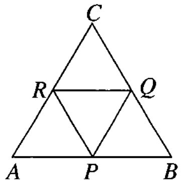
(c) $B\hat{Q}R$. [2]



21. $\triangle ABC$ may be mapped onto $\triangle APQ$ by a transformation H followed by a transformation K. Describe the successive transformations H and K completely. [3]



22. The diagram shows an equilateral triangle ABC of sides 6 cm. P, Q and R are the midpoints of AB, BC and AC, respectively.



Describe completely the transformation which will map

(a) $\triangle ABC$ onto $\triangle APR$, [1]

(b) $\triangle PQR$ onto $\triangle CQR$, [1]

(c) $\triangle PQR$ onto $\triangle PQB$, [1]

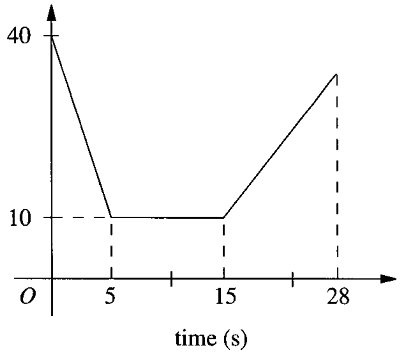
(d) $\triangle APR$ onto $\triangle RQC$. [1]

23. The diagram shows the speed-time graph of an object during a period of 28 seconds.

(a) Calculate the speed of the object when $t = 4$. [2]

(b) Calculate the distance travelled in the first 10 seconds. [2]

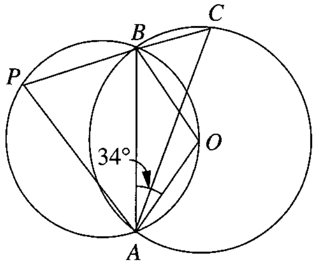
(c) Given that the acceleration of the object between $t = 15$ and $t = 28$ is 2 m/s^2, calculate the speed of the object when $t = 28$. [1]



24. In the diagram, O is the centre of the circle which passes through A, B and C. Given that PBC is a straight line and $B\hat{A}O = 34°$, calculate the size of

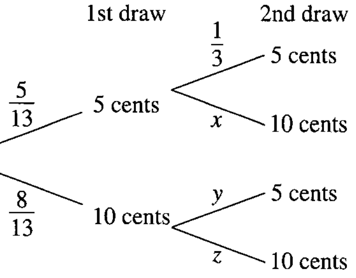
(a) $A\hat{P}B$, [2]

(b) $A\hat{C}B$. [2]



25. A bag contains 13 identical small balls. 5 of the small balls are numbered as 5 cents and the others are numbered as 10 cents.

(a) Two balls are drawn at random, one after the other from the bag. The balls drawn are not replaced. The tree diagram below shows the possible outcomes and the corresponding probabilities.



(i) Find the values of x, y and z. [3]

- **(ii)** What is the probability that the total value of the two coins is
  - **(a)** 15 cents, [1]
  - **(b)** more than 20 cents. [1]
- **(b)** Three more balls are drawn in succession and without replacement, find the probability that the value of the 5 balls is more than 25 cents. [2]

Paper 2 Time: 2 h 30 min

Answer **all** the questions. All working must be clearly shown. The number of marks given is shown in the brackets [ ] at the end of each question or part question.

1. **(a)** Use your calculator to evaluate $\frac{3.795^2}{\sqrt[3]{42.5}+\sqrt{2.86}}$, giving your answer correct to 4 significant figures. [2]

(b) Given that $\frac{1}{a}+\frac{2}{b^2}=\frac{c}{d}$,

(i) find the value of d when $a = 3$, $b = -1$ and $c = 4$, [2]

(ii) express a in terms of b, c and d. [2]

2. The total cost of water per month supplied to a household is calculated as follows.

Fixed charge per household = \$7.50

Cost of each unit of water = \$0.87

Water conservation tax = 20% for each unit of water used

(a) Calculate the total cost for a household using 14.5 units of water in a month. [1]

(b) Calculate the number of units used when the total cost is \$44.04. [1]

(c) Write down the formula connecting the total cost, \$$C$, and n, the number of units used. [2]

To discourage wastage of water, the utility company introduces a multi-tier system of charging water usage.

There is no fixed charge.

For the first twenty units, each unit costs \$0.87.

For the next ten units, each unit costs \$0.98.

For the next ten units, each unit costs \$1.15.

For any subsequent units used, each unit costs \$1.35.

Water conservation tax = 25% for bills less than or equal to \$25 and it will be 30% for the part of bill exceeding \$25. Calculate

(d) the bill for a household using 26 units of water, [1]

(e) the bill for a household using 45 units of water, [2]

(f) the number of units of water used for a household with a total bill of \$78.40. [3]

3. To qualify for entry to the science stream in Secondary Three, a student must have at least two distinction in two of the three subjects namely English, Maths and Science. The probabilities of Joan getting a distinction in English, Maths and Science are $\frac{5}{7}$, $\frac{3}{4}$ and $\frac{5}{6}$ respectively. Find the probability that Joan will

(a) not be able to get any distinction, [1]

(b) be able to get exactly one distinction, [2]

(c) be able to qualify to enter science stream. [2]

4. **Answer the whole of this question on a sheet of plain paper.**

(a) Construct $\triangle ABC$ in which $AB = 14$ cm, $AC = 12$ cm and $BC = 11$ cm. [1]

(b) On the same diagram construct

(i) the locus of point P, on the same side of AB as C, such that area of $\triangle ABP = 28\text{ cm}^2$, [2]

(ii) the locus of point Q which is equidistant from AB and AC, [1]

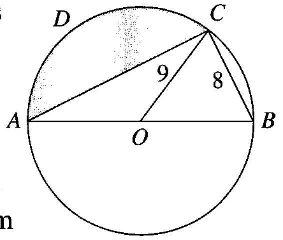
(iii) the locus of point R, on the same side of AB as C, such that $A\hat{R}B = 90°$, [2]

(iv) the locus of point S such that $AS = BS$. [1]

(c) Given that K is a point inside $\triangle ABC$ such that area of $\triangle AKB \geqslant 28\text{ cm}^2$, $A\hat{K}B \geqslant 90°$ and $K\hat{A}B \geqslant K\hat{A}C$ and

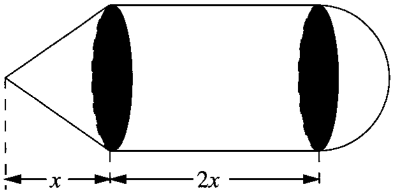
$BK \leqslant AK$. Indicate by shading the region in which K must lie. [2]

5. The diagram shows a circle with centre O and AB is a diameter. Given that C is a point on the circumference such that $BC = 8$ cm and $OC = 9$ cm, calculate



(a) AC, [2]
(b) $A\hat{O}C$, [3]
(c) the area of $\triangle BOC$, [2]
(d) the area of the shaded segment ADC. [3]

6. The diagram shows a container which consists of a cylinder with a cone attached to one end and a hemisphere attached to the other end.



(a) Given that the height of the cone is x cm while the length of the cylinder is $2x$ cm, find the ratio of $\frac{\text{volume of cone}}{\text{volume of cylinder}}$. [2]

(b) If the volume of the cylinder is 485 cm^3 and its height is 12 cm, find the radius of the cylinder. [3]

(c) Find the curved surface area of the cone. [3]

(d) The exterior of the container is to be painted with a coat of paint with a thickness of 0.3 mm, find the volume of paint needed to paint such a container. [4]

[Volume of cone $= \frac{1}{3}\pi r^2 h$,

Volume of sphere $= \frac{4}{3}\pi r^3$,

Curved surface area of cone $= \pi rl$,

Curved surface area of sphere $= 4\pi r^2$.]

7. Answer the whole of this question on a sheet of graph paper.

The distribution of marks gained by a group of 600 candidates in an examination is given in the table below.

Marks	*No of candidates who gained less than this mark*
0	0
10	15
20	38
30	130
40	240
50	380
60	540
70	582
80	600

Using a vertical scale of 2 cm to represent 100 candidates and a horizontal scale of 2 cm to represent 10 marks, plot these values and draw a curve through the points. [4]

(a) Use your graph to estimate
  (i) the median mark, [1]
  (ii) the pass mark which would allow 70% of the candidates to pass the examination, [1]
  (iii) the inter-quartile range. [1]

(b) Grade F9 was awarded to candidates scoring less than 28 marks and Grade A1 to those scoring 66 marks or more.
  (i) A candidate was selected at random from the group. Find the probability that
    (a) the candidate was awarded Grade F9, [1]
    (b) the candidate was awarded either Grade F9 or A1. [2]
  (ii) Two candidates were selected at random from the group. Find the probability that they were both awarded Grade A1. [2]

8. A ship sails from a port A at 09 00 on a bearing of 055° towards port B. It sails at an average speed of 12 kmh^{-1}, reaching port B at 11 15. It rested for 30 minutes and then sails at the same average speed of 12 kmh^{-1} to port C which is 30 km away from port B. At port C the ship took 45 minutes to unload some goods before it sets sail again at an average speed of 14 kmh^{-1} to port D, which is due north of port B. Given that $A\hat{B}C = 80°$ and $B\hat{C}D = 75°$, calculate

(a) the bearing of C from B, [2]
(b) the distance of AC, [3]
(c) the time when the ship reaches D, [2]
(d) the distance of BD, [3]
(e) the bearing of D from A. [2]

9. Answer the whole of this question on a sheet of graph paper.

The following is an incomplete table of values of x and y for the function of $y = 2x + \frac{8}{x} - 7$, correct to 1 decimal place.

x	y
1	3
1.5	1.3
2	1
2.5	1.2
3	1.7
3.5	h
4	3
4.5	3.8
5	4.6
5.5	5.5
6	6.3
6.5	k

(a) Calculate the values of h and k. [1]

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = 2x + \frac{8}{x} - 7$ for $1 \leqslant x \leqslant 6.5$. [3]

(c) By drawing suitable straight lines to your graph, find the solutions of each of the following equations where they exist:

(i) $2x + \frac{8}{x} - 7 = 2.5$ [1]

(ii) $2x + \frac{8}{x} = 12$ [1]

(iii) $x + \frac{4}{x} = 3.5$ [1]

(iv) $x + \frac{8}{x} = 7.5$ [3]

(d) By drawing a suitable straight line, find the gradient of the curve at the point where $x = 4$. [2]

10. Answer either (I) or (II) of this question.

(I) Answer the whole of this question on a sheet of graph paper.

The vertices of $\triangle ABC$ are $A(4, -2)$, $B(6, -2)$ and $C(4, 1)$. The vertices of $\triangle A_1B_1C_1$ are $A_1(6, 4)$, $B_1(6, 2)$ and $C_1(9, 4)$.

(a) Using a scale of 1 cm to represent 1 unit on each axis, draw x- and y-axes for $-4 \leqslant x \leqslant 14$ and $-8 \leqslant y \leqslant 10$. Draw and label $\triangle ABC$ and $\triangle A_1B_1C_1$. [2]

(b) $\triangle ABC$ can be mapped onto $\triangle A_1B_1C_1$ by a single transformation H. Describe fully the transformation H. [2]

(c) Under an enlargement, scale factor 3, centre at (10, 2), $\triangle A_1B_1C_1$ is mapped onto $\triangle A_2B_2C_2$. Draw and label $\triangle A_2B_2C_2$ on the same diagram. State the ratio of the area of $\triangle A_2B_2C_2$: area of $\triangle A_1B_1C_1$. [3]

(d) Under a shear, $\triangle A_1B_1C_1$ is mapped onto $\triangle A_3B_3C_3$ with vertices $A_3(8, -2)$, $B_3(10, -2)$ and $C_3(14, 1)$. Find the invariant line and the shear factor. [3]

(e) Under a translation represented by the column vector $\begin{pmatrix} a \\ b \end{pmatrix}$ followed by a reflection in the line $x + y = 0$, $\triangle ABC$ is mapped onto $\triangle A_4B_4C_4$ with vertices $A_4(0, -2)$, $B_4(0, -4)$ and $C_4(-3, -2)$. Find the values of a and b. [2]

(II) The Central Provident Fund (CPF) is a social security savings plans for Singaporeans and permanent residents. The table below gives the contribution rate in 1999 for the various age groups.

Age group *y (years)*	*Employer's Contribution (% of salary)*	*Employee's Contribution (% of salary)*	*Total Contribution (% of salary)*	*Credited to Ordinary Account*	*Credited to Medisave Account*
$y < 35$	10	20	30	24	6
$35 \leqslant y < 45$	10	20	30	23	7
$45 \leqslant y < 55$	10	20	30	22	8
$55 \leqslant y < 60$	4	12.5	16.5	8.5	8
$60 \leqslant y < 65$	2	7.5	9.5	1.5	8
$y \geqslant 65$	2	5	7	0	7

(a) Mr Lim is 28 years old and he works as a supervisor in a factory earning a monthly income of $1 650.

- **(i)** Calculate his take home pay after deducting CPF. [1]
- **(ii)** How much will be credited into his ordinary account per month? [1]

(b) Madam Alinah is 47 years old and she works as a marketing executive earning $2 380 per month. Calculate the amount that will be credited into her Medisave account from January 1999 to March 2000. [2]

(c) Mr Rajakumar is 56 years old and he works as a lawyer earning $8 500 per month. Calculate the amount credited into his Ordinary account for the whole of 1999 assuming that he also gets 5 months year-end bonus. [2]

(d) Mr Ismail is a retired teacher re-employed by the government to teach Malay in a secondary school. He is 61 years old and earns $4 300 per month, calculate his total contribution into his CPF account for the period from January 1999 to August 1999. [2]

(e) Mr Wong is a 67 year-old retiree working in a fast food outlet earning $780 per month. Calculate his take home pay and the amount credited into his Medisave account per month. [1]

(f) A trading company has a staff strength of 67 consisting of 43 employees aged 55 years old and below, 9 employees aged between 55 and 60 years, 6 employees aged between 60 and 65 and the remaining staff are over 65 years old. Calculate the total monthly CPF contribution amount the company must contribute to all the 67 employees. Assuming each employee earns an average of $2 500 per month. [3]

Answers

Exercise 1a *(Pg 6)*

1. (a) 13 **(b)** 12 **(c)** 80%

2. (a) 12 **(b)** $\frac{7}{50}$ **(c)** 45

3. (b) (i) 82 **(ii)** $\frac{1}{5}$
(iii) 56

4. (b) (i) 33 **(ii)** 6
(iii) 450.7

5. (a) 30, 80, 170, 320, 420, 470, 490
(c) (i) 290 mm **(ii)** 10%
(iii) 131

6. (a) 80, 150, 270, 530, 660, 700, 730
(c) (i) 200 **(ii)** $\frac{1}{5}$
(iii) 23.5

7. (b) (i) 540 **(ii)** 70%
(c) 180, 240, 80, 20

8. (b) (i) 50 **(ii)** $\frac{4}{7}$
(iii) 6.4

Exercise 1b *(Pg 19)*

1. (a) $44.50 **(b)** $16

2. (a) (i) 50 cents
(ii) 22 cents
(b) (i) 720 **(ii)** 440

3. (a) (i) 49.4 **(ii)** 44.7
(b) 27

4. (a) 8 **(c)** 18, 7.5
(d) (i) 30 **(ii)** 8

5. (a) 10, 32, 48, 58, 62
(c) (i) 15 **(ii)** 8
(iii) 6

6. (a) 8, 13, 20, 28, 39, 48, 54
(b) 60, 42, 73, 31
(c) (i) 41.4%
(ii) 17.9%

7. (a) 4, 3; 6, 7; 8, 15; 10, 29; 12, 53; 14, 88; 16, 138, 18, 185; 20, 200
(c) (i) 14.5 **(ii)** 16.5, 11.75
(iii) 171

8. (a) 29, 9; 34, 16; 39, 27; 44, 38; 49, 46; 54, 55
(c) (i) 41 **(ii)** 15
(iii) 41 **(iv)** 32

9. (c) (i) 157 **(ii)** 9
(iii) 55

10. (c) (i) 3.4 **(ii)** 4

11. (a) (i) 152.5 cm
(ii) 11.5 cm
(iii) 6
(b) (i) Subtract 3 cm from the result in (a)(i) to give the correct value of the median
(ii) no adjustment required

12. (a) 19, 87, 138, 156, 159
(c) (i) 1 950 hours
(ii) 400 hours
(e) (i) 149 **(ii)** 132
(f) Brand B; Brand B bulbs last longer in general

Review Questions 1 *(Pg 25)*

1. (a) 49.86
(d) (i) 51 **(ii)** 60
(iii) 28 **(iv)** 180
(v) 45

2. (b) 54.5 g
(e) (i) 54.5 g **(ii)** 6 g
(iii) grade 1: 7.5%, grade 2: 70%, grade 3: 22.5%

3. (b) 47.05 cm
(d) (i) 50 cm **(ii)** 56 cm
(iii) 41 cm **(iv)** 13
(v) 46

4. (a) 12, 16, 6 **(b)** 40.9 cm
(d) (i) 42 cm **(ii)** 21 cm
(iii) $\frac{1}{3}$

5. (a) Mathematics: 38, 63; English: 2, 12, 45, 76, 80
(c) (i) 61 **(ii)** 23
(iii) 9 **(iv)** 7

6. (a) (i) 170 seconds
(ii) 8 seconds
(iii) 20 **(iv)** 174
(b) 16, 26, 28
(d) 169 seconds

Exercise 2a *(Pg 36)*

1. A circle centre *O*, radius 4 cm

4. Two straight lines parallel to *l* and a distance 5 cm from *l*.

11. A circle of radius 4 cm

12. A circle of radius 4 cm

Exercise 2b *(Pg 41)*

1. (a) 9 cm
(c) (ii) 4.8 cm

2. (c) (ii) 6.7 cm

3. (d) 5 cm, 50 m

4. (c) (ii) 693 m²

5. (a) 81° **(d)** 27 m

6. (d) 4.5 m

10. 84 m

Exercise 2c *(Pg 48)*

3. 10.4 cm

5. (b) 3.7 cm

6. 7.7 cm

10. 12.8 cm

12. (d) 9.9 cm

15. (a) 13.2 cm

17. (a) 103°

Review Questions 2 *(Pg 50)*

1. (a) 100° **(e)** 1.2 m

2. (a) 7.7 cm, 25 m
(d) 20 m

3. (a) 040° **(d)** 144 m

4. (a) 10.4 cm

5. (a) (i) 14.9 cm **(ii)** 32°
(b) (iii) 7.6 cm

6. (a) 110°

7. (a) 10.9 cm **(c)** 4 cm

8. (c) 5.7 cm **(e)** 4.4 cm

9. (d) 25 m

10. (d) 22.4 m **(e)** 22.4 m

Exercise 3a *(Pg 62)*

1. $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\overrightarrow{BA} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$,
$\overrightarrow{CD} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\overrightarrow{DC} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$,
$\overrightarrow{EF} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$, $\overrightarrow{FE} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

2. $\mathbf{b} = \mathbf{e} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$,

$\mathbf{c} = \mathbf{f} = \mathbf{n} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$,

$\mathbf{d} = \mathbf{g} = \mathbf{l} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

4. **(a)** $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

(c) $\begin{pmatrix} -7 \\ 9 \end{pmatrix}$ **(d)** $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(e) $\begin{pmatrix} -p \\ -q \end{pmatrix}$ **(f)** $\begin{pmatrix} -3 \\ -7 \end{pmatrix}$

(g) $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ **(h)** $\begin{pmatrix} -8 \\ 2 \end{pmatrix}$

(i) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **(j)** $\begin{pmatrix} -x \\ -y \end{pmatrix}$

5. **(a)** 13 units **(b)** 10 units

(c) $\sqrt{29}$ units **(d)** $\sqrt{50}$ units

(e) 3 units

6. $p = \pm\sqrt{21}$

7. **(a)** 5 units **(b)** ±4

8. 13

9. **(a)** 4.5 units **(b)** 6

10. $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are not parallel

11. $s = -3$, $t = -4$

12. **(a)** $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$

13. **(a)** $\overrightarrow{AB} = \overrightarrow{IJ}$, $\overrightarrow{EF} = \overrightarrow{UV}$, $\overrightarrow{GH} = \overrightarrow{OP}$, $\overrightarrow{MN} = \overrightarrow{ST}$

(b) $\overrightarrow{AB}$ and $\overrightarrow{KL}$, $\overrightarrow{IJ}$ and $\overrightarrow{KL}$, $\overrightarrow{CD}$ and $\overrightarrow{QR}$

14. **(a)** **(i)** $\overrightarrow{IJ} = \overrightarrow{HK} = \overrightarrow{GL} = \overrightarrow{FE} = \overrightarrow{KB}$

(ii) $\overrightarrow{AJ} = \overrightarrow{CL} = \overrightarrow{DE}$

(iii) $\overrightarrow{HI} = \overrightarrow{GH} = \overrightarrow{FG} = \overrightarrow{KJ} = \overrightarrow{LK} = \overrightarrow{EL}$

(iv) $\overrightarrow{BC} = \overrightarrow{AB} = \overrightarrow{CD}$

(v) $\overrightarrow{AK} = \overrightarrow{JH} = \overrightarrow{KG} = \overrightarrow{CE}$

(vi) $\overrightarrow{LB} = \overrightarrow{FL}$

(b) **(i)** $\overrightarrow{LB}$ **(ii)** $\overrightarrow{KJ}$

(iii) $\overrightarrow{KB}$

(c) **(i)** $\overrightarrow{AB}$ and $\overrightarrow{DE}$ are not parallel

(ii) Length $AK \neq$ Length AB or $\overrightarrow{AK}$ and $\overrightarrow{AB}$ are not parallel

15. **(a)** $\overrightarrow{KL}$

(b) $\overrightarrow{RQ} = \overrightarrow{TU}$, $\overrightarrow{QP} = \overrightarrow{ST}$

(c) $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{BC} = \overrightarrow{AD}$

(d) $\overrightarrow{LM} = \overrightarrow{QP}$, $\overrightarrow{MN} = \overrightarrow{RQ}$, $\overrightarrow{NO} = \overrightarrow{SR}$, $\overrightarrow{OP} = \overrightarrow{LS}$

Exercise 3b *(Pg 72)*

1. **(d)** $\overrightarrow{ML}$

3. **(a)** $\overrightarrow{PR}$ **(b)** $\overrightarrow{SR}$ **(c)** $\overrightarrow{SR}$

(d) $\overrightarrow{ST}$ **(e)** $\overrightarrow{PR}$ **(f)** $\overrightarrow{RS}$

4. **(a)** $\overrightarrow{AC}$ **(b)** $\overrightarrow{AD}$ **(c)** $\overrightarrow{AD}$

(d) $\overrightarrow{AA}$ or **O** **(e)** $\overrightarrow{AD}$

(f) $\overrightarrow{CA}$

5. **(a)** **(i)** $\overrightarrow{PR}$ **(ii)** $\overrightarrow{RQ}$

(iii) $\overrightarrow{PQ}$

(b) **(i)** $\overrightarrow{OP}$ **(ii)** $\overrightarrow{OR}$

(iii) $\overrightarrow{RS}$ **(iv)** $\overrightarrow{RP}$

(v) $\overrightarrow{QR}$ **(vi)** $\overrightarrow{SR}$ or $\overrightarrow{PQ}$

(c) **(i)** **a** **(ii)** **a + b**

(iii) **a – b**

6. **(a)** $\overrightarrow{KS}$ **(b)** $\overrightarrow{QS}$ **(c)** $\overrightarrow{PR}$

(d) $\overrightarrow{PS}$ **(e)** $\overrightarrow{PR}$ **(f)** **O**

7. **(a)** $\overrightarrow{CB}$ **(b)** $\overrightarrow{BC}$ **(c)** $\overrightarrow{BA}$

8. **(a)** **b** **(b)** **v** **(c)** **–u**

(d) **c**

9. **(a)** $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} -6 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} -4 \\ 4 \end{pmatrix}$ **(d)** $\begin{pmatrix} a + c \\ b + d \end{pmatrix}$

10. **(a)** $a = 7$, $b = -1$

(b) $a = 2$, $b = -1$

(c) $a = 1$, $b = 4$

11. **(a)** $\begin{pmatrix} 10 \\ 9 \end{pmatrix}$ **(b)** $\begin{pmatrix} 10 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} 5 \\ 11 \end{pmatrix}$ **(d)** $\begin{pmatrix} 5 \\ 11 \end{pmatrix}$

Commutative and associative laws of vector addition.

12. **(a)** $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

13. $a = c$ and $b = d$

14. **(a)** $x = 3, y = 5$

(b) $x = -4, y = -6$

(c) $x = -4, y = 6$

(d) $x = 3, y = 5$

15. **(a)** $\begin{pmatrix} 3 \\ 9 \end{pmatrix} + \begin{pmatrix} -3 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} 5 \\ -7 \end{pmatrix} + \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} x \\ -y \end{pmatrix} + \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

16. **(a)** $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} 9 \\ 0 \end{pmatrix}$ **(d)** $\begin{pmatrix} 9 \\ -1 \end{pmatrix}$

17. **(a)** $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$

(c) $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$ **(d)** $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$

18. **(a)** $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$ **(b)** $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$

19. **(a)** $\begin{pmatrix} 8 \\ 9 \end{pmatrix}$ **(b)** $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$

(c) $\begin{pmatrix} 10 \\ 1 \end{pmatrix}$

20. **(a)** $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 9 \\ 3 \end{pmatrix}$ **(d)** $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

(e) $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ **(f)** $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

21. **(a)** **(i)** 15 **(ii)** 13

(b) **(i)** $\begin{pmatrix} 20 \\ 12 \end{pmatrix}$ **(ii)** $\begin{pmatrix} -10 \\ 12 \end{pmatrix}$

(iii) $\begin{pmatrix} -5 \\ -12 \end{pmatrix}$

Exercise 3c *(Pg 78)*

1. **(a)**, **(b)**, **(d)**, **(e)**, **(f)**

2. **(a)** $\begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}-2\\-1\end{pmatrix}$

(b) $\begin{pmatrix}-6\\-4\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}$

(c) $\begin{pmatrix}16\\-10\end{pmatrix}, \begin{pmatrix}-8\\5\end{pmatrix}$

(d) $\begin{pmatrix}-14\\8\end{pmatrix}, \begin{pmatrix}7\\-4\end{pmatrix}$

(e) $\begin{pmatrix}-1\\4\end{pmatrix}, \begin{pmatrix}1\\-4\end{pmatrix}$

(f) $\begin{pmatrix}9\\2\end{pmatrix}, \begin{pmatrix}-9\\-2\end{pmatrix}$

(g) $\begin{pmatrix}-4\\-18\end{pmatrix}, \begin{pmatrix}2\\9\end{pmatrix}$

(h) $\begin{pmatrix}1\\1.6\end{pmatrix}, \begin{pmatrix}-0.5\\-0.8\end{pmatrix}$

3. **(a)** $\begin{pmatrix}-10\\1\end{pmatrix}$ **(b)** $\begin{pmatrix}-2\\10\end{pmatrix}$

(c) $\begin{pmatrix}10\\11\end{pmatrix}$ **(d)** $\begin{pmatrix}10\\-5\end{pmatrix}$

(e) $\begin{pmatrix}5\\-16\end{pmatrix}$ **(f)** $\begin{pmatrix}24\\-9\end{pmatrix}$

4. **(a)** $\begin{pmatrix}23\\5\end{pmatrix}$ **(b)** $\begin{pmatrix}11\frac{1}{3}\\8\frac{1}{3}\end{pmatrix}$

(c) $\begin{pmatrix}26\\-10\end{pmatrix}$ **(d)** $\begin{pmatrix}38\\-25\end{pmatrix}$

5. **(a)** $\begin{pmatrix}2\\4\end{pmatrix}$ **(b)** $\begin{pmatrix}-18\\18\end{pmatrix}$

(c) $\begin{pmatrix}-32\\-33\end{pmatrix}$ **(d)** $\begin{pmatrix}5\\16\end{pmatrix}$

8. $-1\frac{1}{2}$

9. -4

11. **(a)** $5\mathbf{a} + \mathbf{b}$ **(b)** $\frac{5}{6}\mathbf{u} - \frac{1}{6}\mathbf{v}$

(c) $4\mathbf{u}$

12. **(a)** $x = 2, y = 1$

(b) $x = -7, y = 4$

(c) $x = -1\frac{4}{5}, y = -1$

13. **(a)** $\begin{pmatrix}0\\31\end{pmatrix}$ **(b)** 10 units

(c) $x = 22, y = -4$

14. **(a)** 5.4 units **(b)** $\begin{pmatrix}-2\\-5\end{pmatrix}$

(c) $p = 5\frac{1}{2}, q = \frac{2}{3}$

15. **(a)** $\begin{pmatrix}16\\-15\end{pmatrix}$

(b) **(i)** 13 units

(ii) 18 units

(c) -48

16. **(a)** $\begin{pmatrix}-1\\30\end{pmatrix}$ **(b)** -4.5

(c) 5.8 units

17. **(a)** **(i)** 045° **(ii)** 3 km

(b) **(i)** 225° **(ii)** 15 km

18. **(a)** **(i)** 13 km **(ii)** 112.4°

(b) **(i)** 13 km **(ii)** 157.6°

19. $k = \frac{2}{5}, s = -30, t = 12\frac{1}{2}$

20. $t = \frac{1}{4}, a = 16, b = -12$

21. $s = 7.5, t = 7$

22. $a = 2, b = 1$

Exercise 3d *(Pg 83)*

1. **(a)** $\mathbf{v} - \mathbf{u}$ **(b)** $\frac{1}{2}\mathbf{u}$

(c) $\frac{1}{2}\mathbf{v}$

(d) $\frac{1}{2}(\mathbf{v} - \mathbf{u})$; $\overrightarrow{BC}$ and $\overrightarrow{MN}$ are parallel.

2. **(a)** $2\mathbf{a}$ **(b)** $\mathbf{b}$

(c) $-2\mathbf{b}$ **(d)** $-3\mathbf{a}$

(e) $\mathbf{a} + \mathbf{b}$ **(f)** $2\mathbf{a} + \mathbf{b}$

(g) $2\mathbf{a} - \mathbf{b}$

(h) $2\mathbf{a} + 2\mathbf{b}$ or $2(\mathbf{a} + \mathbf{b})$

(i) $2\mathbf{b} - \mathbf{a}$ **(j)** $3\mathbf{a} - \mathbf{b}$

(k) $3\mathbf{a} + \mathbf{b}$

(l) $-2\mathbf{a} - 3\mathbf{b}$ or $-(2\mathbf{a} + 3\mathbf{b})$

(m) $3(\mathbf{a} + \mathbf{b})$ **(n)** $-2(\mathbf{a} + \mathbf{b})$

(o) $5\mathbf{a} + 2\mathbf{b}$

3. **(a)** $\frac{4}{3}\mathbf{q}$ **(b)** $\mathbf{q} - \mathbf{p}$

(c) $\mathbf{p} - \frac{4}{3}\mathbf{q}$

4. **(a)** $\mathbf{a} + 2\mathbf{b}$ **(b)** $-\frac{8}{3}\mathbf{b}$

(c) $\frac{2}{3}\mathbf{b} - \mathbf{a}$

5. **(a)** $-\frac{1}{2}\mathbf{q}$ **(b)** $\mathbf{p} - \mathbf{q}$

(c) $\mathbf{p} + \frac{1}{2}\mathbf{q}$ **(d)** $\frac{1}{2}\mathbf{q} - \mathbf{p}$

6. **(a)** $\mathbf{v} - \mathbf{u}$ **(b)** $\frac{2}{5}(\mathbf{v} - \mathbf{u})$

(c) $\frac{3}{2}\mathbf{u} + \mathbf{v}$

(d) $\frac{1}{5}(3\mathbf{u} + 2\mathbf{v})$

(e) $\frac{1}{2}\mathbf{u} + \mathbf{v}$

7. **(a)** $\mathbf{u} + \frac{3}{2}\mathbf{v}$ **(b)** $\mathbf{u} + \frac{1}{2}\mathbf{v}$

(c) $\frac{1}{4}\mathbf{u} + \frac{9}{8}\mathbf{v}$ or $\frac{1}{8}(2\mathbf{u} + 9\mathbf{v})$

(d) $\frac{9}{8}\mathbf{v} - \frac{1}{4}\mathbf{u}$ or $\frac{1}{8}(9\mathbf{v} - 2\mathbf{u})$

8. **(a)** **(i)** $\frac{2}{3}\mathbf{b}$ **(ii)** $\frac{2}{3}\mathbf{a}$

(iii) $\frac{2}{3}\mathbf{a} + \mathbf{b}$

(iv) $\mathbf{a} - \mathbf{b}$ **(v)** $\frac{1}{3}(\mathbf{a} - \mathbf{b})$

(b) **(i)** $\frac{1}{3}$ **(ii)** $\frac{1}{18}$

9. **(a)** **(i)** $-3\mathbf{a} - 7\mathbf{b}$

(ii) $-2(\mathbf{a} + 4\mathbf{b})$

(iii) $\mathbf{a} - 11\mathbf{b}$

(c) $4(\mathbf{b} - \mathbf{a})$

(d) **(i)** $\frac{3}{4}$ **(ii)** $\frac{3}{4}$

(iii) $\frac{1}{3}$

10. **(a)** **(i)** $2(\mathbf{b} - \mathbf{a})$ **(ii)** $\mathbf{b} - \mathbf{a}$

(iii) $\frac{12}{7}(\mathbf{a} + \mathbf{b})$ **(iv)** $3\mathbf{a}$

(v) $4\mathbf{b}$

(b) $4\mathbf{b} - 3\mathbf{a}$ **(c)** $\frac{3}{7}$

(d) **(i)** $\frac{3}{7}$ **(ii)** $\frac{2}{7}$

Exercise 3e *(Pg 87)*

1. **(a)** $\begin{pmatrix}4\\7\end{pmatrix}$ **(b)** $\begin{pmatrix}-2\\5\end{pmatrix}$

(c) $\begin{pmatrix}6\\-1\end{pmatrix}$ **(d)** $\begin{pmatrix}-4\\-9\end{pmatrix}$

2. **(a)** $\begin{pmatrix}-1\\-2\end{pmatrix}$ **(b)** $\begin{pmatrix}0\\7\end{pmatrix}$

(c) $\begin{pmatrix}1\\-5\end{pmatrix}$

3. **(a)** $\begin{pmatrix}-4\\6\end{pmatrix}$ **(b)** $\begin{pmatrix}2\\5\end{pmatrix}$

(c) $\begin{pmatrix}6\\-1\end{pmatrix}$

4. **(a)** $Q(-3, 8)$, $R(-1, 10)$

(b) 1, $\begin{pmatrix}2\\2\end{pmatrix}$

5. **(a)** $B(1, 4)$, $D(3, -6)$

(b) $\overrightarrow{BC} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} -4 \\ -10 \end{pmatrix}$

6. $P(-1, 4)$, $Q(3, -4)$, $S(-7, 2)$

7. **(a)** (2, 2) **(b)** (0, –2)

8. **(a)** **(i)** $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ **(iv)** $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

(b) $\frac{1}{2}$

9. **(a)** $\begin{pmatrix} 6 \\ -10 \end{pmatrix}$ **(b)** (7, –8)

(c) (2, 5)

10. **(a)** $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ **(b)** (3, –2)

11. **(a)** $\begin{pmatrix} t + 3 \\ 4 \end{pmatrix}$ **(b)** 29

(c) –10 or 4

12. **(a)** $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$ **(b)** $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$

(c) (3, 10)

Exercise 3f *(Pg 89)*

1. $\frac{1}{2}\mathbf{a} - \mathbf{b}$

2. $\frac{1}{5}(3\mathbf{a} + 2\mathbf{b})$

3. **(a)** $4\mathbf{q} - 9\mathbf{p}$

(b) $6\mathbf{p} - 6\mathbf{q}$ or $6(\mathbf{p} - \mathbf{q})$

(c) $3\mathbf{p} + 2\mathbf{q}$

4. $\frac{1}{8}(3\mathbf{a} + 2\mathbf{b})$

5. $\overrightarrow{AB} = 5(\mathbf{v} - \mathbf{u})$, $\overrightarrow{UV} = 15(\mathbf{v} - \mathbf{u})$,

$\overrightarrow{VA} = 5(\mathbf{u} - 3\mathbf{v})$,

$\overrightarrow{UX} = \frac{15}{4}(\mathbf{v} - 3\mathbf{u})$,

$\overrightarrow{XB} = \frac{5}{4}(\mathbf{v} - 3\mathbf{u})$

6. **(a)** $15(\mathbf{b} - \mathbf{a})$,

(b) $\frac{15}{4}(\mathbf{b} - \mathbf{a})$

(c) $\frac{15}{4}(3\mathbf{a} + \mathbf{b})$

(d) $5(3\mathbf{a} + \mathbf{b})$

7. **(a)** $4(\mathbf{a} + \mathbf{b})$ **(b)** $2(\mathbf{a} + 2\mathbf{b})$

(c) $4(2\mathbf{b} - \mathbf{a})$

8. **(a)** $4(5\mathbf{q} - 2\mathbf{p})$ **(b)** $5\mathbf{q} - 2\mathbf{p}$

(c) $6\mathbf{p} + 5\mathbf{q}$ **(d)** $3(2\mathbf{p} - \mathbf{q})$

9. **(a)** **(i)** $4\mathbf{q} - 3\mathbf{p}$

(ii) $4(\mathbf{q} - \mathbf{p})$

(iii) $\mathbf{p} + \mathbf{q}$ **(iv)** $3\mathbf{q} - 4\mathbf{p}$

(v) $\frac{3}{4}(4\mathbf{p} - 3\mathbf{q})$

(c) **(i)** $\frac{3}{4}$ **(ii)** $\frac{9}{16}$

10. **(a)** **(i)** $\frac{2}{3}\mathbf{b}$ **(ii)** $\frac{2}{3}\mathbf{b} - 2\mathbf{a}$

(iii) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

(iv) $\frac{1}{2}\mathbf{a} - \frac{1}{6}\mathbf{b}$

(b) $\frac{3}{1}$ or 3

Review Questions 3 *(Pg 91)*

1. **(a)** $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 12 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$ **(d)** $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$

(e) $\begin{pmatrix} -6 \\ -1 \end{pmatrix}$ **(f)** $\begin{pmatrix} 9 \\ -1 \end{pmatrix}$

2. **(a)** **(i)** $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

(c) **(i)** 4.1 units

(ii) 5.8 units

3. **(a)** ±4 **(b)** 4, 16

4. **(a)** **(i)** $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\sqrt{34}$ units

(ii) $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\sqrt{17}$ units

(iii) $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$, $\sqrt{17}$ units

(b) $\triangle ABC$ is isosceles, $8\frac{1}{2}$ unit^2

5. **(a)** $2(k - 1)\mathbf{p}$ **(b)** $10\mathbf{p} + 3\mathbf{q}$

6. $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, 7.3 units; $2\lambda + \mu = 11$,

$-3\lambda + 4\mu = -11$, $\lambda = 5$, $\mu = 1$

7. $a = -6$, $b = 8$

8. **(a)** $4\mathbf{b} - 3\mathbf{a}$ **(b)** $\mathbf{b} - 3\mathbf{a}$

(c) $\frac{1}{3}(3\mathbf{a} + 2\mathbf{b})$

(d) $\frac{2}{3}(3\mathbf{a} + 2\mathbf{b})$

9. **(a)** $3\mathbf{p} + \mathbf{q}$ **(b)** $4(\mathbf{q} - \mathbf{p})$

(c) $\frac{1}{2}(7\mathbf{q} - 3\mathbf{p})$

(d) $7\mathbf{q} - 3\mathbf{p}$

10. **(a)** **(i)** 5 units

(ii) $\begin{pmatrix} 0 \\ 19 \end{pmatrix}$

(iii) $a = 5$, $b = 1$

(b) **(ii)** *AB* is parallel to *DC*, *DC* is 3 times as long as *AB*

(iii) $2(2\mathbf{s} - \mathbf{t})$

(iv) $-\mathbf{t}$

11. **(a)** $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$ **(b)** $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

12. **(a)** 6 units

(b) **(i)** $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

13. **(a)** $\begin{pmatrix} -1 \\ 13 \end{pmatrix}$ **(b)** –6

(c) 8 units

14. **(a)** **(i)** $2\mathbf{a}$ **(ii)** $\mathbf{a} - \mathbf{b}$

(iii) $4\mathbf{a} + 2\mathbf{b}$

(iv) $6\mathbf{a}$

(b) **(i)** $\frac{1}{2}$ **(ii)** $\frac{1}{9}$

(iii) $\frac{2}{3}$

15. **(a)** **(i)** $\mathbf{p} - \mathbf{q}$ **(ii)** $\frac{1}{3}(\mathbf{p} + 2\mathbf{q})$

Exercise 4a *(Pg 99)*

1. **(a)** **(i)** (3, –4) **(ii)** (–1, –3)

(iii) (3, –3) **(iv)** (–3, 4)

(v) (3, 2) **(vi)** $(p, -q)$

(b) **(i)** (–3, 4) **(ii)** (1, 3)

(iii) (–3, 3) **(iv)** (3, –4)

(v) (–3, –2) **(vi)** $(-p, q)$

(c) **(i)** (4, 3) **(ii)** (3, –1)

(iii) (3, 3) **(iv)** (–4, –3)

(v) (–2, 3) **(vi)** (q, p)

2. (–1, 11)

3. (–1, –5), No

4. (1, 2)

5. $p = -1$, $q = 3$

6. (2, –2)

7. (–2, 1), (6, 1), (–6, 1), (10, 1)

8. $y = x$, $x = 0$

9. **(a)** $x = 2$ **(b)** $y = 4$
(c) $y = x + 1$ **(d)** $x + y = 2$
(e) $x + y = 1$ **(f)** $y = 2x - 2$
10. (1, –2), (2, 1)
11. (3, 10)
12. **(a)** $y = -3x - 2$
(b) $y = -3x + 2$
(c) $y = -3x + 14$
13. **(a)** (3, 10) **(b)** (8, 3)
14. (–2, 2)
15. **(a)** $y = -x - 3$ **(b)** $x + y = 3$
(c) $x + y = 9$ **(d)** $y = -x - 1$
16. (5, 2), (5, 2), Yes, (3, 3)
17. (4, 5), (4, –1), No, (4, 2)

Exercise 4b *(Pg 102)*

3. **(a)** (3, –3) **(b)** (7, 6)
(c) (9, –2)
5. (2, 0), 180°
6. **(a)** **(i)** (6, 3) **(ii)** 180°
(b) **(i)** (5, 2) **(ii)** 90°
(c) (2, 2), (0, 0), (2, –1)
7. **(a)** 6 **(b)** (4.5, 1)
(c) (1, 3.5)
8. **(a)** 77.5° **(b)** 31.5°
9. (4, 6), 90° clockwise
10. **(a)** (6, 5) **(b)** (7, 0)
11. 120° anticlockwise rotation about *O*, 240° anticlockwise rotation about *O*
12. $x + y = 2$
13. $y = x + 4$

Exercise 4c *(Pg 106)*

1. $P'(4, 1)$, $Q'(10, 3)$, $R'(5, -2)$
2. (–5, 0), $\begin{pmatrix}-4\\5\end{pmatrix}$
3. $\begin{pmatrix}-7\\-4\end{pmatrix}$, (2, 10)
4. **(a)** (4, 8) **(b)** $p = q = 0$
(c) $h = 6$, $k = 12$
(d) (–2, –4)
5. $x = 6$, $y = 1$, $h = -12$, $k = -2$
6. **(a)** (4, 10) **(b)** (13, –4)
(c) (10, 0) **(d)** (10, 0)
(e) (1, 14)
7. **(a)** $y = x - 1$ **(b)** (0, 2), 180°
(c) a translation $\begin{pmatrix}7\\-2\end{pmatrix}$
(d) A 90° anticlockwise rotation about (0, 0)

Review Questions 4 *(Pg 107)*

1. **(a)** (–2, 1) **(b)** (–7, 4)
2. (0, –7)
3. **(a)** $2y = 5x - 28$
(b) (1, –1)
4. **(a)** translation $\begin{pmatrix}-6\\0\end{pmatrix}$
(b) a reflection in $x = 6$
(c) 180° rotation about (6, 3)
(d) 90° clockwise rotation about (6, 0)
(e) 90° clockwise rotation about (6, 6)
5. **(a)** (–2, –5) **(b)** (5, 2)
(c) (–2, –5)
6. $y + x = 0$
7. **(a)** $x + y = 8$ **(b)** $y = x + 5$
(c) $x + y = 4$
8. **(a)** (3, 2) **(b)** (–12, 5)
(c) (–3, 4) **(d)** (–1, 4)
9. $m = 2$, $c = -5$
10. a translation of 9 cm along *AC*.
11. **(a)** (3, 1) **(b)** (5, 2)
(c) (7, 3)
12. **(a)** **(i)** (–5, 3) **(ii)** (4, 7)
(b) **(i)** (4, –3) **(ii)** (–3, 2)
(c) **(i)** (–5, 2) **(ii)** (2, 5)

Exercise 5a *(Pg 114)*

1. **(a)** $A'(4, 4)$ **(b)** $B'(10, 6)$
(c) $C'(4, 2)$ **(d)** $D'(4.5, 5)$
2. **(a)** (7, 3), $k = 2$
(b) (0, 1), $k = 2$
(c) (4, 3), $k = -\frac{1}{3}$
(d) (5.5, 2.5), $k = \frac{1}{3}$
5. **(a)** (7, 6), 2 **(b)** (2, 1), 3
(c) (4, 6), $1\frac{1}{2}$ **(d)** (4, 6), $-\frac{1}{3}$
(e) (3, 2), 2 **(f)** (4, 5), –2
6. **(a)** (–1, 0) **(b)** 2
7. **(a)** $P(-6, 5)$, $Q(0, -3)$
(b) 10 units
8. $A(1, 1)$, $B(5, 2)$, $C(2, 3)$
9. $A(1, 1)$, $C(0, 3)$
10. **(a)** (1, 1)
(b) 135 units^2
11. 20 cm
13. 6 cm
14. **(a)** –3, centre *O*

Exercise 5b *(Pg 121)*

1. **(a)** A stretch with the *x*-axis invariant and stretch factor 4.
(b) A stretch with the *y*-axis invariant and stretch factor $2\frac{1}{2}$.
2. **(a)** A two-way stretch with *x*-axis invariant, stretch factor 2 and *y*-axis invariant, stretch factor 2.
(b) A two-way stretch with *x*-axis invariant, stretch factor 3 and *y*-axis invariant, stretch factor 2.
6. **(a)** (1, 0), (4, 0), (3, 4)
(b) (3, 0), (12, 0), (9, 2)
9. 18 units^2
11. 3 units^2
12. **(a)** 3 **(b)** (1, 6)
(c) enlargement scale factor 2 centre at origin.

Exercise 5c *(Pg 128)*

1. **(a)** 1 **(b)** 2
2. **(a)** 3 **(b)** $1\frac{1}{2}$
4. **(a)** $1\frac{1}{2}$ **(b)** $1\frac{1}{2}$
(c) $1\frac{1}{2}$ **(d)** 2
5. (4, 2), (8, 2), (8, 4)
6. $A'(4, 0)$, $B'(2, -1)$, $C'(-2, -1)$, $D'(-6, -3)$
7. $k = 3$
8. **(a)** $x = 1$ **(b)** (–2, 7)
9. **(a)** (2, 1) **(b)** (3, 1)
10. $B'(-1, -1)$, $C'(3, -1)$
11. (12, 4)
12. $P'(2, 0)$, $Q'(-3, 1)$
13. **(a)** *P* is a shear with $y = 0$ as invariant line and shear factor 5.
(b) (39, 7)
14. **(a)** (5, 1)
(b) $A'(7, 2)$, $B'(-5, -2)$, $C'(-2, -2)$
15. *Q* is the set of points on the line $y = x + 3$, (0, 3).
17. **(a)** $y = 1$ **(b)** 2
18. **(a)** $x = -2$ **(b)** 1
19. **(a)** $y = 0$ **(b)** –2

Exercise 5d *(Pg 142)*

1. **(a)** (–5, 3) **(b)** (–1, 3)
2. **(a)** (–3, –2) **(b)** (1, 3)
3. $A_1(5, 5)$, $A_2(5, 11)$; $k = 6$
4. A reflection in $y = 4$ followed by an enlargement scale factor $1\frac{1}{2}$ and centre (1, 4).
5. A half-turn about the origin.
6. **(a)** (6, 11) **(b)** (2, 5)
(c) (2, 5) **(d)** (–8, –4)
(e) (–3, 1)
7. A reflection in the line $y + x = 0$.
9. (3, –2), (6, –1)
10. $B(1, 8)$, $C(2, 3)$; $D(-5, -1)$
12. (3, 3)
13. (6, 12), $y + x = 12$; same area
14. **(b)** reflection in the line *POR*
15. $A'(1, 1)$, $B'(3, 2)$, $C'(4, -1)$; $P(5, 0)$, $Q(5, 3)$, $R(8, 3)$, $S(8, 0)$
16. $C_1(12, 8)$, $C_2(-28, -8)$
17. **(a)** 90° anticlockwise rotation about origin
(b) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$
(c) $P(3, 11)$
(d) **(i)** *A*
(ii) **(a)** $k = 9$ **(b)** $1\frac{1}{2}$
(c) (0, –3) **(d)** 4 : 9
18. **(a)** *B*; 2 **(b)** 4
(c) An anticlockwise rotation of 180° about *M* followed by an enlargement with centre *B* and scale factor 2.
19. **(a)** **(i)** 2 **(ii)** (–10, –4)
(b) A reflection in the y-axis
(c) A reflection in the line $y = x$
(d) –3
(e) A clockwise rotation of 90° about (2, –4)
20. **(a)** $-\frac{1}{2}$ **(b)** 16 units^2
(c) (1, 0) **(d)** (5, 3)
21. **(b)** Reflection in the line $y = x$
(d) A reflection in the x-axis.

Review Questions 5 *(Pg 146)*

1. $p = 7$, $m = 6$, $n = -1$
4. **(a)** 4 units^2 **(b)** 8 units^2
(c) 8 units^2

5. A stretch with $x = 2$ as the invariant line and stretch factor 3.
7. *H* is a reflection in the line $y = 1$. *K* is a stretch with the y-axis as invariant line and stretch factor 2.
8. *H* is a reflection in the line $x = 4$. *K* is an enlargement centre at (4, 2) and scale factor 2.
9. **(a)** A 90° anticlockwise rotation about (1, 2).
(b) A shear with $y = 2$ as invariant line and shear factor 2.
(c) A stretch with y-axis as invariant line and stretch factor 2.
(d) A reflection in the line $y = 2$.
10. **(a)** A shear with $x = 1$ as the invariant line and shear factor 1.
(b) A stretch with $y = 3$ as invariant line and stretch factor 2.
(c) A 90° clockwise rotation about (4, 2).
(d) An enlargement centre at (5, 1) scale factor –1 or a 180° rotation about (5, 1).
11. **(a)** 4 **(b)** 1
(c) 16
12. **(a)** 1 : 2 **(b)** 4 : 1
13. **(a)** (1, 1) **(b)** 3
(c) (4, 4) **(d)** (2, 3)
14. **(a)** (–4, 0), (–4, –4), (–12, –4)
(b) 2 **(c)** $\frac{1}{4}$
15. **(a)** 105 cm **(b)** 25 cm
16. **(a)** (6, 2) **(b)** 8
(c) $h = -4$, $k = 2$
17. A reflection in the line $y = x - 1$; $y = x - 1$, $3\frac{1}{2}$ units^2
18. **(a)** (–3, 4), $x = 0$
(b) (–4, –3)
(c) reflection in $y + x = 0$
19. **(a)** An enlargement scale factor 4, centre *A*
(b) A translation parallel to *AP* with length *AP*
(c) A 180° rotation about *P*

(d) An enlargement scale factor 2, centre at *A*
(e) An enlargement scale factor 3, centre at *X* where $2HX = XK$
(f) An enlargement scale factor 2, centre at *B*.
20. **(a)** $\begin{pmatrix} 9 \\ -3 \end{pmatrix}$ **(b)** (3, 4)
(c) centre (10, 11), $k = -\frac{1}{2}$
(d) $1\frac{1}{2}$
21. **(a)** **(i)** –1 **(ii)** $y + x = 2$
(iii) (4, 2)
(b) (–1, –3)
(c) **(i)** (–2, 2) **(ii)** (8, 0)
(d) **(i)** (4, 1) **(ii)** 2 units
22. **(b)** **(ii)** $k = 4$ **(c)** $y + x = 0$
(e) $A_4(-3, 4)$, $B_4(-6, 4)$, $C_4(-6, 1)$
23. **(a)** 90° anticlockwise rotation about (–1, 0)
(b) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ **(c)** $y = x - 1$
(d) $k = 3$, $B_3(2, 8)$
24. **(b)** An enlargement scale factor 3, centre at (0, 3)
(c) A reflection in the line $y = x$
(d) $k = 3$, (20, 6)
25. **(a)** $\begin{pmatrix} -5 \\ -5 \end{pmatrix}$
(b) **(i)** $y + x = 0$
(ii) $x + y + 5 = 0$
(c) **(i)** (4, 1) **(ii)** –2
(iii) $\frac{1}{4}$
(d) A shear parallel to the x-axis, shear factor $1\frac{1}{2}$.

Exercise 6a *(Pg 164)*

1. $\frac{3}{8}$
2. $\frac{1}{5}$
3. **(a)** $\frac{2}{5}$ **(b)** $\frac{4}{15}$ **(c)** $\frac{1}{5}$
4. **(a)** $\frac{2}{11}$ **(b)** $\frac{4}{11}$ **(c)** 0

5. **(a)** $\frac{1}{52}$ **(b)** $\frac{3}{13}$ **(c)** 0

6. **(a)** $\frac{1}{9}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{5}$

7. $\frac{1}{3}$

8. **(a)** $\frac{3}{20}$ **(b)** $\frac{25}{33}$

9. $\frac{2}{3}$

10. **(a)** $\frac{1}{2}$ **(b)** 0 **(c)** 1
(d) $\frac{3}{10}$ **(e)** $\frac{1}{5}$ **(f)** $\frac{7}{10}$

11. $\frac{x}{15}$, 10

12. **(a)** $\frac{1}{4}$ **(b)** $\frac{3}{8}$ **(c)** $\frac{1}{6}$
(d) $\frac{5}{24}$

Exercise 6b *(Pg 170)*

1. **(a)** $\frac{2}{9}$ **(b)** $\frac{1}{5}$

2. **(a)** $\frac{1}{2}$ **(b)** $\frac{1}{3}$

3. **(a)** **(i)** $\frac{2}{3}$ **(ii)** $\frac{1}{3}$
(iii) $\frac{3}{13}$ **(iv)** $\frac{12}{13}$
(b) **(i)** $\frac{6}{19}$ **(ii)** $\frac{3}{38}$
(iii) $\frac{15}{19}$

4. **(a)** **(i)** $\frac{2}{5}$ **(ii)** $\frac{1}{8}$
(b) **(i)** $\frac{2}{39}$ **(ii)** $\frac{1}{13}$

5. $\frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}$

6. **(b)** **(i)** $\frac{1}{6}$ **(ii)** $\frac{5}{6}$
(iii) $\frac{1}{4}$ **(iv)** $\frac{3}{4}$
(v) $\frac{5}{12}$

7. **(a)** 12, 13, 12, 13, 14, 13, 15; 28, 35, 32, 48, 36, 45, 54
(b) **(i)** $\frac{2}{9}$ **(ii)** $\frac{7}{9}$
(iii) $\frac{5}{9}$
(c) **(i)** $\frac{4}{9}$ **(ii)** $\frac{2}{3}$
(iii) $\frac{8}{9}$

8. **(a)** 0, 1, 2, 3, 4, 5, 2, 3, 5, 6, 2, 3, 4, 5, 6, 7, 3, 4, 5, 6, 7, 8, 4, 6, 7, 8, 9, 5, 6, 7, 8, 9, 10
(b) 36
(c) **(i)** $\frac{1}{9}$ **(ii)** $\frac{17}{36}$
(iii) $\frac{19}{36}$ **(iv)** $\frac{1}{2}$
(v) $\frac{1}{2}$
(d) sum of 7

9. **(a)** 25
(b) **(i)** $\frac{1}{5}$ **(ii)** $\frac{1}{5}$
(iii) $\frac{4}{5}$ **(iv)** $\frac{6}{25}$
(c) $\frac{2}{5}$

10. **(a)** 2, 3, 4, 5, 6, 2, 4, 8, 10, 12
(b) **(i)** $\frac{1}{4}$ **(ii)** $\frac{3}{4}$
(iii) $\frac{1}{3}$ **(iv)** $\frac{5}{6}$
(v) $\frac{1}{3}$

11. **(a)** 1, 2, 3, 4, 5, 1, 0, 2, 3, 2, 1, 0, 1, 2, 3, 3, 2, 1, 0, 1, 2, 4, 3, 2, 1, 0, 1, 5, 3, 2, 1
(b) **(i)** $\frac{5}{18}$ **(ii)** $\frac{5}{6}$
(iii) $\frac{1}{2}$ **(iv)** $\frac{4}{9}$
(v) $\frac{1}{3}$

12. **(b)** **(i)** $\frac{1}{4}$ **(ii)** $\frac{1}{2}$

Exercise 6c *(Pg 177)*

1. **(a)** **(i)** $\frac{1}{4}$ **(ii)** $\frac{1}{2}$

2. **(a)** $\frac{1}{6}$ **(b)** $\frac{1}{3}$

3. **(a)** $\frac{2}{3}$ **(b)** $\frac{2}{9}$ **(c)** $\frac{2}{9}$

4. **(b)** **(i)** $\frac{1}{6}$ **(ii)** $\frac{2}{3}$
(iii) $\frac{1}{2}$ **(iv)** $\frac{3}{4}$
(v) $\frac{2}{3}$ **(vi)** $\frac{5}{12}$
(vii) $\frac{1}{4}$ **(viii)** $\frac{1}{2}$

5. **(a)** $\frac{1}{8}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{2}$
(d) $\frac{1}{4}$ **(e)** $\frac{11}{16}$ **(f)** $\frac{3}{4}$
(g) $\frac{1}{4}$

6. **(a)** $\frac{5}{11}$ **(b)** $\frac{4}{11}$ **(c)** $\frac{9}{11}$
(d) $\frac{4}{11}$ **(e)** $\frac{7}{11}$ **(f)** $\frac{6}{11}$
(g) $\frac{3}{11}$ **(h)** $\frac{9}{11}$

7. **(a)** $\frac{7}{15}$ **(b)** $\frac{1}{3}$ **(c)** $\frac{4}{5}$
(d) $\frac{1}{5}$

8. **(a)** $\frac{3}{17}$ **(b)** $\frac{2}{17}$ **(c)** $\frac{5}{17}$
(d) $\frac{10}{17}$ **(e)** $\frac{13}{17}$ **(f)** $\frac{8}{17}$

9. **(a)** $\frac{1}{8}$ **(b)** $\frac{3}{8}$ **(c)** $\frac{1}{2}$

10. **(a)** $\frac{5}{6}$ **(b)** $\frac{1}{6}$

11. **(a)** $\frac{5}{14}$ **(b)** $\frac{11}{14}$ **(c)** $\frac{3}{14}$

12. **(a)** $\frac{7}{15}$ **(b)** $\frac{17}{30}$ **(c)** $\frac{8}{15}$
(d) $\frac{13}{30}$

Exercise 6d *(Pg 187)*

1. **(a)** $\frac{5}{9}, \frac{5}{9}, \frac{4}{9}, \frac{5}{9}, \frac{4}{9}$
(b) **(i)** $\frac{20}{81}$ **(ii)** $\frac{40}{81}$
(iii) $\frac{5}{9}$

2. **(a)** $\frac{6}{10}, \frac{6}{10}, \frac{4}{10}; \frac{6}{10}, \frac{4}{10}$
(b) **(i)** $\frac{6}{10} \times \frac{6}{10}$
(ii) $2\left(\frac{6}{10} \times \frac{4}{10}\right)$

3. **(a)** $\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{4}, \frac{3}{4}; \frac{1}{4}, \frac{1}{4}, \frac{3}{4}$
(b) **(i)** $\frac{3}{4}$ **(ii)** $\frac{1}{4}$
(c) **(i)** $\frac{3}{8}$ **(ii)** $\frac{9}{16}$
(iii) $\frac{15}{16}$ **(iv)** $\frac{1}{16}$

4. **(a)** **(i)** $\frac{1}{15}$ **(ii)** $\frac{11}{30}$
(iii) $\frac{2}{5}$ **(iv)** $\frac{3}{10}$
(v) $\frac{7}{10}$
(b) $\frac{3}{10}, \frac{1}{2}; \frac{1}{3}, \frac{1}{5}, \frac{3}{10}, \frac{1}{2}; \frac{5}{12}, \frac{1}{5}, \frac{3}{10}, \frac{1}{2}$
(c) **(i)** $\frac{43}{120}$ **(ii)** $\frac{7}{60}$
(iii) $\frac{3}{8}$ **(iv)** $\frac{1}{5}$

5. **(a)** $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}; \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}; \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}$
(b) **(i)** $\frac{5}{108}$ **(ii)** $\frac{1}{72}$
(iii) $\frac{1}{36}$ **(iv)** $\frac{5}{18}$
(v) $\frac{7}{24}$

6. $\frac{3}{8}$
(a) $\frac{2}{7}$ **(b)** $\frac{2}{7}, \frac{5}{7}; \frac{5}{8}, \frac{3}{7}, \frac{4}{7}$
(c) **(i)** $\frac{3}{28}$ **(ii)** $\frac{5}{14}$
(iii) $\frac{15}{28}$

7. **(a)** $\frac{5}{8}, \frac{3}{8}; \frac{1}{3}, \frac{3}{4}, \frac{1}{4}$
(b) **(i)** $\frac{1}{4}$ **(ii)** $\frac{1}{2}$
(iii) $\frac{5}{12}$ **(iv)** $\frac{11}{12}$

8. **(a)** $\frac{5}{9}$ **(b)** $\frac{16}{81}$ **(c)** $\frac{40}{81}$

9. **(a)** $\frac{1}{5}$ **(b)** $\frac{3}{4}$
(c) **(i)** $\frac{1}{5}$ **(ii)** $\frac{7}{20}$
(iii) $\frac{13}{20}$ **(iv)** $\frac{19}{20}$

10. **(a)** $\frac{3}{5}$
(b) **(i)** $\frac{1}{3}$ **(ii)** $\frac{8}{15}$
(iii) $\frac{2}{3}$

11. **(a)** $\frac{2}{3}$ **(b)** $\frac{5}{22}$ **(c)** $\frac{5}{11}$

12. **(a)** **(i)** $\frac{7}{30}$ **(ii)** $\frac{2}{5}$
(iii) $\frac{11}{30}$
(b) $\frac{1}{10}$

13. **(a)** **(i)** $\frac{3}{7}$ **(ii)** $\frac{10}{21}$
(b) **(i)** $\frac{5}{21}$ **(ii)** $\frac{20}{273}$

14. **(a)** $\frac{3}{10}$ **(b)** $\frac{1}{15}$
(c) $\frac{2}{15}$ **(d)** $\frac{1}{9}$

15. **(a)** $\frac{1}{5}$
(b) **(i)** $\frac{1}{10}$ **(ii)** $\frac{2}{5}$
(iii) $\frac{3}{10}$ **(iv)** $\frac{3}{5}$

16. **(a)** $\frac{2}{3}$
(b) **(i)** $\frac{1}{15}$ **(ii)** $\frac{1}{6}$
(iii) $\frac{1}{30}$

17. **(a)** **(i)** $\frac{3}{11}$ **(ii)** $\frac{21}{110}$
(b) $\frac{7}{55}$

Review Questions 6 *(Pg 192)*

1. **(a)** $\frac{1}{6}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{4}$
(d) $\frac{1}{2}$

2. **(a)** $\frac{1}{12}$ **(b)** $\frac{1}{4}$

3. **(a)** $\frac{1}{26}$ **(b)** $\frac{1}{104}$

4. **(a)** $\frac{1}{26}$ **(b)** $\frac{1}{26}$ **(c)** $\frac{1}{24}$

5. **(a)** $\frac{6}{7}$ **(b)** $\frac{1}{49}$
(c) $\frac{12}{49}$ **(d)** $\frac{216}{343}$

6. **(a)** $\frac{4}{5}$ **(b)** $\frac{2}{15}$ **(c)** $\frac{1}{5}$

7. **(a)** **(i)** $\frac{40}{87}$ **(ii)** $\frac{38}{87}$
(b) **(i)** $\frac{1}{3}$ **(ii)** $\frac{8}{15}$
(iii) $\frac{13}{15}$

8. **(a)** $\frac{3}{10}$ **(b)** $\frac{31}{45}$ **(c)** $\frac{11}{120}$
(d) $\frac{5}{36}$ **(e)** $\frac{1}{24}$

9. **(a)** 1, 1; 3, $\frac{3}{4}; \frac{1}{3}, \frac{2}{3}$
(b) $\frac{5}{9}$

10. **(a)** $\frac{1}{2}$ **(b)** $\frac{1}{8}$ **(c)** $\frac{7}{16}$
(d) $\frac{7}{64}$

11. **(a)** **(i)** $\frac{5}{8}$ **(ii)** $\frac{3}{8}$
(b) **(i)** $\frac{1}{16}$ **(ii)** $\frac{1}{64}$
(iii) $\frac{7}{32}$ **(iv)** $\frac{17}{32}$
(v) $\frac{7}{16}$

12. **(a)** $\frac{27}{55}$ **(b)** $\frac{18}{55}$

Revision Exercise 7.1a *(Pg 198)*

1. **(a)** 0.03 **(b)** 0.034 6
(c) 3.456×10^{-2}
2. **(a)** 4.0 **(b)** 4.03
3. **(a)** 0.000 074 **(b)** 7.4×10^{-5}
4. **(a)** 173 min
(b) 01 08 the next day
5. **(a)** 1.556×10^{6}
(b) $2.386\ 8 \times 10^{6}$
(c) 2×10^{2}
(d) $1.216\ 8 \times 10^{10}$
(e) $4.745\ 52 \times 10^{13}$
(f) 2×10^{-2}
6. 0.3
7. 1.4×10^{6}
8. **(a)** 1.92 **(b)** 10.90
(c) 63.17
9. **(a)** 3 805.2 **(b)** 453
10. $403.20
11. $220
12. $85
13. 25%
14. $308
15. 386
16. **(a)** 34 **(b)** $171.30
17. 62.9%

18. \$89.04
19. **(a)** 41 min **(b)** 59 min
20. 9 km/h
21. 10 yrs 6 mth
22. 90
23. 700 boys, 500 girls
24. 259
25. $7\frac{1}{2}$ days
26. **(a)** 2.2
(b) $\frac{3}{20}, \frac{2}{15}, \frac{1}{8}, 2^{-\pi}$
27. \$120, \$48, \$12
28. $3\frac{1}{3}\%$
29. $1\frac{1}{2}$ yrs
30. \$8 200

Revision Exercise 7.1b *(Pg 199)*

1. 4% decrease
2. \$440
3. \$4 140
4. \$80
5. 22.14%
6. \$3 900, \$3 300
7. 10.5%
8. 20%
9. 6%
10. $3\frac{1}{3}$ days
11. $7\frac{1}{5}$ days
12. **(a)** \$38.25
(b) \$35.04, \$40.53
13. 45.05%, \$7 335, 10.14%
14. **(a)** \$336 **(b)** 10%
(c) 3%
15. **(a)** 250 000
(b) 3 000, 100, 5
(c) 2 400, 80, 4
(d) 100 000
(e) 1 436, 48
16. **(a)** S\$31 000 **(b)** S\$34 720
(c) RM78 022.50
(d) gain 56%
17. \$2 076.50; 24.2%
18. **(a)** 80 min **(b)** 105 km/h
(c) 22, 6 **(d)** 4.3×10^5
(e) 14.48%
19. **(a)** **(i)** \$224 **(ii)** \$1.50
(iii) \$50 **(iv)** \$8.40
(b) \$8; $42\frac{1}{2}\%$

20. **(a)** **(i)** US\$360
(ii) US\$1 = 5.1F
(iii) £600
(iv) **(a)** US\$1.63 = £1
(b) 8%
(b) £25

Revision Exercise 7.2 *(Pg 205)*

1. **(a)** **(i)** $3x + 4$
(ii) $3x + 2k$
(b) $3x$ **(c)** $5x$
2. **(a)** **(i)** $5x$ **(ii)** kx
(b) **(i)** $8x + 0.8$
(ii) $hx + 0.1\ h$
(c) $yx + zx + 0.1z$
3. **(a)** **(i)** 12 **(ii)** $7 + x$
(b) **(i)** 11 **(ii)** $4 + y$
(c) **(i)** 35 **(ii)** $11 + 2z$
4. **(a)** **(i)** $x + 4$ **(ii)** $x + h$
(b) **(i)** $x + y + 8$
(ii) $x + y + k$
(c) **(i)** $2x + y + 18$
(ii) $2x + y + 2z$
5. **(a)** 120 km **(b)** $60x$ km
(c) 5 km **(d)** y km
6. **(a)** $3x$ km **(b)** xy km
(c) $\frac{7x}{60}$ km **(d)** $\frac{xz}{60}$ km
7. **(a)** $\frac{10}{v}$ h **(b)** $\frac{x}{v}$ h
(c) $\frac{y}{1\,000\ v}$ h
8. **(a)** $\frac{1}{x}$ **(b)** $\frac{2}{x + 5}$
(c) $\frac{3(2x + 5)}{x(x + 5)}$
9. **(a)** $\frac{1}{x}$ **(b)** $\frac{4}{x + 6}$
(c) $\frac{2(x + 3)}{x(x + 6)}$
10. **(a)** $2x + 25$ **(b)** $x + 28$
(c) $-23x - 3$ **(d)** $18x - 1$
(e) $18a + 9$ **(f)** $-7k - 26$
(g) $9c - 21a + 5$
(h) $7q$ **(i)** $4y^2 - 3x^2$
(j) $8ab$ **(k)** $8x^2 - 8y^2$
(l) $7x^2 - 16xy$
11. **(a)** $2a + 12$ **(b)** $4y - x + 3z$
(c) $4x - 4y - 3z$
(d) $-8a + 4b - 2c$
(e) $10b - 11a$
(f) $8z - 5y$
(g) $2ab + 3ac - 3bc - 5a^2$
(h) $11ab - 4ac - 2a + 2b - 2c$
$- bc$

(i) $ab + 5bc$
(j) $22b - 3c - 9a$
(k) $3b$
(l) $5x + 12y$
12. **(a)** -4 **(b)** 3 **(c)** -5
13. **(a)** 1 **(b)** 81 **(c)** -200
14. **(a)** -3 **(b)** 16 **(c)** 96
15. **(a)** 3 **(b)** $\frac{y^2 + a}{2}$
16. **(a)** $\pm\frac{1}{\sqrt{2}}$ **(b)** $\frac{y^2}{2(x + 1)}$
17. **(a)** $4\,106\frac{2}{3}$ **(b)** $r = \sqrt{\frac{3V}{\pi h}}$
18. **(a)** $\frac{3}{4}$ **(b)** $x = \frac{k}{1 - 2y}$
19. $y = \frac{k - 2a}{a - 3}$
20. **(a)** $\frac{16}{3}t - 3$
(b) $t = \frac{9}{16}, x = \frac{1}{8}, y = 2\frac{3}{8}$
21. **(a)** $12x$
(b) $x = \frac{1}{4}, a = 7\frac{3}{4}, b = -4\frac{1}{2}$
22. **(a)** $a = -13, b = 35$
(b) $6t^2 + 6t + 14$
23. **(a)** $-1\frac{2}{3}$ **(b)** $\frac{1 - y}{1 + y}$
24. $a = \frac{-7b}{11}$
25. $\frac{2ac}{3c - a}$
26. **(a)** 6 **(b)** 3
(c) 5 **(d)** 2
(e) $\frac{4}{7}$ **(f)** $-1\frac{1}{5}$
(g) 3 **(h)** $2\frac{2}{3}$
27. **(a)** $-1\frac{1}{9}$ **(b)** $-5\frac{1}{2}$
(c) 12 **(d)** $\frac{7}{9}$
(e) $-5\frac{1}{5}$ **(f)** -14
(g) $\frac{9}{13}$ **(h)** 5
(i) 45 **(j)** $2\frac{9}{10}$
(k) $5\frac{3}{4}$ **(l)** $8\frac{9}{11}$
28. **(a)** $b = \frac{1}{3}(2a + c)$
(b) $y = \frac{3}{5}(x + 1)$

(c) $y = \dfrac{ax + c}{h + k}$

(d) $x = \dfrac{ay^2 + 2}{y^2 - 1}$

(e) $x = \pm\sqrt{4b^2 + ak}$

(f) $y = \dfrac{h(a + b)}{2k}$

(g) $y = \pm\sqrt{2x - h}$

(h) $r = \pm\sqrt{\dfrac{3A}{\pi h}}$

29. (a) (2, 3) **(b)** (2, –1)

(c) $\left(\frac{1}{2}, \frac{1}{3}\right)$ **(d)** (1, –1)

(e) (–1, –2) **(f)** (5, –1)

(g) (3, –2) **(h)** (4, –3)

(i) (7, 9)

30. (a) $9(x + 2)(x - 2)$

(b) $8x(x + 2)$

(c) $(2x - 3y)(2x + 3y)$

(d) $(x - 12)(x + 12)$

(e) $(x - 2y)^2$

(f) $3xy(x^2 - 3y)$

(g) $(3x - y)(3x + y)$

(h) $(3 - x)(3 + x)$

(i) $(5 - 8k)(5 + 8k)$

(j) $(a - b + c)(a - b - c)$

(k) $(3x + 4y - 3z)(3x + 4y + 3z)$

(l) $(a + 2b + 2c)(a + 2b - 2c)$

(m) $(a - b + x - y)(a - b - x + y)$

(n) $(x + 2)(x - 6)$

(o) $(x + 8)(x - 19)$

(p) $(x + 3)(x - 17)$

(q) $(2a - 1)(a + 2)$

(r) $(3a - 2)(a + 3)$

(s) $(4k - 7)(k + 2)$

(t) $(5x + 1)(x + 2)$

(u) $(3a - 5)(2a - 7)$

(v) $(7a - 1)(a + 10)$

(w) $(4a - 5)(2a - 7)$

(x) $(3x - 1)(3x + 11)$

(y) $(10x - 3)(x - 7)$

(z) $(11x - 2)(x + 7)$

31. (a) $(x + 3)(x + y)$

(b) $(a - c)(b - c)$

(c) $(x - b)(a + k)$

(d) $(3 + c)(x + c)$

(e) $(a - k)(x - h)$

(f) $(5x + y)(a + b)$

(g) $(3x + 2)(y - 4)$

(h) $(5c - d)(4a - 3k)$

(i) $(2a + b)(3a - 4k)$

(j) $(2a - 1)(a^3 + 2)$

(k) $(x - 1)(x^2 + 1)$

(l) $(5x + 7)(x^2 + 1)$

32. (a) 5 or –14 **(b)** 12 or –13

(c) 4 or 7 **(d)** $\frac{7}{3}$ or $\frac{23}{2}$

(e) 3 or –7 **(f)** 3 or 5

(g) 3 or 16 **(h)** 7 or 25

(i) 1 or 7 **(j)** –4 or –5

(k) $\frac{1}{2}$ or 4 **(l)** $\frac{1}{3}$ or 3

(m) $\frac{1}{4}$ or –2 **(n)** $-\frac{1}{5}$ or –5

(o) $\frac{4}{3}$ or $\frac{3}{2}$ **(p)** 2 or $-\frac{1}{9}$

33. 70¢, \$1.80

34. (a) (10, 32) **(b)** (3, –1)

(c) (7, –13) **(d)** (5, 6)

(e) $\left(3, -\frac{1}{2}\right)$ **(f)** (56, –17)

(g) (8, –10) **(h)** (–1, –1)

35. $\dfrac{100y + z}{x}, \dfrac{100kx}{100y + z}$

36. (a) $x = 3, y = 1$

(b) they are parallel lines

37. (a) 10 **(b)** 242 **(c)** 5

(d) $-9\frac{1}{2}$ **(e)** $-8\frac{6}{7}$

38. $6\frac{4}{5}$

Revision Exercise 7.3 *(Pg 212)*

1. (a) $21x^7$ **(b)** $6x^{\frac{5}{6}}$

(c) $8x^{7\frac{1}{2}}$ **(d)** $4x^4$

(e) $3x^{4\frac{1}{2}}$ **(f)** $\frac{1}{32}x^{13}$

(g) $49x^{-2}$ **(h)** $\frac{1}{8}x^4$

(i) $21x^3$ **(j)** $x^{\frac{1}{3}}y^{\frac{1}{4}}$

(k) $x^{\frac{1}{3}}y^{4\frac{1}{2}}$ **(l)** $2x^7y$

2. (a) $\frac{1}{9}$ **(b)** 8 **(c)** $\frac{1}{16}$

(d) $\frac{1}{243}$ **(e)** 1 **(f)** 125

(g) $2\frac{10}{27}$ **(h)** $\frac{1}{16}$ **(i)** 9

(j) $\frac{1}{216}$ **(k)** 4 **(l)** $\frac{5}{32}$

(m) 416 **(n)** 9 **(o)** $4\frac{1}{2}$

(p) $\frac{2}{13}$ **(q)** 192 **(r)** $\frac{-88}{147}$

3. (a) 6.25×10^6 **(b)** 6.62×10^8

(c) 3.72×10^{10}

(d) 6.602×10^{15}

(e) 6.68×10^{-2}

(f) 5.57×10^{-7}

(g) 1.669×10^{-5}

(h) 2.208×10^{-6}

(i) 5×10^{15} **(j)** 9×10^{-8}

4. (a) 270 **(b)** $7\frac{1}{2}$

(c) $y = 3\frac{1}{2}x^{\frac{2}{3}}, 31\frac{1}{2}$

5. (a) $\dfrac{17x}{14}$ **(b)** $\dfrac{7 - 2x}{12}$

(c) $\dfrac{43 - x}{20}$ **(d)** $\dfrac{5x - 6}{4x}$

(e) $\dfrac{5x - 7}{2x - 3}$ **(f)** $\dfrac{2(10a - 1)}{5(5a - 1)}$

(g) $\dfrac{5x + 1}{(x - 1)(x + 2)}$

(h) $\dfrac{7x - 8}{(x + 1)(2x - 1)}$

(i) $\dfrac{-a - 9}{(3a - 1)(2a - 3)}$

(j) $\dfrac{a^2 - 2a + 3}{(1 - 2a)(4 + a)}$

(k) $\dfrac{x^2 - 7x - 5}{(x + 1)(x - 2)}$

(l) $\dfrac{3x^2 + 14x - 6}{(x - 3)(x + 4)}$

(m) $\dfrac{5p^2 + p + 12}{(p - 3)(p + 1)}$

(n) $\dfrac{2a^2 - 2a + 1}{(2a - 3)(a + 1)}$

(o) $\dfrac{3y - x}{x(x - y)}$

(p) $\dfrac{15x^2 - 3xy - y^2}{(2x - y)(3x - y)}$

6. (a) $\frac{3}{5}$ **(b)** $\frac{4}{5}$

(c) $3\frac{9}{10}$ **(d)** $\pm\frac{2}{3}$

(e) 15 **(f)** –1 or 4

(g) 5 **(h)** –5

(i) $\frac{1}{3}$ **(j)** $-\frac{1}{3}$ or 5

(k) $-\frac{3}{8}$ or 2 **(l)** $1\frac{1}{5}$

7. $4\frac{1}{2}$

8. $y = \frac{7}{8}, y = \frac{7}{z^3}$

9. $1\frac{7}{9}$, 2, $y = \frac{16}{x^2}$

10. $2\frac{1}{2}$, -3

11. $y = \frac{32}{27}(x + 1)^2$, $18\frac{26}{27}$

12. **(a)** $\frac{2}{5}$ **(b)** -3 **(c)** $\frac{1}{17}$

13. **(a)** 33 cm **(b)** 1.2 kg
  (c) $L = 12 + \frac{3m}{100}$

14. **(a)** $3\frac{1}{2}$ **(b)** $-\frac{4}{7}$
  (c) $-\frac{1}{4}$ **(d)** -4
  (e) $1\frac{1}{2}$ **(f)** 20

15. **(a)** $b = \frac{-c - ax^2}{x}$
  (b) $l = \frac{8gt^2}{\pi^2}$
  (c) $c = \frac{6a}{2ak - 2 - ab}$
  (d) $x = \frac{-5k - 9}{12}$
  (e) $u = \pm\sqrt{v^2 - 2as}$
  (f) $a = \frac{bx^3}{1 + x^3}$

16. **(a)** $\frac{3}{x + 1}$
  (b) $\frac{3x + 1}{x^2 - 4}$
  (c) $\frac{x^2 - 5x + 8}{2x^2 + x - 6}$
  (d) $\frac{5}{x^2 - x - 2}$

17. **(a)** –0.27, 3.77
  (b) –1.70, 1.37
  (c) 0.57, 2.18
  (d) 0.23, 1.43
  (e) –1.54, 0.87
  (f) –0.87, 1.54

18. 17.56 cm

19. 1.94 cm

20. **(a)** 2.69 or –0.19
  (b) 3.62 or –1.29
  (c) –1.65 or 0.85
  (d) 1.59 or –3.59
  (e) 2.12 or –0.79
  (f) 0.29 or –1.00
  (g) 2.28 or –1.61
  (h) –0.15 or –45.85
  (i) 0.69 or –1.13
  (j) 0.07 or –1.34

21. $x^2 + 3.4x - 125 = 0$, 13.01 cm

22. $5t^2 + 7t - 100 = 0$, 3.83 sec

23. $x^2 + 2.4x - 156 = 0$, 50.20 cm

24. 30 m/min

25. 32.23 km/h, 37.23 km/h

26. 4, 9

27. $\frac{3}{5}$

28. $\frac{380}{x}, \frac{380}{x - 5}, \frac{380}{x - 5} - \frac{380}{x} = \frac{5}{12}$
70 km/h, 65 km/h

29. 2 h

30. **(a)** (–3, 0), (–5, 0)
  (b) (0, 1), (0, 15)

31. **(a)** **(i)** $29 **(ii)** 350
  (b) **(i)** $8, fixed or basic cost
    (ii) $0.07, unit cost of electricity
    (iii) $85

32. **(a)** IV **(b)** $\frac{1}{4}$ times
  (c) **(i)** $F = \frac{36}{x^2}$
    (ii) 0.36

33. **(a)** **(i)** $\frac{11}{x}$ cm
    (ii) $\frac{11}{x + 3}$ cm
  (b) $\frac{11}{x} - \frac{11}{x + 3} = 2$
  (c) 2.83 or –5.83
  (d) 1.89 cm

Revision Exercise 7.4 *(Pg 218)*

1. **(a)** 7.96 cm **(b)** 19.79 cm

2. **(a)** 54°
  (b) **(i)** 14 cm **(ii)** 92.4 cm²

3. 90°

4. **(a)** 201 cm² **(b)** 18 cm²

5. **(a)** **(i)** 134.1 cm
    (ii) 4 289.9 cm²
    (iii) 1 169.4 cm²
  (b) 37 536.4 cm³

6. **(a)** 5.5 cm **(b)** 19.25 cm²
  (c) 154 cm² **(d)** 194.5 cm²

7. **(a)** 614.34 cm²
  (b) 24 573.6 cm³
  (c) 1 930.4 cm²

8. **(a)** 1 344 cm³ **(b)** $436\frac{1}{3}$ m³
  (c) 385 m³ **(d)** 2 310 cm³

9. **(a)** **(i)** 3 696 cm³
    (ii) $718\frac{2}{3}$ cm³
  (b) 9 cm
  (c) 2 kg (nearest kg)
  (d) 1 615 cm²

10. **(a)** 1 584 cm³ **(b)** 6.9 cm
  (c) 9 cm

11. **(a)** 6. 8 cm
  (b) 214 cm³; 15 kg

12. **(a)** 8 cm **(b)** 551 cm²

13. **(a)** **(i)** 2.1 m²
    (ii) 340.3 kg
  (b) 2.52 m³ **(c)** 8.6 cm
  (d) 5 040 **(e)** $2.10

14. **(a)** 4 646 400 cm³
  (b) 8 800 cm³ **(c)** 528

15. **(a)** 27 440 cm³
  (b) 52.4%
  (c) 406

16. **(a)** 1.28 kg
  (b) 810.6 cm³, 5.3 cm
  (c) 5.9 cm, 11 cm

17. **(a)** $3\,490\frac{2}{3}$ cm³
  (b) 1.4 kg
  (c) 1 219 cm²
  (d) $29 252

18. **(a)** 4 800 cm³ **(b)** 7.5 cm
  (c) 2 400 cm³ **(d)** 31°
  (e) 10.29 cm

19. 1 983 cm³; 21 217 g

20. **(a)** 1 741.7 cm³
  (b) 896.7 cm²

Revision Exercise 7.5 *(Pg 224)*

1. **(a)** 45° **(b)** 30° **(c)** 15°

2. **(a)** 120° **(b)** 144 **(c)** 156

3. **(a)** 72 **(b)** 45 **(c)** 30

4. **(a)** 36 **(b)** 90 **(c)** 20

5. **(a)** 160° **(b)** 150°

6. 140°

7. 12

8. 36

9. **(a)** 90° **(b)** 40° **(c)** 62°

10. 129

12. **(a)** 41 **(b)** 105°

13. **(a)** 8 **(b)** 16

14. **(a)** 26° **(b)** 102° **(c)** 128°

15. **(a)** 72° **(b)** 66° **(c)** 66°

16. **(a)** 97° **(b)** 253°

17. **(a)** 56° **(b)** 62°

18. **(a)** 77° **(b)** 18°

19. **(a)** 38° **(b)** 79° **(c)** 71°

20. **(a)** 30° **(b)** 150°

21. **(a)** 108° **(b)** 60° **(c)** 120°

22. **(a)** 4 : 1 **(b)** 420 litres

23. **(a)** 16 cm **(b)** $11\frac{23}{27}$ kg

24. **(a)** 17 cm
(b) 7 812.5 cm^3

25. 432.64 cm^3

26. 1 890 cm^3

27. 675 cm^2

28. **(a)** 3 375 : 1
(b) 225 : 1; 75 000 kg

29. 3 850 m^2, 616 cm^2, 25 : 4, 32 cm^2

30. 204 m; $1.44 million; 2 g

31. **(a)** **(i)** 3 393.36 cm^3
(ii) 4 807.26 cm^3
(b) **(i)** 41 cm, 1 159.4 cm^2
(ii) 1 347.92 cm^2
(c) **(i)** 1 : 8 **(ii)** 600.91 cm^3

32. **(a)** 12 cm
(b) **(i)** 4 : 9 **(ii)** 8 : 27
(c) $3.24

33. **(a)** 9 cm
(b) **(i)** $\frac{1}{3}$ **(ii)** 16

34. **(a)** 45 cm^2 **(b)** 25 cm^2

35. 24 cm^2, $21\frac{3}{5}$ cm^2

36. 50 cm^2

37. **(a)** 2.4 cm **(b)** 3 : 2
(c) 9 : 16

38. **(a)** 73.5 cm^2 **(b)** 50 cm^2

39. **(a)** 36, 10; 30, 12
(b) 100
(c) 2°

40. **(a)** 17 **(b)** 27

41. **(a)** 12.5 cm **(b)** 7.5 cm
(c) 27 : 125

Revision Exercise 7.6 *(Pg 230)*

1. **(a)** 7 **(b)** –2 **(c)** 7

2. **(a)** 8 **(b)** 11 **(c)** 8

3. 5

4. 4, 5, 6

5. 6, 7, 8, 9, 10, 11

6. 11, 13, 17, 19, 23

7. 6

8. (1, 2), (2, 1)

9. (1, 3), (2, 1)

10. **(a)** 11 **(b)** –10 **(c)** 7

11. **(a)** $x \leqslant 2\frac{1}{9}$ **(b)** $x < 2\frac{1}{2}$
(c) $x \geqslant 5\frac{4}{5}$ **(d)** $x \geqslant \frac{3}{4}$

12. **(a)** 24 **(b)** –90
(c) 89 **(d)** –64

13. 5, 7, 9, 11

14. **(a)** 1 **(b)** $y = 5x - 7$

15. $k = -1, y = -\frac{1}{2}x - 4\frac{1}{2}$

16. 10

17. **(a)** (4.5, 3) **(b)** $\sqrt{117}$
(c) $y = 3x - 10.5$

18. $h = 1, k = 6, y = \frac{2}{3}x + 5\frac{1}{3}$

19. (2, 3)

20. **(a)** 26 **(b)** 9 **(c)** $\frac{9}{25}$

21. **(a)** 1.1 **(b)** 13
(c) 60 **(d)** –130

22. **(a)** 28 cm **(b)** 61.75 cm^2

23. 5, 6, 7, 8, 9

24. –5, –3, –2

25. **(a)** 13 **(b)** 6
(c) $-4 \leqslant x \leqslant 4$

26. **(a)** $t = -\frac{1}{3}$ **(b)** $y + 2x = 9$

27. **(a)** $\frac{1}{2}$ **(b)** $2y = x - 2$
(c) (2, 0) **(d)** 27 units^2
(e) 13.42 **(f)** 4.02 units

28. $4y + 6x = 1$

29. **(a)** $k = 4$ **(b)** 9.49
(c) $3y = 2x - 12$

30. **(a)** $h = 4, k = 5$
(b) $2\sqrt{5}$, $\sqrt{185}$
(c) 43 units^2

31. –15

32. $5y = 7x + 14$

34. **(a)** $\sqrt{17}$, $\sqrt{68}$, $\sqrt{85}$
(b) 17 units^2
(c) 3.69
(d) $\left(1, 1\frac{1}{2}\right)$

35. **(a)** $M\left(2, 5\frac{1}{2}\right)$
(b) $2y = 19x - 27$

36. 4, –6

37. **(a)** $P(6, -6), Q(5, 3), R(-3, 7), S\ (-2, -2)$

38. **(a)** (–9, 9) **(b)** $t = 6$
(c) 9.22 **(d)** $y = 3x + 21$
(e) 22.5

39. **(a)** $h = -4, k = 10$
(b) $y + 2x = 9\frac{1}{3}$
(c) $21\frac{7}{9}$

40. **(a)** 15 units **(b)** (12, 9)
(c) 48 units^2 **(d)** 8.54 units

41. **(a)** (7.6, 0)
(b) $y = 1\frac{1}{2}x - 11\frac{2}{5}$
(c) $t = 11\frac{1}{10}$ **(d)** 22.8 units^2

42. **(a)** $B(0, 8), Q(0, 16), P(-15, 0)$
(b) $15y = 16x + 240$
(c) 21.93 units
(d) 10.94 units

Revision Exercise 7.7 *(Pg 236)*

1. **(a)** $A(-1, 0), B(0, 2), C(2, 0)$
(b) $x = \frac{1}{2}$

3. **(a)** (0, –5) **(b)** $k = 2$

4. $k = -5, p = 4$

5. **(a)** $y = x, x + y = 0$
(b) Reflection in the line $y + x = 0$

6. **(a)** 12 **(b)** 0.6 m/s^2
(c) 30 m

7. **(a)** 40 **(b)** –0.8 m/s^2
(c) 360 m

8. **(a)** $1\frac{2}{3}$ m/s^2 **(b)** 812.5 m
(c) 70

9. **(a)** 24 m/s **(b)** 22
(c) $15\frac{2}{11}$ m/s

10. **(a)** 0 m/s^2 **(b)** –4.8 m/s^2
(c) 242.24 m **(d)** 12.1 m/s

11. **(a)** 40 m **(b)** 120 m

12. **(a)** **(i)** 36 m **(ii)** 14.4 m/s
(iii) $2\frac{2}{3}$ s **(iv)** 6 s
(b) **(i)** 2 m/s^2
(ii) 19 s
(c) **(i)** $T_1 = 10, T_2 = 16$
(ii) constant speed of A for first 10 sec.

13. **(b)** **(i)** 1.7 or –4.7
(ii) $-3 \leqslant x \leqslant 1$
(iii) 0.4 or –4.4
(c) (–1, –5)

14. 37, 23, 25; 22, 3.75; 5

15. **(a)** 6.4 s **(b)** 0.68 m/s^2

16. 18, 26, 10
(a) 26.5
(b) 3.6 or –2.3
(c) $-1.7 \leqslant x \leqslant 3$
(d) –14

17. –5, –3, –9, 25
(a) **(i)** –1.4 **(ii)** $x \geqslant -1.4$
(b) 3 **(c)** 3.4
18. **(a)** $h = 0.17$, $k = 1.33$
(c) **(i)** –2 or 0.33
(ii) 1.9
(iii) $x \leqslant -2.1$ or $x \geqslant 0.7$
19. 5.3, 4, 0.4
(b) –4
(c) $x^2 - 15x + 12 = 0$
(d) 1.4
20. **(a)** **(i)** Area = $30x$,
(ii) height = $\frac{250}{x}$
(d) **(i)** $13.5 \leqslant x \leqslant 36.5$
(ii) 1 840 **(iii)** 10.9 cm
21. **(a)** **(i)** $\frac{40}{x}$ m **(ii)** $(x - 4)$ m
(iii) $\left(\frac{40}{x} - 2\right)$ m
(d) **(i)** 6.3 m **(ii)** 8.9 m
(iii) 4.9 m by 2.5 m
22. **(a)** 8, 5.7, 2, 1.4, 0.7, 0.5
(c) –0.5 **(d)** –1
(e) 0.22
23. **(a)** $h = 3.7$, $k = 1.7$
(c) –1.9
(d) **(i)** 2.2 **(ii)** 1.4
(iii) 2.4 **(iv)** $x < 1.2$
24. **(b)** $n = 4\,500$
(c) **(i)** 1 500
(ii) the rate of production of bacteria per hour at $t = 5.5$
(d) **(ii)** $t = 4.94$ h
(e) $k = 50$
25. **(a)** **(i)** $p = 0$ **(iii)** 6.05
(b) **(i)** $q = 2.83$
(c) **(i)** $x = 0.76, 2.7$
(ii) $0.35 < x < 2.35$
26. **(a)** **(i)** $p = 12$
(b) 480
(c) **(i)** –0.03
(ii) The unit cost is decreasing at 3 cents per book
(d) **(ii)** $210 \leqslant x \leqslant 690$
27. **(b)** 6.5 m
(c) The particle stopped moving
(d) **(i)** 1.16
(ii) The gradient represents the speed of the particle

(e) 1.62 m/s
(f) **(ii)** 2.15 s
28. **(a)** $k = 3.9$ **(c)** –4.4
(d) $x = 1.9$ or 5.3
(e) **(ii)** $2.1 \leqslant x \leqslant 5.6$

Revision Exercise 7.8 *(Pg 246)*

1. **(a)** 27° **(b)** 36° **(c)** 54°
2. **(a)** 61° **(b)** 96°
3. **(a)** 49° **(b)** 27°
4. **(a)** 43° **(b)** 47°
5. **(a)** 64° **(b)** 32°
(c) 32° **(d)** 26°
6. **(a)** 56° **(b)** 62° **(c)** 56°
7. **(a)** 132° **(b)** 26°
(c) 24° **(d)** 40°
8. **(a)** 32° **(b)** 124° **(c)** 24°
9. 2.31 cm
10. **(a)** 85 cm² **(b)** 12.85 cm
11. 13.23 cm
12. 9.60 cm
13. **(a)** 28° **(b)** 56°
(c) 86° **(d)** 28°
14. 3 cm
15. **(a)** 120° **(b)** 70°
(c) 20° **(d)** 40°
16. **(a)** 50° **(b)** 25°
17. 114° **18.** 61°
19. **(a)** 96° **(b)** 84°
20. 40°
21. 62°
22. $x = 10$, $y = 5$; 50°
23. **(a)** 24° **(b)** 114° **(c)** 68°
24. **(a)** 35° **(b)** 65° **(c)** 30°
25. **(a)** 11 **(b)** 55° **(c)** 23°

Revision Exercise 7.9 *(Pg 253)*

1. **(a)** 54.1° **(b)** 7.53 cm
(c) 9.51 cm, 3.54 cm
2. **(a)** 7 cm **(b)** 6 cm
(c) 15 cm
3. **(a)** 24 cm
(b) **(i)** $\frac{12}{13}$ **(ii)** $-\frac{12}{13}$
(iii) $-\frac{12}{5}$ **(iv)** $\frac{5}{13}$
4. **(a)** 8.12 m **(b)** 13.3°
5. **(a)** $\frac{4}{5}$ **(b)** $\frac{4}{5}$
6. $1\frac{11}{20}$
7. 53.6 m
8. 35°
9. 5.07 cm, 9.81 cm

10. 38.7°
11. 692.8 cm²
12. **(a)** 13 cm **(b)** 37.4 cm²
13. **(a)** 12.65 cm **(b)** 103.3°
14. **(a)** 32.2° or 147.8°
(b) 3.20 cm or 10.57 cm
15. 10.46 cm, 33°, 17.79 cm
16. **(a)** 4.85 cm **(b)** 33.3°
17. **(a)** 8.94 cm **(b)** 83.6°
18. 23.42 cm
19. **(a)** 30° **(b)** 4 cm
(c) 3.46 cm
20. **(a)** 42 km **(b)** 82°
(c) 138.2°
21. **(a)** 59.7° **(b)** 120.3°
22. **(a)** 25 cm **(b)** 48.6°
(c) 29.15 cm
(d) **(i)** 49.9° **(ii)** 26.05 cm
(iii) 149.4 cm²
23. 502.8 m, 153.5°
24. **(a)** 22.7 cm **(b)** 337.2 cm²
25. 312 cm², 20.1 cm
26. **(a)** 056.1° **(b)** 12.7 m
(c) 56.0 m **(d)** 939.3 m²
27. **(a)** 6.50 km **(b)** 334.4°
(c) 718.9 ha
28. **(a)** 126.6 m **(b)** 39.7°
(c) 30.1° **(d)** 158.6°
(e) 9 211 m²
29. **(a)** 32.3° **(b)** 9.0 cm
(c) 15.5 cm; 25.6 cm²
30. **(a)** **(i)** 111° **(ii)** 61.7°
(iii) 082.7°
(b) **(i)** 57.4 m
(ii) 20.7°
(c) 44.9 m
31. **(b)** $NC = 347$ m
(c) **(i)** 42.5 m
(ii) 29.6°
32. **(a)** 20.7 m **(b)** 77.6 m
(c) 1 760 m²
(d) 45.3 m **(e)** 24.6°
33. **(a)** **(i)** 108° **(ii)** 40.9 m
(b) **(i)** 60° **(ii)** 30°
(iii) 14.9 m
(c) 27.7 m

Revision Exercise 7.10 *(Pg 260)*

1. **(a)** 86, 80 **(b)** $(1 - n)^3$
2. **(a)** 35, 48
(b) $\sqrt{21}$, $\sqrt{34}$
(c) $1\frac{1}{6}$, $1\frac{5}{12}$

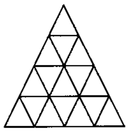
3. **(a)** $8^2 - 8 = 56$
(b) 11

4. **(a)** $l = 60$, $m = 25$, $n = 36$
(b) $T = 2\sqrt{SP}$ or $T^2 = 4SP$
(c) 196
(d) 112 is not a perfect square.
(e) 4 442 is not a multiple of 4.

5. **(a)** 3, 4, 24; 20; 4, 5, 6, 120; 70, 5, 6, 7
(b) $3S = M$
(c) (i) 330
(d) (i) 8 120 **(ii)** 3 080
(e) $S = \frac{1}{3}n(n + 1)(n + 2)$
(f) 6 663 is not an even number.

6. **(a)** $p = 4$, $q = 5$; p and q are the next two terms of the sequence 0, 1, 2, 3, …
(b) $r = 14$, $s = 20$
(c) $v = n - 3$
(d) $d = \frac{nv}{2}$; $d = \frac{n(n - 3)}{2}$
(e) 405

7. **(a)**



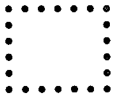
;
$a = 30$, $b = 18$, $c = 16$, $d = 15$
(b) $e = 45$, $f = 30$, $g = 25$, $h = 21$
(c) $T = n^2$
(d) $M = T + P - 1$
(e) 144, 198
(f) 3 335 is not a multiple of 3
(g) 623 is not a perfect square; 625

8. **(a)** $a = 25$, $b = 27$, $c = 29$, $d = 31$, $e = 33$, $f = 35$, $g = 37$
(b) 1, 4, 16, 25, 36; 31; 27, 64, 125, 216
(c) $M = N^2$; 625
(d) $S = N^3$; row 36
(e) $T = M - N + 1$; $T = N^2 - N + 1$
(f) (i) 381 **(ii)** 40

9. **(a)** $6^4 - 16 = 4 \times 8 \times (6^2 + 4)$
(b) $8^4 - 16 = 6 \times 10 \times (8^2 + 4)$
(c) $n^4 - 16 = (n - 2)(n + 2)(n^2 + 4)$
(d) $50\,609 = 13 \times 17 \times 229$

10. **(a)** 12, 13
(b) (i) 15, 36, 21
(ii) $x = 3n$, $y = (n + 1)^2$, $z = (n + 1)^2 - 3n$
(c) 9 901

11. **(a)**



(b) 14, 18, 22
(c) (i) 42 **(ii)** 2 002
(d) 17 **(e)** $x = 4n + 2$

Revision Exercise 7.11 *(Pg 268)*

1. **(a)** 88°
2. **(a)** 35°
3. **(a)** 575 m
4. **(a)** 5.4 cm **(b)** 6.1 cm
5. 10.1 cm
6. 8.3 cm
7. **(c)** 3.5 cm
8. 6.7 cm
9. **(a)** 9.9 cm
10. **(a)** 7.7 cm
11. **(a)** 8 cm
12. **(a)** 132°

Revision Exercise 7.12 *(Pg 273)*

1. **(a)** $\overrightarrow{OM} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
(b) $\overrightarrow{ON} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

2. $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$,
$\overrightarrow{MN} = \begin{pmatrix} \frac{5}{2} \\ \frac{7}{2} \end{pmatrix}$, $\overrightarrow{XY} = \begin{pmatrix} -\frac{5}{2} \\ -\frac{7}{2} \end{pmatrix}$;
$MN = XY$ and $MN \parallel YX$

3. **(a)** $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$,
$\overrightarrow{AD} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$; $BC \parallel AD$, $\frac{2}{5}$
(b) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

4. **(a)** 8.25 **(b)** (6, 4)
(c) 6.32

5. **(a)** $\begin{pmatrix} 3 \\ 12 \end{pmatrix}$
(b) (i) 11.18 **(ii)** 12.37

6. **(a)** 10 **(b)** ±10

7. **(a)** $y = -4$ **(b)** $x = 3\frac{1}{5}$

8. **(a)** $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1.5 \\ -3.5 \end{pmatrix}$
(c) (–2, –5)

9. **(a)** 13 units **(b)** $\begin{pmatrix} 2 \\ 27 \end{pmatrix}$
(c) $p = 6.5$, $q = -5$

10. **(a)** $\begin{pmatrix} 2 \\ 19 \end{pmatrix}$ **(b)** –5
(c) 8 units

11. **(a)** 26 units **(b)** (–2, 43)
(c) $\begin{pmatrix} -2.5 \\ 6 \end{pmatrix}$

12. **(a)** $\begin{pmatrix} -9 \\ 12 \end{pmatrix}$
(b) $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$, 13 units
(c) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ **(d)** $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$

13. **(a)** 5 units
(b) (i) $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
(iii) $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ **(iv)** $\begin{pmatrix} -6 \\ -10 \end{pmatrix}$

14. **(a)** trapezium **(b)** –(**a** + **b**)

15. **(a)** $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ **(b)** (9, 5)
(c) (i) $\sqrt{37}$ **(ii)** $-\frac{1}{6}$
(iii) $y = -\frac{1}{6}x + 2$
(iv) $\left(3, \frac{3}{2}\right)$

16. **(a)** $-\frac{3}{4}\mathbf{p}$
(b) (i) $\frac{3}{4}\mathbf{p} + \mathbf{q}$
(ii) $\mathbf{p} - \mathbf{q}$
(iii) $\frac{1}{4}(\mathbf{p} + 3\mathbf{q})$

17. **(b)** $QR \parallel PS$ and $QR = 3PS$
(c) $3(\mathbf{a} + 2\mathbf{b})$
(d) $\frac{3}{2}\mathbf{a}$

18. **(a)** (i) $\mathbf{b} - \mathbf{a}$ **(ii)** $\frac{1}{2}(\mathbf{b} - \mathbf{a})$
(iii) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

(c) **(i)** **a + b**
(ii) **b**
(iii) **a**; parallelogram

19. **(b)** $\overrightarrow{OP} = -2(\mathbf{t} + \mathbf{s})$,
$\overrightarrow{OQ} = \frac{8}{3}\mathbf{t} - 2\mathbf{s}$,
$\overrightarrow{OR} = -\mathbf{t} + 4\mathbf{s}$,
$\overrightarrow{OU} = \frac{2}{3}\mathbf{t} - \frac{3}{2}\mathbf{s}$

20. **(a)** **(i)** $2(\mathbf{a} + 3\mathbf{b})$
(ii) $\mathbf{a} + 3\mathbf{b}$
(iii) $\frac{3}{2}(\mathbf{b} - \mathbf{a})$
(iv) $\mathbf{b} - \mathbf{a}$
(b) *PN* is parallel to *ML*
(c) trapezium
(d) **(i)** $\frac{1}{3}$ **(ii)** $\frac{9}{4}$

Revision Exercise 7.13 *(Pg 280)*

1. **(a)** $\frac{2}{15}$ **(b)** $\frac{2}{5}$ **(c)** $\frac{2}{5}$
(d) $\frac{8}{15}$ **(e)** $\frac{8}{45}$

2. **(b)** $\frac{17}{24}$

3. **(a)** $\frac{25}{64}$ **(b)** $\frac{9}{64}$ **(c)** $\frac{15}{32}$

4. **(a)** $\frac{2}{9}$ **(b)** $\frac{1}{15}$ **(c)** $\frac{2}{9}$
(d) $\frac{14}{45}$

5. **(a)** **(i)** $\frac{7}{12}$ **(ii)** $\frac{5}{6}$
(b) **(i)** $\frac{7}{22}$ **(ii)** $\frac{7}{22}$
(iii) $\frac{15}{22}$

6. $\frac{x}{x + 36}$, $x = 24$

7. **(a)** **(i)** $\frac{1}{2}$ **(ii)** $\frac{11}{18}$
(iii) $\frac{1}{9}$ **(iv)** $\frac{8}{9}$
(b) **(i)** $\frac{13}{90}$ **(ii)** $\frac{38}{63}$

8. **(a)** $\frac{2}{3}$
(b) **(i)** $\frac{33}{299}$ **(ii)** $\frac{266}{299}$

9. **(a)** $\frac{41}{50}$ **(b)** $\frac{7}{50}$ **(c)** $\frac{7}{25}$
(d) $\frac{17}{50}$

10. **(a)** $\frac{1}{14}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{3}{14}$
(d) $\frac{3}{7}$ **(e)** $\frac{1}{2}$

11. **(a)** $\frac{35}{88}$ **(b)** $\frac{23}{44}$ **(c)** $\frac{21}{44}$

12. **(a)** $\frac{3}{5}$ **(b)** $\frac{2}{5}$ **(c)** $\frac{29}{45}$
(d) $\frac{16}{45}$

13. **(a)** **(i)** $\frac{7}{36}$ **(ii)** $\frac{4}{81}$
(b) **(i)** $\frac{1}{5}$ **(ii)** $\frac{5}{33}$

14. **(a)** $\frac{1}{6}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{12}$

15. **(a)** $\frac{1}{6}$ **(b)** $\frac{19}{78}$
(c) $m = 10, n = 6$

16. **(a)** $\frac{8}{15}, \frac{7}{14}, \frac{4}{14}, \frac{3}{14}; \frac{8}{14}, \frac{3}{14};$
$\frac{3}{15}, \frac{8}{14}, \frac{4}{14}, \frac{2}{14}$
(b) **(i)** $\frac{4}{35}$ **(ii)** $\frac{68}{105}$
(iii) $\frac{4}{15}$ **(iv)** $\frac{13}{35}$

17. **(a)** $\frac{5}{8}, \frac{4}{9}, \frac{4}{9}, \frac{5}{9}$
(b) $\frac{1}{6}$ **(c)** $\frac{25}{72}$

18. **(a)** $\frac{1}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{6}$
(b) **(i)** $\frac{3}{5}$ **(ii)** $\frac{43}{120}$

19. **(a)** **(i)** $\frac{1}{6}$ **(ii)** $\frac{1}{2}$
(b) $\frac{11}{18}$ **(c)** $\frac{4}{15}$

20. $x = 12, y = 24$
(a) $\frac{103}{295}$ **(b)** $\frac{24}{295}$

21. **(a)** 2, 10, 18, 3, 9, 21, 33, 5, 25, 45, 55, 21, 63, 11, 55, 121, 39, 65, 117
(b) **(i)** $\frac{5}{6}$ **(ii)** $\frac{1}{6}$
(iii) $\frac{4}{9}$ **(iv)** $\frac{5}{18}$
(v) $\frac{1}{9}$

Revision Exercise 7.14 *(Pg 291)*

1. **(a)** 27
(b) **(i)** \$13 500
(ii) \$3 240

2. **(a)** 18 000 tonnes

3. **(a)** $11\frac{1}{9}\%$
(b) 2.8 million tonnes
(c) 135°

4. **(a)** **(i)** $52\frac{7}{9}\%$
(ii) $16\frac{2}{3}\%$
(b) **(i)** \$38 000
(ii) \$12 000
(c) $x = 70$

5. **(a)** **(i)** 182 **(ii)** 35.2%
(b) school bus

6. 60, 35, 20, 5

7. **(a)** 46 **(b)** 45.5
(c) 45

8. **(a)** **(i)** 4 **(ii)** 5.5
(iii) 7
(b) $a = 33$

9. **(a)** 75.25 cm **(c)** $\frac{2}{5}$
(d) $\frac{8}{39}$

10. **(a)** 20, 25, 37
(b) 62.6 cm **(d)** $\frac{17}{550}$

11. **(b)** **(i)** 13 hours
(ii) 13 hours
(iii) 13.5 hours

12. **(b)** **(i)** 6 marks
(ii) 5 marks
(iii) 4.44 marks

13. **(a)** $A = 14, B = 18$
(c) **(i)** 4.25 km
(ii) 2.35 km
(d) $\frac{11}{20}$ **(e)** $\frac{5}{44}$ **(f)** $\frac{1}{3\,160}$

14. **(a)** 27 km **(b)** 17 km
(c) $\frac{9}{13}$

15. **(a)** **(i)** 53 **(ii)** 12
(b) 67 **(c)** $\frac{81}{95}$

16. **(c)** **(i)** 80.4 mm
(ii) 22 mm
(d) $\frac{51}{100}$ **(e)** $\frac{11}{51}$ **(f)** $\frac{1}{330}$

17. **(c)** **(i)** 51 cm **(ii)** 17 cm
(d) **(i)** $\frac{15}{28}$ **(ii)** $\frac{9}{56}$
(e) **(i)** $\frac{9}{14}$ **(ii)** $\frac{5}{16}$
18. **(c)** 90 girls
(d) **(i)** 3.5 km
(ii) 1.2 km
(e) **(i)** $\frac{2}{7}$ **(ii)** 5
(f) $\frac{1}{4\ 900}$
19. **(c)** **(i)** 62 **(ii)** 22
(iii) 10 **(iv)** 7
20. **(a)** 19, 87, 138, 156, 159
(c) **(i)** 1 975
(ii) 1 425
(e) **(i)** 148 bulbs
(ii) 138 bulbs
(f) Brand B is a better buy because it has a larger median than Brand A.

Revision Exercise 7.15 *(Pg 302)*

4. **(a)** (18, 7) **(b)** (–8, –5)
(c) (7, 4)
5. **(a)** $y = -2x - 1$
(b) $y = -2x + 1$
(c) $y = -2x + 9$
6. (15, –7)
7. **(a)** **(i)** (3, 4)
(ii) $k = 5$, $E(-1, 0)$
(b) $L(9, 4)$, $M(6, 2)$, $N(3, 2)$
(c) $G(0, 4)$, $H(-2, 2)$, $J(4, 2)$
(d) A 90° clockwise rotation about (4, 4)
8. **(a)** $A'(2, 2)$, $B'(2, -2)$, $C'(-2, -2)$, $D'(-2, 2)$
(b) $A'(-1, -1)$, $B'(-1, 1)$, $C'(1, 1,)$, $D'(1, -1)$
9. $h = 7$, $k = 6$, $p = 7$, $q = 4$
10. **(a)** (4, 7) **(b)** (–3, –7)
(c) (–3, 5)
11. **(a)** translation $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$
(b) reflection in the y-axis
(c) 180° rotation about (0, 3)
13. **(a)** 90° clockwise rotation about the origin
(b) 180° rotation about (1, –3)
(c) 90° anticlockwise rotation about the origin
14. 90° anticlockwise rotation about the origin
15. $k = 7$, (1, 5)
16. **(a)** reflection in the y-axis
(b) reflection in the x-axis
(c) 180° rotation about (0, 0)
(d) 90° clockwise rotation about (0, 0)
17. **(a)** translation of 3 cm parallel to AC
(b) reflection in the line BC and BE
(c) reflection in the line BC followed by a 60° anticlockwise rotation about C
(d) an enlargement centre A, scale factor $\frac{1}{2}$
18. $A_1(-2, 4)$, $B_1(-3, 7)$, $C_1(-5, 6)$, $A_2(1, 3)$, $B_2(4, 10)$, $C_2(3, 10)$.
19. **(a)** a shear with the x-axis as the invariant line and shear factor –2.
(b) an enlargement centre at (1, 1) and scale factor 2.
(c) $x + y = 4$
(d) a 90° anticlockwise rotation about (–1, 2)
20. P is a reflection in the x-axis, Q is a 90° clockwise rotation about (3, 0)
21. **(a)** (2, 5)
(b) (5, –28)
(c) 14 cm²
22. **(a)** 180° rotation about O
(b) translation parallel to AO and length AO
(c) 120° anticlockwise rotation about O
(d) 60° clockwise rotation about O
23. **(a)** reflection in the x-axis.
(b) reflection in the line $x + y = 0$.
(c) reflection in the line $y = x$.
(d) 90° anticlockwise rotation about (0, 0).
24. **(a)** 8 **(b)** 16
(c) 144 **(d)** 144
25. **(b)** $P(7, -2)$, $Q(-1, 0)$
(c) A shear with $x = 1$ as the invariant line and shear factor 1.
(d) $x + y = 2$
(e) centre at (–4, 1), 90° anticlockwise.
26. **(a)** $x + y = 2$
(b) $P(0, -2)$, $Q(-2, -1)$
(c) **(i)** $y = 1$ **(ii)** 2
(d) $L(-2, -1)$, $M(0, 0)$
(e) $A_2(-8, 2)$, $B_2(-6, -2)$
27. **(b)** Shear with $x = 0$ as invariant line and shear factor $1\frac{1}{2}$
(c) A 90° anticlockwise rotation about (1, 4)

Specimen Paper A *(Pg 308)*

Paper 1

1. 4
2. 185
3. **(a)** $1\frac{1}{27}$ **(b)** $\frac{1}{16}$
4. **(a)** $\frac{7}{10}$ **(b)** $\frac{2}{5}$
5. **(a)** 00 21 **(b)** $151.20
6. **(a)** $-24\frac{2}{3}$ **(b)** –3
7. **(a)** 2.98×10^5 km/s
(b) 2.5×10^{-6} s
8. **(a)** $14p^{\frac{20}{3}}$ **(b)** 7
(c) 9 km
9. **(a)** 90°
(b) **(i)** 84° **(ii)** 42°
10. $2^2 \times 3 \times 7^3$, 21
11. **(a)** $6t(t - 3)$
(b) $(x - 2)(x + 7)$
(c) $(3a - b)(3a + b)$
12. **(a)** $\frac{1}{2}$
(b) **(i)** $\frac{x}{2}$ **(ii)** $\frac{1}{25}$
13. **(a)** Yes; They are equiangular.
(b) No; $\frac{7.4}{6.0} \neq \frac{4.5}{2.5} = \frac{9.0}{5.0}$
14. **(a)** 100° **(b)** 1.584 km²
15. **(a)** $\frac{13\ 440}{12\ 000} \times \frac{14}{20} \times 15$
(b) $345
16. **(a)** $\frac{17x - 3}{14}$
(b) $x = 2$, $y = -3$
17. **(a)** $S\hat{R}P = S\hat{Q}R$, $R\hat{S}P = R\hat{S}Q$
(b) **(i)** $3\frac{1}{2}$ cm **(ii)** $\frac{1}{4}$
18. **(a)** $u = v + ts^2$ **(b)** 3×10^4

19. **(a)** $\begin{pmatrix} 8 \\ -6 \end{pmatrix}$ **(b)** (1, 4)
(c) trapezium
(d) $k = -1$ or 11
20. **(a)** $2r - 1$ **(c)** 3 cm
21. **(a)** **(i)** $1\frac{1}{3}$ ms^{-2}
(ii) $\frac{2}{3}$ ms^{-2}
(iii) 1.8 km
(iv) 2.25 km
(b) 120
22. **(a)** 18, 14, 16 **(b)** 28.1 m
23. **(a)** $s_3 = 36 = 6^2$,
$s_4 = 100 = 10^2$,
$s_5 = 225 = 15^2$,
(b) $s_n = \left[\frac{n(n+1)}{2}\right]^2$
(c) 20th square

Paper 2

1. **(a)** **(i)** 270 m **(ii)** 49.4°
(b) **(i)** \$111 792
(ii) 85.0%
(iii) \$45 120
(iv) \$69.31
2. **(a)** **(ii)** 76.6 kg
(b) **(i)** 46% **(ii)** 984
3. **(a)** 466 m **(b)** 52 465 m^2
(c) 14.3°
4. **(c)** 304°, 346°
(d) 10 20
5. **(b)** $x = 45$, $y = 30$;
$12x = 540$, $12y = 360$
(c) \$3 198 **(d)** 31.98%
6. **(a)** 12.5 mm^3
(b) **(i)** $\sqrt[3]{\frac{3V}{4\pi}}$ **(ii)** 1.44 mm
(c) 1.1×10^{-4} mm
7. **(a)** 337.5°
(b) 60.7 km
(c) 17 55 or 5.55 p.m.
(d) 43.4°
(e) 41.9 km
(f) 2 531 km^2
8. **(b)** **(ii)** **(a)** 1 : 1 **(b)** 1 : 2
(c) 1 : 4
(iii) 1 : 16
(c) **(ii)** 180° rotation about (0, 0) and enlargement scale factor –1, centre (0, 0)
(d) **(ii)** 1 : 1

9. **(a)** **(i)** Probability
$= \frac{5}{10} \times \frac{5}{9} + \frac{5}{10} \times \frac{4}{9}$
$= \frac{5}{18} + \frac{4}{18} = \frac{1}{2}$;
(ii) Probability
$= \frac{9}{18} \times \frac{9}{17} + \frac{9}{18} \times \frac{8}{17}$
$= \frac{9}{34} + \frac{8}{34} = \frac{1}{2}$.
(b) $a = n$; $b = n - 1$; $p = 2n$;
$q = 2n - 1$;
$r = 2(2n - 1)$ or $4n - 2$
(c) $\frac{5}{19}$ **(d)** $\frac{35}{46}$
(e) $c = 500$, $d = 499$
10. **(I)** **(a)** $\frac{35}{x^2}$
(c) 62.3, 55.7, 53, 59.3
(e) **(i)** 5.4 cm
(ii) 51
(iii) 2.1 cm
(iv) 60.6 cm^2
(II) **(a)** 12, 23, 38, 48, 53
(c) **(i)** 37 **(ii)** 52
(d) **(i)** $\frac{1}{5}$ **(ii)** $\frac{19}{60}$
(e) **(i)** $\frac{96}{295}$ **(ii)** $\frac{1\,681}{3\,600}$

Specimen Paper B *(Pg 315)*

Paper 1

1. **(a)** $1\frac{1}{8}$ **(b)** $1\frac{1}{4}$
2. **(a)** 93, 142 **(b)** $2^n + 1$
3. 6
4. **(a)** 21 35 on Sunday
(b) 76 km/h
5. **(a)** $3(2x + y)(2x - y)$
(b) $(x + y)(2x - 3)$
6. 25 : 27
7. **(a)** $-\frac{4}{5}$ **(b)** $\frac{yc + a^3}{y - 1}$
8. **(a)** 10 km **(b)** 50 cm^2
9. $x = 91°$, $y = 44°$, $z = 64°$
10. $x = 2$, $y = -3$
11. 0.5%
12. **(a)** 150° **(b)** 330°
13. \$450 000
15. **(a)** 41
(b) 5, 6, 7 as these are the most common sizes worn by school children

16. 21 units^2
17. **(a)** 37° **(b)** 16°
18. **(a)** $A\left(-1\frac{1}{2}, 0\right)$, $C(0, 15)$
(b) $x = \frac{1}{2}$
19. **(a)** **(i)** $\frac{1}{25}$ **(ii)** $\frac{2}{5}$
(b) **(i)** $\frac{1}{12}$ **(ii)** 1
20. **(a)** 24 **(b)** 80
(c) $\frac{1}{5}$
21. **(a)** $-\frac{3}{4}$ **(b)** (5, 5)
22. 4 cm
23. **(a)** 4.96 cm **(b)** 6.8 cm
24. **(a)** $10(\mathbf{q} - \mathbf{p})$ **(b)** $5(6\mathbf{q} - \mathbf{p})$
(e) $s = \frac{3}{5}$, $t = \frac{1}{5}$
(f) $\frac{1}{4}$

Paper 2

1. **(a)** 39.79 cm **(b)** 4.07 cm
(c) 10 h 46 min
2. **(a)** $1^3 + 2^3 + \ldots + 6^3 + 7^3$
$= 784$
$= (1 + 2 + \ldots + 6 + 7)^2$
(b) 441 **(c)** 10
(d) $(1 + 2 + \ldots + 49 + 50)^2$
$= \left[\frac{50}{2}(1 + 50)\right]^2$
$= 1\,275^2 = 1\,625\,625$
3. **(a)** 50.0° **(d)** 17.5°
4. **(a)** 20.2 cm **(b)** 171.4 cm^2
(c) 32.2° **(d)** 23.3°
5. **(a)** **(i)** $\frac{x}{4}$, $\frac{2}{x}$
(ii) $\frac{2}{y}$, $\frac{x^2}{8} + \frac{4}{y^2}$; 2
(b) **(i)** 110° **(ii)** 49°
(iii) 49°
6. **(a)** $\frac{3}{5}, \frac{2}{5}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{5}{8}$
(b) $\frac{7}{10}$ **(c)** $\frac{91}{100}$
7. **(a)** 5 h 45 min
(b) RM6 608 S\$2 891.90
(c) **(i)** $\frac{200}{x}$ **(ii)** $\frac{200}{x + 5}$
(iv) 30.9 km/h

8. **(a)** 0.3, 1.1, –4.5
(c) **(i)** 2.7
(ii) 3.5 or –1.1
(iii) 3.1 or –0.7
(iv) –0.8
9. **(a)** $a = 8, b = 24, c = 24, d = 8$
(b) 8
(c) 27
(d) **(i)** $12(n - 2)$
(ii) $6(n - 2)^2$
(iii) $(n - 2)^3$
10. **(I)** **(c)** (–2, 0) 90° clockwise
(d) 15 units^2
(f) 1
(II) **(a)** 12, 23, 38, 48, 53, 60
(c) **(i)** 37 cm
(ii) 11.5
(iii) $\frac{7}{15}$
(d) $\frac{5}{6}$
(e) **(i)** $\frac{107}{295}$ **(ii)** $\frac{1}{15}$

Specimen Paper C *(Pg 321)*

Paper 1

1. **(a)** 3.0×10^{-3} **(b)** 10
2. **(a)** 37°C **(b)** $x = -3$
3. **(a)** 0.314, –0.314, –31.4, $-3\frac{1}{7}$
(b) 5
4. **(a)** $-\frac{4}{5}$ **(b)** –5
5. 120
6. $x = 5, y = -2$
7. 600, 400
8. 40.6%
9. **(a)** $\sqrt{34}$ **(b)** –2
10. $h = 26, k = -2\frac{1}{2}$

11. **(a)** $x(x - 2)(x + 3)$
(b) $9(2 + a)(2 - a)$
12. $7.74 billion
13. **(a)** $y = \frac{9}{4}(x + 1)^2$ **(b)** 36
14. 8 : 27
16. 34°
17. **(a)** 4 500 **(b)** 45
18. **(a)** $-\frac{7}{78}$ **(b)** 19.2 cm
19. 16 000 people
20. **(a)** 108° **(b)** 28°
(c) 64°
21. Reflection in the line AQ; enlargement centre at A, scale factor 1.5
22. **(a)** An enlargement with scale factor $\frac{1}{2}$ and centre at A
(b) A reflection in line QR
(c) A reflection in line PQ
(d) A translation of 3 cm along AC
23. **(a)** 16 m/s **(b)** 175 m
(c) 36 m/s
24. **(a)** 68° **(b)** 56°
25. **(a)** **(i)** $x = \frac{2}{3}, y = \frac{5}{12}, z = \frac{7}{12}$
(ii) **(a)** $\frac{20}{39}$ **(b)** 0
(b) $\frac{1\,286}{1\,287}$

Paper 2

1. **(a)** 2.780
(b) **(i)** $1\frac{5}{7}$
(ii) $a = \frac{b^2 d}{cb^2 - 2d}$

2. **(a)** $22.64 **(b)** 35
(c) $C = 7.50 + 0.87n \times 1.2$
(d) $28.28 **(e)** $57.84
(f) 56.72
3. **(a)** $\frac{1}{84}$ **(b)** $\frac{1}{8}$
(c) $\frac{145}{168}$
5. **(a)** 16.12 cm **(b)** 127.2°
(c) 32.25 cm^2 **(d)** 57.68 cm^2
6. **(a)** $\frac{1}{6}$ **(b)** 3.587 cm
(c) 78.77 cm^2 **(d)** 12.9 cm^3
7. **(a)** **(i)** 45 **(ii)** 52
(iii) 21
(b) **(i)** **(a)** $\frac{1}{6}$ **(b)** $\frac{7}{30}$
(ii) $\frac{13}{2\,995}$
8. **(a)** 315° **(b)** 36.71 km
(c) 16 45 **(d)** 33.46 km
(e) 024.3°
9. **(a)** $h = 2.3, k = 7.2$
(c) **(i)** 1.1 or 3.7
(ii) 5.2 or 0.8
(iii) no solution
(iv) 1.3 or 6.2
(d) 1.5
10. **(I)** **(b)** H is a 90° clockwise rotation about point (8, 0)
(c) 9 : 1
(e) $a = -2, b = 2$
(II) **(a)** **(i)** $1 320
(ii) $396
(b) $571.20
(c) $12 282.50
(d) $3 268
(e) $741, $54.60
(f) $12 400

APPENDIX A

(Sample worksheet for IT Open Tool, Geometer's Sketch Pad)

Shearing Effect on a Triangle

1. Use ▨ to draw lines AB, BC and AC to form a triangle ABC.
2. Use ↖ to select point A and, with shift-key on, select line segment BC, choose *Parallel Line* from Construct Menu to draw a line parallel to line BC.
3. Use ⊡ to select two points D and E on the line parallel to BC.
4. Use ↖ to select the line m and choose *Hide Line* from Display Menu. Next use ▨ to join the points D and E as a line segment n.
5. Use ⊡ to select another point F on the line segment. Use ▨ to join BF and CF.
6. Use ↖ to select point A and, with shift-key on, select points B and C, choose Construct Menu and select *Polygon Interior*.
7. Repeat the above for the triangle FBC.
8. Use ↖ to select triangle ABC by clicking on the inside of the triangle ABC. Select *Area* from Measure Menu. The area of triangle ABC will be displayed. Repeat the same for triangle FBC. What do you notice about the area of the two triangles? ______________________________
9. Use ↖ to select point A and, with shift-key on, select line BC and choose *Distance* from Measure Menu. The perpendicular distance from A to line BC will be shown.
10. Use ↖ to select point A and, with shift-key on, select point F. Choose *Distance* from Measure Menu to measure the length of AF. Choose *Calculate* from Measure Menu to calculate the ratio of AF over distance of A to BC. This gives the shear factor.
11. Use ↖ to select point A, move it about and observe how the various values change. Repeat the same for points B, C, D, E and F. What do you notice about the shear factor? ____________
12. Use ↖ to select F and, with shift-key on, select line segment DE. Choose *Animate* from Display Menu. It will enable the point F to move along line DE either *Slowly*, *Normally* or *Quickly*. Select *Slowly* and observe how the various values change.

Area FBC = 23.5 cm^2

Area ABC = 23.5 cm^2

FGB 15.2 cm

FC = 6.7 cm

Distance A to $\overline{BC}$ = 4.8 cm

AF = 12.0 cm

$$\frac{AF}{(\text{Distance A to } \overline{BC})} = 2.49$$