New Syllabus

Mathematics 3

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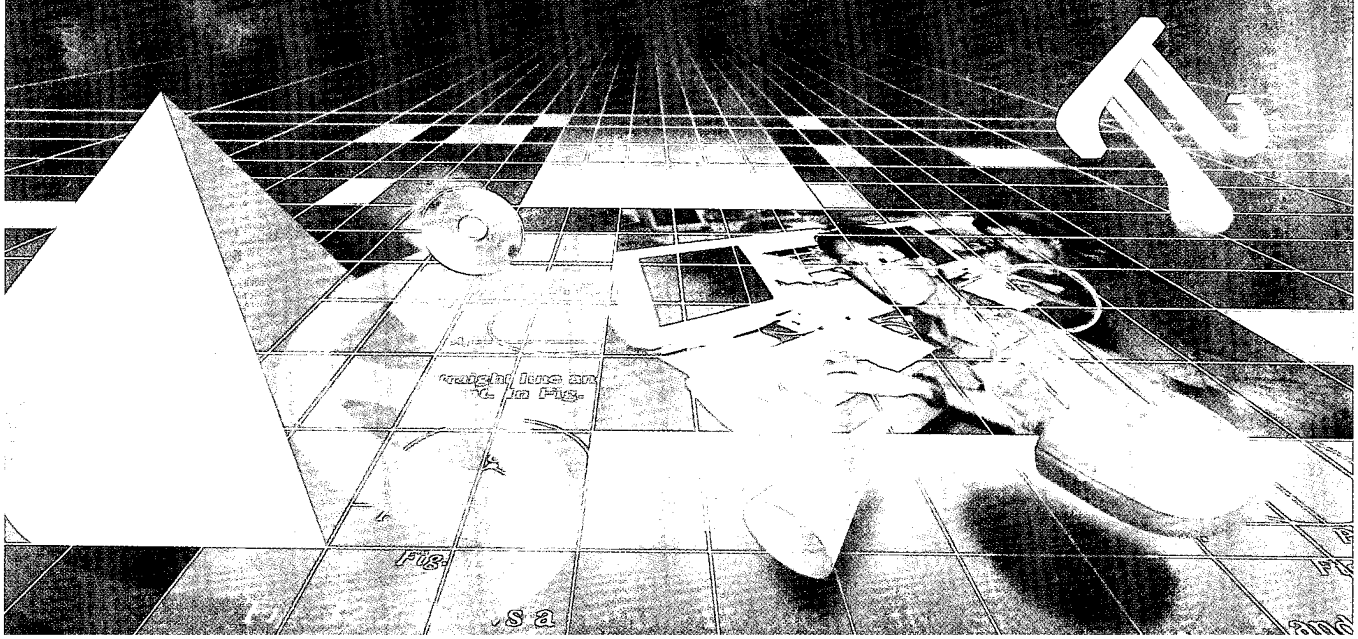

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PREFACE

New Syllabus Mathematics is a series of four books. These books follow the Mathematics syllabus for Secondary Schools, implemented from 2001 by Ministry of Education, Singapore. The whole series covers the complete syllabus for the Singapore-Cambridge GCE 'O' Level Mathematics.

The fifth edition of New Syllabus Mathematics 3 retains the goals and objectives of the previous edition, but has been revised to meet the requests of users of the fourth edition and to keep materials up-to-date as well as to give students a better understanding of the contents.

All topics are comprehensively dealt with to give students a firm grounding in the subject. Explanations of concepts and principles are concise and written in clear language with supportive illustrations and examples. Examples and exercises have been carefully graded to aid students in progressing within, as well as up, each level. Those exercises marked with a ✶ are either tricky or involve more calculations. Innovative sections called "Problem Solving" and "Exploration", placed at the end of the chapter, contain more difficult and challenging questions requiring students to apply their knowledge and experience in solving them.

Numerous revision exercises are provided at appropriate intervals to enable students to recapitulate what they have learnt. In addition, there are mid-year and end-of-year examinations specimen papers.

Important features which have been revised in this edition to facilitate learning are:

- an interesting introduction at the beginning of each chapter complete with photographs or graphics
- brief specific instructional objectives for each chapter
- in-class activities
- activities and interesting information in the marginal text ("Down Memory Lane", "Investigate", "Back In Time", "For Your Information", "Check This Out!", "Library Corner", "Just For Fun". "Problems", "It's A Fact", "Are You Game Enough?", " IT" as well as clip-notes.)

Problem-solving heuristics taught in Book 1 and 2 are introduced at appropriate sections of this book to reinforce problem-solving skills. In addition, questions which call for these skills are also set in the margin for students to do at their own pace and time.

Ample opportunities are also provided for mathematical investigative and communicative activities.

It is hoped that these features will help students learn mathematics with more zest and excel in the subject.

CONTENTS

CHAPTER 1

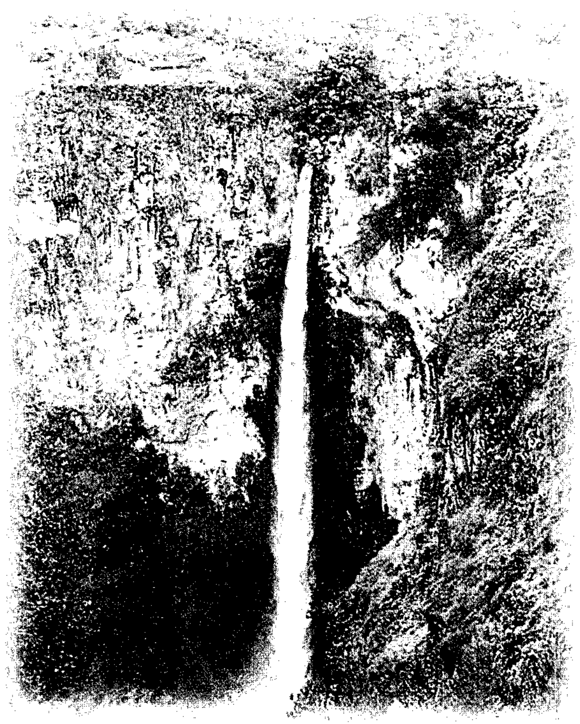
Solutions to Quadratic Equations

In this chapter, you will learn how to solve

- △ quadratic equations by factorisation;
- △ quadratic equations by using "completing the square" method;
- △ quadratic equations by using formula;
- △ problems that can be reduced to quadratic equations.

Preliminary Problem

The majestic Sipiso-piso waterfall near Lake Toba in Indonesia falls down a stiff ravine to the riverbed. Do you know that the height, h metres, of a particular droplet of water above the river bed, t seconds after leaving the top of the cliff, can be represented by $h = 125 + 2t - 5t^2$? If we want to find out how long it takes that droplet of water to fall to the riverbed, we must solve the equation $125 + 2t - 5t^2 = 0$.



Solving Quadratic Equations by Factorisation

We will begin this chapter by revising what we have learnt in Lower Secondary.

Example 1

Solve the following equations by factorisation:

(a) $2x^2 = 7x$ *(b)* $3x^2 - 5x - 8 = 0$
(c) $(2x - 1)(x - 2) = 5$

Solution

(a)
$$2x^2 = 7x$$
$$2x^2 - 7x = 0$$
$$x(2x - 7) = 0$$
i.e. $x = 0$ or $2x - 7 = 0$

$\therefore$ $x = 0$ or $x = 3\frac{1}{2}$

(b)
$$3x^2 - 5x - 8 = 0$$
$$(3x - 8)(x + 1) = 0$$
i.e. $3x - 8 = 0$ or $x + 1 = 0$

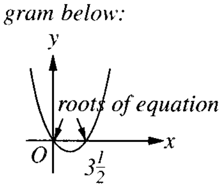
$\therefore$ $x = 2\frac{2}{3}$ or $x = -1$

(c)
$$(2x - 1)(x - 2) = 5$$
$$2x^2 - 5x + 2 - 5 = 0$$
$$2x^2 - 5x - 3 = 0$$
$$(2x + 1)(x - 3) = 0$$
i.e. $2x + 1 = 0$ or $x - 3 = 0$

$\therefore$ $x = -\frac{1}{2}$ or $x = 3$

Notes: *It is important that you do not cancel x on both sides, such as the equation shown in Example 1(a), as this will result in the loss of the root 0.*

Graphically, the roots of the equation correspond to places where the graph of $y = 2x^2 - 7x$ cuts the x-axis, as shown in the diagram below:



NB: Forming an equation with given roots is the reverse process of solving an equation.

Example 2

Find the equation in x whose roots are 2 and $-\frac{3}{4}$.

Solution

Since $x = 2$ or $x = -\frac{3}{4}$

$x - 2 = 0$ or $4x + 3 = 0$

$\therefore$ The equation is $(x - 2)(4x + 3) = 0$

i.e. $4x^2 - 5x - 6 = 0$

For your information: *A parabola is a curve similar to the trajectory traced out by a ball when it is projected at an angle in the air and falls to the ground.*

Exercise 1a

1. Solve the following quadratic equations by factorisation:

(a) $x^2 - 5x = 0$	**(b)** $4x^2 = 7x$
(c) $6t^2 = t(t - 4)$	**(d)** $5y^2 = y(y + 3)$
(e) $a^2 + 9a = 0$	**(f)** $3h^2 = h(5 - 2h)$
(g) $x^2 - 2x + 1 = 0$	**(h)** $7a + a^2 - 18 = 0$
(i) $2z^2 + 5z - 3 = 0$	**(j)** $c^2 + 2c = 35$
(k) $8p - 16 - p^2 = 0$	**(l)** $4 - 3b - b^2 = 0$
(m) $12 - a - a^2 = 0$	**(n)** $10t^2 - t = 2$
(o) $y^2 - 22y + 96 = 0$	**(p)** $12a^2 - 16a - 35 = 0$

2. Form a quadratic equation in x with the given roots for each of the following:

(a) 2, 3 **(b)** 3, −4 **(c)** −5, 6 **(d)** $5, \frac{1}{2}$

(e) $\frac{2}{3}, -\frac{4}{5}$ **(f)** $-\frac{7}{8}, \frac{5}{6}$ **(g)** $-2\frac{1}{2}, -1\frac{3}{4}$ **(h)** $-1\frac{1}{2}, \frac{2}{3}$

Back in Time

Both linear and quadratic equations had been dealt with for over two thousand years. The early Chinese and Babylonians made use of equations to solve everyday problems such as the sharing of an inheritance. No general method or formula was available to solve equations at that time.

Solution by Completing the Square

Sometimes the roots of a quadratic equation cannot be obtained by simple factorisation. So, a more general method is used to obtain the roots of the quadratic equation. This method is based on the fact that any quadratic equation may be written in the form $(x + p)^2 = q$, where p and q are **real numbers**.

For Your Information

The word 'quadratic' comes from the Latin word 'quadratum', which means 'a squared figure'.

*The **roots** of a quadratic equation are the solutions to the equation.*

Example 3

Solve (a) $(x + 2)^2 = 16$ and (b) $(x - 3)^2 = 5$.

Solution

(a) $(x + 2)^2 = 16$

Taking the square root of each side:

$x + 2 = 4$ or $x + 2 = -4$

$\therefore$ $x = 2$ or $x = -6$

(b) $(x - 3)^2 = 5$

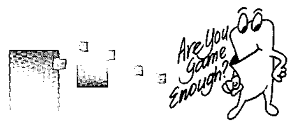
Taking the square root of each side:

$x - 3 = \sqrt{5}$ or $x - 3 = -\sqrt{5}$

$x = 3 + \sqrt{5}$ or $x = 3 - \sqrt{5}$

$x = 5.24$ or $x = 0.76$ (correct to 2 dec. places)

To write a given quadratic equation in the form $(x + a)^2 = b$, we first consider the identity learnt in Book 2, which is $\boldsymbol{x^2 + 2ax + a^2 = (x + a)^2}$.



John has to bring 4 horses, A, B, C and D, across a river. The times taken by the horses to swim across the river are as follows:

***Horse A** - 1 min*
***Horse B** - 2 min*
***Horse C** - 5 min*
***Horse D** - 10 min*

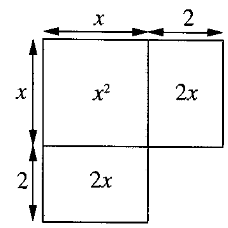
Given that John can take at most two horses across the river each time, he decides to adopt the following strategy:

(a) Take horses A and B across first.
(b) Come back with horse A.
(c) Take horses A and C across.
(d) Come back with horse A.
(e) Take horses A and D across.
The total time taken will be = (2 + 1 + 5 + 1 + 10) = 19 minutes.

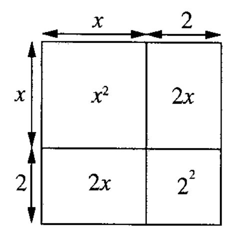
Note that the time taken to cross the river each time depends on the slower horse and that John has to use a horse each time. What do you think about John's strategy? Can you reduce the total time taken with a better strategy?

In the quadratic expression $x^2 + 2ax$, the coefficient of x^2 is 1 and the constant term is 0. What must we add to this expression to make it a *perfect square*?

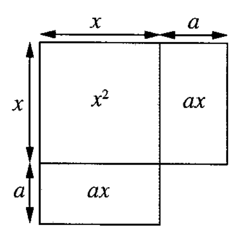
The following diagrams illustrate the idea of 'completing the square':



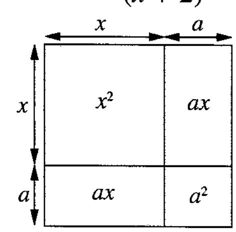
Total area = $x^2 + 4x$



Complete the square by adding 2^2.
Area of square $= x^2 + 4x + 2^2$
$= (x + 2)^2$



Total area = $x^2 + 2ax$



Complete the square by adding a^2.
Area of square $= x^2 + 2ax + a^2$
$= (x + a)^2$

In general, to make $(x^2 + kx)$ a perfect square, $\left(\frac{k}{2}\right)^2$ must be added to it.
Thus $x^2 + kx + \left(\frac{k}{2}\right)^2 = \left(x + \frac{k}{2}\right)^2$.

Notice that the quantity to be added is the square of half the coefficient of x, provided the **coefficient of x^2 is 1**, i.e. to make $(x^2 - 8x)$ a perfect square, $\left(-\frac{8}{2}\right)^2$ must be added to it. The result is

$$x^2 - 8x + \left(-\frac{8}{2}\right)^2 = x^2 - 8x + (-4)^2$$
$$= (x - 4)^2$$

Example 4

What must be added to each of the following expressions to obtain a perfect square?

(a) $x^2 + 5x$ *(b)* $a^2 + 2ka$ *(c)* $c^2 - 4c$

Solution

(a) $x^2 + 5x$
The coefficient of x is 5. Half of this is $\frac{5}{2}$.

$$x^2 + 5x + \left(\frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)^2$$

$\therefore$ $\left(\frac{5}{2}\right)^2$ must be added.

(b) $a^2 + 2ka$
This is a quadratic expression in a.
The coefficient of a is $2k$.
Half of this is k.
$a^2 + 2ka + k^2 = (a + k)^2$
$\therefore$ k^2 must be added.

(c) $c^2 - 4c$
This is a quadratic expression in c.
The coefficient of c is -4.
Half of this is -2.
$c^2 - 4c + (-2)^2 = (c - 2)^2$
$\therefore$ $(-2)^2$ must be added.

For Your Information

'Completing the square' is an innovative way of solving a quadratic equation, by dividing both sides of the equation by the coefficient of x^2 and then adding a constant term, in order to express the equation as a single squared term.

Exercise 1b

1. Solve the following equations, giving your answers correct to 2 decimal places where necessary:

(a) $(x + 1)^2 = 9$ **(b)** $(2x + 1)^2 = 16$
(c) $(3x + 2)^2 = 49$ **(d)** $\left(2x + \frac{3}{4}\right)^2 = \frac{49}{25}$
(e) $(5x - 4)^2 = 81$ **(f)** $(3x + 7)^2 = \frac{25}{49}$
(g) $(x - 4)^2 = 17$ **(h)** $(x + 3)^2 = 11$
(i) $(2x - 3)^2 = 23$ **(j)** $(3x + 2)^2 = 43$
(k) $(5x - 7)^2 = 74$ **(l)** $(3 + 7x)^2 = 65$

For Your Information

For coefficients of x^2 that are greater than 1, divide the equation throughout by that particular x^2-coefficient, so that, ultimately, the coefficient of x^2 equals 1.

2. What must be added to each of the following expressions to obtain a perfect square?

(a) $x^2 + 7x$ **(b)** $x^2 - 3x$ **(c)** $x^2 + \frac{7}{2}x$
(d) $x^2 - 1.8x$ **(e)** $a^2 + 2.4a$ **(f)** $c^2 + 4\frac{2}{3}c$
(g) $y^2 - \frac{4}{5}y$ **(h)** $v^2 - 3\frac{1}{2}v$ **(i)** $b^2 - 10kb$
(j) $g^2 - 5kg$ **(k)** $h^2 + 3mh$ **(l)** $k^2 - 1\frac{1}{3}k$
(m) $d^2 + 10xd$ **(n)** $k^2 - 5xk$ **(o)** $m^2 - 5n^2m$

The following examples demonstrate how quadratic equations are solved using completing the square method.

Example 5

Solve $x^2 + 8x + 9 = 0$, giving your answer correct to 3 significant figures.

Solution

$x^2 + 8x + 9 = 0$
i.e. $x^2 + 8x = -9$
$x^2 + 8x + (4)^2 = -9 + (4)^2$
$(x + 4)^2 = 7$
$\therefore$ $(x + 4) = +\sqrt{7}$ or $(x + 4) = -\sqrt{7}$
$x = -4 + \sqrt{7}$ or $x = -4 - \sqrt{7}$
$x = -1.35$ or $x = -6.65$ (correct to 3 sig. fig.)

Notes

*The result would be that the LHS of the equation now becomes a **perfect square**.*

Example 6

Solve $3x^2 - 4x - 6 = 0$, giving your answer correct to 2 decimal places.

*Add $\left(-\frac{2}{3}\right)^2$ to both sides of the equation, making use of '**completing the square**' method.*

Solution

$$3x^2 - 4x - 6 = 0$$
$$x^2 - \frac{4}{3}x - \frac{6}{3} = 0$$
$$x^2 - \frac{4}{3}x = 2$$
$$x^2 - \frac{4}{3}x + \left(-\frac{2}{3}\right)^2 = 2 + \left(-\frac{2}{3}\right)^2$$
$$\left(x - \frac{2}{3}\right)^2 = 2\frac{4}{9}$$
$$\therefore \quad x - \frac{2}{3} = +\sqrt{\frac{22}{9}} \quad \text{or} \quad x - \frac{2}{3} = -\sqrt{\frac{22}{9}}$$
$$x = 2.23 \quad \text{or} \quad x = -0.90 \quad \text{(correct to 2 dec. places)}$$

Example 7

Solve $2x^2 - 7x + 3 = 0$.

Solution

Divide the equation throughout by 2.

Add $\left(-\frac{7}{4}\right)^2$ to both sides of the equation.

$$2x^2 - 7x + 3 = 0$$
$$x^2 - \frac{7}{2}x + \frac{3}{2} = 0$$
$$x^2 - \frac{7}{2}x = -\frac{3}{2}$$
$$x^2 - \frac{7}{2}x + \left(-\frac{7}{4}\right)^2 = -\frac{3}{2} + \left(-\frac{7}{4}\right)^2$$
$$\left(x - \frac{7}{4}\right)^2 = \frac{25}{16}$$
$$\therefore \quad x - \frac{7}{4} = +\sqrt{\frac{25}{16}} \quad \text{or} \quad x - \frac{7}{4} = -\sqrt{\frac{25}{16}}$$
$$x = \frac{7}{4} + \frac{5}{4} \quad \text{or} \quad x = \frac{7}{4} - \frac{5}{4}$$
$$x = 3 \quad \text{or} \quad x = \frac{1}{2}$$

NB: This equation can be solved more quickly by factorisation:

$$2x^2 - 7x + 3 = 0$$
$$(2x - 1)(x - 3) = 0$$
$$\therefore \quad x = \frac{1}{2} \quad \text{or} \quad x = 3$$

Example 7 above shows that the easier method to solve a given quadratic equation is by factorisation. The method of completing the square is to be used only when factorisation does not appear to be a better method.

Example 8

Solve $x^2 + 4x + 8 = 0$.

Solution

$$x^2 + 4x + 8 = 0$$
$$x^2 + 4x = -8$$
$$x^2 + 4x + (2)^2 = -8 + (2)^2$$
$$(x + 2)^2 = -4$$
$$\therefore \quad x + 2 = +\sqrt{-4} \quad \text{or} \quad x + 2 = -\sqrt{-4}$$

There is no real number whose square is -4. The square root of a negative number is an **imaginary number**. Hence, $\sqrt{-4}$ is said to be an imaginary number.

Thus, $x^2 + 4x + 8 = 0$ has no real roots. In other words, the roots of $x^2 + 4x + 8 = 0$ are **complex**. Graphically, the curve $y = x^2 + 4x + 8$ does not cut the x-axis (see Fig. 1.1).

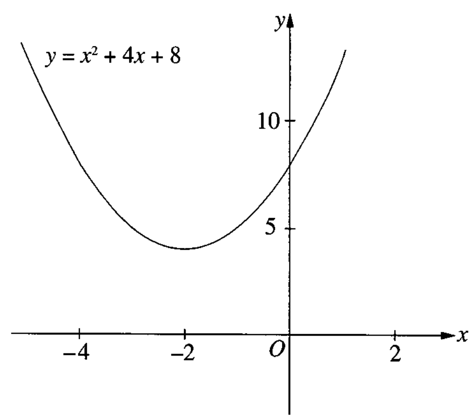


Fig. 1.1

For every point on the number line, there is a corresponding number. These numbers are known as ***real numbers****.*

$\sqrt{-4}$, an imaginary number, does not belong to the real number system. In fact, a real number is a complex number when the imaginary part is zero. However, a complex number is not necessarily a real number.

Exercise 1c

Solve the following equations by factorisation where possible or by completing the square. If the answers involve decimal places, give them correct to 2 decimal places. If an equation has no real roots, indicate that this is so.

(a) $x^2 + 2x + 3 = 0$
(b) $5x^2 - 4x - 2 = 0$
(c) $2x^2 + 7x + 2 = 0$
(d) $4x^2 = 5x - 21$
(e) $2x^2 + 5x - 3 = 0$
(f) $3x^2 + 8x + 2 = 0$
(g) $7x^2 - 28x + 15 = 0$
(h) $5x^2 + 12x + 3 = 0$
(i) $2x^2 + 3x - 4 = 0$
(j) $x^2 - 12x + 36 = 0$
(k) $5x^2 + 30x - 18 = 0$
(l) $3x^2 - 4x + 7 = 0$
(m) $3x^2 + x - 2 = 0$
(n) $x^2 - 11x - 26 = 0$
(o) $3x^2 + 5x - 2 = 0$
(p) $2x - 3x^2 - 4 = 0$
(q) $x^2 - 7x - 30 = 0$
(r) $(2x + 3)(x - 2) - x(x + 1) = 0$
(s) $x^2 - 6x - 16 = 0$
(t) $2x^2 - x - 3\frac{1}{2} = 0$
(u) $x^2 - 16x - 10 = 0$
(v) $(2x + 1)(5x - 4) - (3x - 2)^2 = 0$
(w) $x^2 - 2x - 5 = 0$
(x) $5x^2 - 8x - 30 = 0$
(y) $4x(3x - 1) - 2 = (2x - 1)(5x + 1)$
(z) $5x^2 - 16x + 2 = 0$

General Solution to a Quadratic Equation

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$.

Now, we shall use the method of completing the square to derive a formula for the solution to all quadratic equations.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Notes: Divide both sides of the equation by ***a*** *so that the coefficient of x^2 on the LHS is equal to 1.*

Hence, if $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 9

Solve $4x^2 + 3x - 2 = 0$, giving your answer correct to 3 decimal places.

Solution

Here, $a = 4$, $b = 3$ and $c = -2$.

$$\therefore \quad x = \frac{-3 \pm \sqrt{3^2 - 4(4)(-2)}}{2 \times 4} = \frac{-3 \pm \sqrt{41}}{8}$$

$$= 0.425 \quad \text{or} \quad -1.175 \quad \text{(correct to 3 dec. places)}$$

Notes: The graph of a quadratic function whose equation has ***repeated roots*** *touches the x-axis at exactly 1 point.*

The x-coordinate of the point of intersection is the root of the equation.

Example 10

Solve $9x^2 - 42x + 49 = 0$.

Solution

Here, $a = 9$, $b = -42$ and $c = 49$.

$$x = \frac{-(-42) \pm \sqrt{(-42)^2 - 4(9)(49)}}{2 \times 9} = \frac{42 \pm \sqrt{1\,764 - 1\,764}}{18} = \frac{42}{18} = 2\frac{1}{3}$$

$\therefore$ the equation has the repeated roots of $2\frac{1}{3}$.

Example 11

Solve $5x^2 - 10x + 13 = 0$.

Solution

Here, $a = 5$, $b = -10$ and $c = 13$.

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(5)(13)}}{2 \times 5}$$

$$= \frac{10 \pm \sqrt{100 - 260}}{10}$$

$$= \frac{10 \pm \sqrt{-160}}{10}$$

$\sqrt{-160}$ is an **imaginary** number. Thus, there are no real roots for $5x^2 - 10x + 13 = 0$.

From the examples given earlier, we can conclude that the nature of the roots of a quadratic equation depends on $b^2 - 4ac$.

If $b^2 - 4ac$ is **positive**, then the equation has two real and **distinct** roots (see Example 9).

If $b^2 - 4ac$ is **zero**, then the equation has **equal** roots, i.e. the roots are real and repeated (see Example 10).

If $b^2 - 4ac$ is **negative**, then the equation has no real roots, i.e. the roots are **complex** (see Example 11).

Since $b^2 - 4ac$ discriminates the nature of the roots of a quadratic equation, it is known aptly as the **discriminant** of the equation.

Example 12

Solve the equation $\frac{x-5}{2x} = \frac{x-4}{3}$.

Solution

$$\frac{x-5}{2x} = \frac{x-4}{3}$$

$$3(x-5) = 2x(x-4)$$

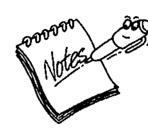
$$3x - 15 = 2x^2 - 8x$$

$$2x^2 - 11x + 15 = 0$$

Here, $a = 2$, $b = -11$ and $c = 15$.

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(2)(15)}}{2 \times 2} = \frac{11 \pm \sqrt{121 - 120}}{4}$$

$$\therefore \quad x = \frac{12}{4} = 3 \quad \text{or} \quad x = \frac{10}{4} = 2.5$$



Multiply the equation throughout by a common denominator. In this case, it would be 3(2x).

The above equation can also be solved by simple factorisation:

$$2x^2 - 11x + 15 = 0$$
$$(2x - 5)(x - 3) = 0$$
$$\therefore \quad x = 2\frac{1}{2} \quad \text{or} \quad x = 3$$

Check: Substitute the answers $x = 3$ and $x = 2.5$ back into the original equation. Are both answers acceptable?

Exercise 1d

Use any method to solve the following equations, giving your answers correct to 2 decimal places where necessary. If an equation has no real roots, state this is so.

1. $x^2 - 5x + 7 = 0$
2. $(3x - 4) = (4x - 3)^2$
3. $3x^2 - 5x - 2 = 0$
4. $2x^2 + 2x = 1$
5. $3x^2 + 8x - 2 = 0$
6. $2x^2 - 8x + 5 = 0$
7. $10x^2 - 12x = 15$
8. $x^2 + 5x - 24 = 0$
9. $(x - 1)(x + 3) - 12 = 0$
10. $(x - 1)(x - 1) = 12$
11. $x(2x + 7) - 3(2x + 7) = 0$
12. $3(x - 2)^2 = x(x - 2)$
13. $3(3x - 2)^2 = -2(3x - 2)$
14. $4(x + 3)^2 = 25(x - 2)^2$
15. $(4x - 3)^2 + (4x + 3)^2 = 26$
16. $(x - 1)^2 - 2x = 0$
17. $(x + 1)(x - 1) = 20$
18. $2(x^2 + 5) = 5(x + 2)$
19. $\frac{x+1}{x+4} = \frac{2x-1}{x+6}$
20. $\frac{x}{6} + \frac{6}{x} = \frac{4}{x} + \frac{x}{4}$
21. $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{10}{3}$
22. $\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} = 0$
23. $(x + 4)(x - 1) + (x + 5)(x + 2) = 6$
24. $(2x + 4)^2 - (4x - 6)^2 = (3x - 1)^2$
25. $(x + 5)^2 + (2x - 1)^2 - 67 = (x + 5)(2x - 1)$

For Your Information

In the graph of $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$,
(a) if $a > 0$, the parabola has a lowest point in terms of the y-axis and is curved upwards like ∪.
(b) if $a < 0$, the parabola has a highest point and is curved downwards like ∩.

Problems Involving Quadratic Equations

We can use the technique of solving quadratic equations to solve many problems that eventually deal with quadratic equations.

Example 13

The product of two consecutive positive odd numbers is 195. Find the numbers.

Solution

Method 1: **Using an equation**

Let x be one number and $x + 2$ the other number.

$$x(x + 2) = 195$$
$$x^2 + 2x - 195 = 0$$
$$(x + 15)(x - 13) = 0$$
$$\therefore \quad x = -15 \quad \text{or} \quad x = 13$$

Since the numbers must be positive, $x = 13$ and $x + 2 = 15$.

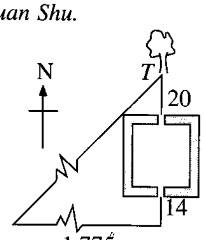
$\therefore$ the two consecutive positive odd numbers are 13 and 15.

Method 2: Listing all possible solutions

We can find the solution by listing the possible numbers: 5, 7, 9, 11, 13, 15, 17, 19, ...
Thus, 13 and 15 are the numbers ($13 \times 15 = 195$).
Which of these two methods do you think is more convenient?

Back in Time

Problems involving quadratic equations were found systematically recorded in "Jiu Zhang Suan Shu", a famous mathematical book in ancient China. The diagram below shows one of the problems exemplified in Jiu Zhang Suan Shu.



The figure above shows a square courtyard of unknown dimensions with two gates at the southern and northern walls. The gates are at the centre of the walls. By walking 20 steps out of the northern gate, one reaches a tree T. By walking 14 steps out of the southern gate and turning west to walk a further 1 775 steps, the tree T is as shown in the diagram. Find the dimensions of the courtyard in terms of the number of steps. Can you use the idea of quadratic equations to solve this problem?

Example 14

The manager of Reliable Car Rental paid a total of $1.12 million for a number of Certificates of Entitlement (COEs) at the onset of the Asian economic crisis in 1997. One year later he found that the successful bidding price of COE has gone down by $11 000 each and that he could get 12 more COEs by paying an extra $40 000 for them, compared to the previous year. Find the number of COEs the manager purchased in 1997 and the price of each COE then.

Solution

Let x be the number of COEs the manager purchased.

The price of one COE was $\$\frac{1\,120\,000}{x}$

The number of COEs the manager could purchase a year later was $(x + 12)$ and the price of each COE in that year would be $\$\frac{1\,160\,000}{x+12}$.

Now we have $\frac{1\,120\,000}{x} - \frac{1\,160\,000}{x+12} = 11\,000$

i.e. $1\,120\,000(x + 12) - 1\,160\,000x = 11\,000x(x + 12)$

$1\,120\,000x + 13\,440\,000 - 1\,160\,000x = 11\,000x^2 + 132\,000x$

Dividing both sides by 1 000 and rearranging the equation, we have

$$11x^2 + 132x + 1\,160x - 1\,120x - 13\,440 = 0$$
$$11x^2 + 172x - 13\,440 = 0$$
$$(x - 28)(11x + 480) = 0$$

$\therefore \quad x = 28 \quad$ or $\quad x = \frac{-480}{11}$, which is not applicable in this case.

Thus, the manager bought 28 COEs and the price of each COE was $\$\frac{1\,120\,000}{28}$, i.e. $40 000 each.

Check: Does $x = 28$ satisfy the original equation?

Example 15

A man travels a distance of 196 km by train and returns in a car which travels 21 km/h faster than the train. If the total journey takes 11 hours, find the speeds of the train and the car respectively.

Solution

Let the speed of the train be x km/h.

Then the speed of the car is $(x + 21)$ km/h.

Time taken for the train journey = $\frac{196}{x}$ h

Time taken for the car journey = $\frac{196}{x + 21}$ h

Because the journey takes 11 hours,

$$\frac{196}{x} + \frac{196}{x + 21} = 11$$

$$\frac{196(x + 21) + 196x}{x(x + 21)} = 11$$

$$196x + 4\,116 + 196x = 11x^2 + 231x$$

$$11x^2 - 161x - 4\,116 = 0$$

$$x = \frac{-(-161) \pm \sqrt{(-161)^2 - 4(11)(-4\,116)}}{2 \times 11} = \frac{161 \pm 455}{22}$$

$$x = 28 \quad \text{or} \quad x \approx -13.36$$

$\therefore$ the speed of the train is 28 km/h (negative answer is *inadmissible* in this case) and that of the car is $(28 + 21) = 49$ km/h.

In many cases, a quadratic equation can have both positive or negative roots. For quantitites like speeds, lengths or even ages, one should be careful to choose appropriate answers.

In general, you can use the following procedures to solve problems leading to quadratic equations:
- *(a) Use a variable to represent the unknown.*
- *(b) Form an equation with the given conditions.*
- *(c) Solve the equation.*
- *(d) Accept the relevant solution and reject the irrelevant one.*
- *(e) Check the solution against the question to test its validity.*

Exercise 1e

1. The product of two consecutive positive integers is 156. Find the numbers.
2. The product of two consecutive odd numbers is 255. Find the numbers.
3. The product of two consecutive positive even numbers is 528. Find the numbers.
4. The perimeter and area of a rectangle are 22 cm and 30 cm^2 respectively. Find the length and breadth of the rectangle.
5. Find two consecutive positive odd numbers given that the difference between their reciprocals is $\frac{2}{63}$.
6. The altitude of a triangle is 6 cm greater than its base. If its area is 108 cm^2, find its base.
7. A wire of length 100 cm is cut into two parts and each part is bent to form a square. If the sum of the areas of the squares is 425 cm^2, find the lengths of the sides of the two squares.
8. A wire of length 200 cm is cut into two parts and each part is bent to form a square. If the area of the larger square is 9 times that of the smaller square, find the perimeter of the larger square.

9. A boat travels 12 km upstream and back in 1 hour 45 minutes. If the speed of the current is 3 km/h throughout, find the speed of the boat in still water, giving your answer correct to 3 significant figures.

10. The sides of a right-angled triangle are x cm, $(x + 1)$ cm and $(x + 3)$ cm. Show that $x^2 - 4x - 8 = 0$ and solve this equation to find the lengths of the sides of the triangle. Give your answers correct to two decimal places.

11. A group of students are on a sightseeing tour. The total fare is $120 and this is to be shared equally among the students. If two more students join the tour, each will pay $2 less. Find the original number of students in the group.

12. In May 1994, Mr Chauhan changed 1 140 Indian Rupees into German Marks when the rate of exchange was x Rupees = 1 Mark.

(a) Write down an expression, in terms of x, for the number of Marks he received.

In July, Mr Chauhan again changed 1 140 Rupees into Marks. The rate of exchange was then $(x + 1)$ Rupees = 1 Mark.

(b) Write down an expression, in terms of x, for the number of Marks he received this time.

(c) Given that he received 3 Marks less in July than he did in May, form an equation in x and show that it reduces to $x^2 + x - 380 = 0$.

(d) Solve this equation to find the rate of exchange in May 1994. *(C)*

13. A light aircraft flies from A to B, 450 km away, and back from B to A in a total time of $5\frac{1}{2}$ hours. Suppose that during the whole journey there is a constant wind blowing from A to B. If the speed of the aircraft in still air is 165 km/h, find the speed of the wind.

14. A car travels at a constant speed from Q to R, a distance of 300 km apart. If the driver increases the speed by 5 km/h, the journey will take 2 hours less. Find the original speed of the car.

15. A motorist makes a journey from Singapore to Malacca to visit his in-laws, a journey of 240 km, at an average speed of x km/h. Write down an expression for the number of hours taken for the journey.

On his return journey from Malacca to Singapore, his average speed is reduced by 6 km/h due to heavy traffic flow at the Causeway. Write down an expression for the time taken for the return journey.

If the return journey takes 20 minutes longer, form an equation in x and solve it to find the average speed for each journey, giving your answer correct to two decimal places.

16. In 1998, the members of a club paid a total of $42 in subscription fees. In 1999, the membership increased by 20 and the subscription fees were subsequently reduced by 10 cents per member. The total subscription fees for 1999 was $45. How many members were there in 1998?

17. The numerator of a fraction is 2 less than its denominator. When both numerator and denominator are increased by 3, the fraction is increased by $\frac{3}{20}$. Find the original fraction.

18. The lengths of the sides of a right-angled triangle are $(3x - 1)$ cm, $5x$ cm and $(5x + 2)$ cm. Find the lengths of each side of the triangle and its area.

19. The length and breadth of a rectangle are $(x + 4)$ cm and x cm respectively. Write down expressions for

(a) the perimeter of the rectangle;

(b) the length of the side of a square with the same perimeter.

If the sum of the areas of the square and the rectangle is 94 cm^2, find x.

★20. There are two rectangular plots of land, L_1 and L_2. The difference between the length and width of L_1 is 3 m. Also, the length of L_1 is 8 m more than that of L_2; the width of L_1 is 5 m less than that of L_2 and the area of L_2 is 250 m^2 more than half that of L_1. Find the lengths and widths of L_1 and L_2.

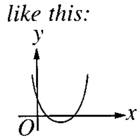
★21. A rectangular tank can be filled with water by two pipes in $11\frac{1}{9}$ minutes. If the larger pipe alone takes 5 minutes less to fill the tank than the smaller pipe, find the time each pipe takes to fill the tank.

★22. There is a two-digit number such that the sum of its digits is 6 while the product of the digits is $\frac{1}{3}$ the original number. Find this number.
(*Hint:* Let x be one of the digits.)

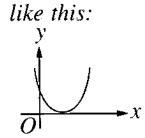
For Your Information

(a) Given a function (or a curve) $y = ax^2 + bx + c$ and $a > 0$,

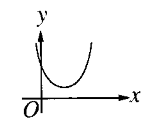
(i) when $b^2 - 4ac > 0$, the graph of the function looks like this:



(ii) when $b^2 - 4ac = 0$, the graph of the function looks like this:

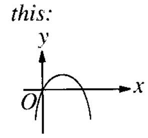


(iii) when $b^2 - 4ac < 0$, the graph of the function looks like this:

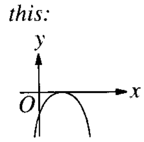


(b) Given a function $y = ax^2 + bx + c$ and $a < 0$,

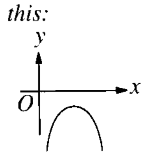
(i) when $b^2 - 4ac > 0$, the graph of the function looks like this:



(ii) when $b^2 - 4ac = 0$, the graph of the function looks like this:



(iii) when $b^2 - 4ac < 0$, the graph of the function looks like this:



Summary

There are three common ways to solve quadratic equations:

(i) factorisation
(ii) completing the square
(iii) general formula

The general formula can be used to solve all quadratic equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$. The solution of the equation is then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When $b^2 - 4ac > 0$, the equation has two real and distinct roots. When $b^2 - 4ac = 0$, the equation has two real and equal roots. And when $b^2 - 4ac < 0$, the equation has complex roots (i.e. no real roots).

Review Questions 1

1. Solve the following quadratic equations, giving your answers correct to 3 significant figures where necessary. If an equation has no real roots, state this is so.

(a) $5x^2 + 15x + 1 = 0$ **(b)** $2a^2 + 3a - 2 = 0$
(c) $176 - 3x - 35x^2 = 0$ **(d)** $x^2 - 7x + 15 = 0$
(e) $7x^2 + 7x + 10 = 0$ **(f)** $3x^2 - 2x - 4 = 0$
(g) $2x^2 + 7x + 1 = 0$ **(h)** $5x^2 + 5x + 2 = 0$
(i) $4x^2 - 3x + 3 = 0$
(j) $(3x + 2)(4x - 3) - 10x(x + 1) = 0$
(k) $(2x - 3)(3x + 1) - 2x^2 + 22x = 0$
(l) $\frac{x}{x+3} + \frac{2x}{x-3} - \frac{45}{4(x^2-9)} = 0$
(m) $\frac{3(x^2-11)}{5} - \frac{2(x^2-60)}{7} = 36$

2. It takes Meihua two hours more to complete a 50-km journey than it takes Ailin to complete a 40-km journey. If the average speed of Meihua for the journey is 5 km/h less than Ailin, calculate the average speed of each girl.

∗3. A number of people were to share equally the expenses of \$900 for a bus trip to Johor. Due to unforeseen circumstances, five of them were unable to make it and the remaining people had to pay an extra \$2.50 each. How many people were supposed to have shared the expenses originally?

∗4. A rectangle has an area of 288 cm². If the width is decreased by 1 cm and the length increased by 1 cm, the area would decrease by 3 cm². Find the original dimensions of the rectangle.

5. **(a)** A shop sells two types of radio, Hiblast and Megadef. At a sale, all prices are reduced by 15%.

(i) A Hiblast radio usually sells for \$41. Find its reduced price.

(ii) A Megadef radio costs \$51 at the sale. Find its usual price.

(b) **(i)** Harry assembles Hiblasts and produces one every x minutes. Write down an expression, in terms of x, for the number he produces in an hour.

(ii) Marion assembles Megadefs and takes two minutes longer than Harry to produce each one. Write down an expression, in terms of x, for the number of Megadefs she produces in an hour.

(iii) Both Harry and Marion produce a total of 11 radios in an hour together. Form an equation and show that it reduces to $11x^2 - 98x - 120 = 0$.

(iv) Solve this equation and hence find how many Megadefs Marion produces in an hour.

(C)

Problem Solving

Example 16

Mr Gan borrowed a weighing scale in order to measure the weights of 5 boys. The scale, however, can only measure weights of more than 80 kg. Each boy weighs less than 80 kg, so Mr Gan arranged in such a way that only two boys be weighed at a time and that each boy be weighed together with one of the other 4 boys only once, thus making a total of 10 weighings. The following are the results of the 10 readings in kg:

86, 92, 96, 100, 104, 109, 110, 113, 119, 127.

What are the individual weights of the 5 boys?

Solution

Method: Use equations and logical reasoning

Let the weight, in kg, of each of the 5 boys be a, b, c, d and e, in ascending order (i.e., a is the lightest and e the heaviest.)

Obviously, the 2 lightest boys together weigh 86 kg and the 2 heaviest boys together weigh 127 kg.

i.e. $a + b = 86$ ——— (1)

and $d + e = 127$ ——— (2)

Similarly, we can reason out that $a + c = 92$ ——— (3)

and that $c + e = 119$ ——— (4)

When we add up the 10 weights, we get

$(86 + 92 + 96 + 100 + 104 + 109 + 110 + 113 + 119 + 127)$ kg
$= 1\ 056$ kg

Since each boy is weighed a total of 4 times, the average total weight of the 5 boys is

$\left(\frac{1\ 056 \text{ kg}}{4}\right) = 264$ kg, i.e., $a + b + c + d + e = 264$ ——— (5)

Substituting equations (1) and (2) into equation (5), we have

$$86 + c + 127 = 264$$
$$\therefore \quad c = 51$$

Knowing c, we can solve equations (3) and (4), giving $a = 41$ and $e = 68$ respectively.

Knowing a and e, we can solve equations (1) and (2), giving $b = 45$ and $d = 59$ respectively.

Thus, the weights of the 5 boys are 41 kg, 45 kg, 51 kg, 59 kg and 68 kg.

Exploration

1. Given that the roots of the equation $2x^2 - 6x - k = 0$ differ by 5, find the value of k.

2. Find the value of k for which the equation $x^2 + 4x + (1 + k)^2 = 2kx$ has equal roots.

3. The roots of the quadratic equation $3x^2 + bx + c = 0$ are a and $\frac{1}{a}$. Find the value of c. If given further that $a^2 + \frac{1}{a^2} = 7$, find the possible values of b.

Problems

1. *Miss Lim intends to measure the weights of 5 children by using a large scale. The scale, however, can only measure weights of more than 100 kg at a time. As none of the children weighs more than 50 kg, she has to arrange for three children to weigh in at a time. To accomplish the task, she can make a total of 10 weighings with groups of three different children every time. The following 10 readings, in kg, were recorded:*

   109, 110, 114, 117, 118, 122, 123, 125, 126, 130

   What is the weight of each child?

2. *Four boys A, B, C and D have different weights. The combined weight of A and B is greater than that of C and D. The combined weight of B and D is less than that of A and C. D is heavier than C. Arrange the weights of the four boys in order, from the lightest to the heaviest.*

3. *Four coloured balls (white, black, green and red) are of different weights. Three of them are weighed together at each time and their combined weights are given below:*

4. Solve the equation $15 \sin^2 x + 4 = 17 \sin x$ for values of x between 0° and 90°.

5. Solve the equation $3 \tan x + \frac{4}{\tan x} = 13$ for values of x between 0° and 90°.

6. The line $y = 2x + k$ is a tangent to the curve $x^2 + y^2 = 5$. Find the possible values of k. (*Hint:* the tangent meets the curve at a point.)

7. At a party everybody shakes hand with one another only once. How many handshakes will there be if the number of people at the party is

(a) 2 **(b)** 3 **(c)** 4 **(d)** 5
(e) 6 **(f)** 7 **(g)** 8 **(h)** n?

8. Study the number pattern below

$$1 + 2 = 3 = \frac{2}{2} \times 3$$

$$1 + 2 + 3 = 6 = \frac{3}{2} \times 4$$

$$1 + 2 + 3 + 4 = 10 = \frac{4}{2} \times 5$$

$$1 + 2 + 3 + 4 + 5 = 15 = \frac{5}{2} \times 6$$

⋮ ⋮

(a) What is the value of

(i) $1 + 2 + 3 + \ldots + 99$; **(ii)** $1 + 2 + 3 + \ldots + n$?

(b) If $1 + 2 + 3 + \ldots + (x + 1) = 125\ 250$, find x.

Weight of (black + white + red) balls = 60 g.
Weight of (black + green + red) balls = 80 g.
Weight of (white + black + green) balls = 70 g.
Weight of (white + green + red) balls = 90 g.
What is the weight of each coloured ball?

CHAPTER 2

Indices and Algebraic Manipulation

In this chapter, you will learn

- △ about the laws of indices;
- △ to manipulate formulae;
- △ how to simplify algebraic expressions;
- △ to solve equations involving algebraic expressions.

Preliminary Problem

According to figures released by the Inland Revenue Authority of Singapore (IRAS), a total of 559 000 Singaporeans paid approximately $\$2.62 \times 10^9$ in income taxes in the year 1997. If this amount were to be paid in 5-cent coins, which has a mass of 1.56 g each, how many such coins will be needed? How heavy will the coins weigh? Big numbers are usually dealt with using indices.

In Book 1, we used 5^4 to represent $5 \times 5 \times 5 \times 5$. The digit 5 in 5^4 is known as the **base** and the digit 4, which indicates the number of equal factors, is known as the **power** or **index** (plural: **indices**). In general, a^n means the n^{th} power of a (read as 'a to the power of n').

There are a few important laws on indices and we shall investigate them by considering a^n where a is a real number ($a \neq 0$) and n is any integer.

Notes

a^n ← *index*, a ← *base*

A factor is one of the numbers that make up a number or expression by multiplication.

Example 1

Simplify (a) $a^2 \times a^3$; (b) $a^5 \times a^2$.

Solution

(a) $a^2 \times a^3$

$= (a \times a) \times (a \times a \times a)$

2 factors + 3 factors = 5 factors

$= a^5$

(b) $a^5 \times a^2$.

$= (a \times a \times a \times a \times a) \times (a \times a)$

5 factors + 2 factors = 7 factors

$= a^7$

In general, $a^m \times a^n = (a \times a \times \ldots \times a) \times (a \times a \times \ldots \times a)$

m factors + n factors = $(m + n)$ factors

$= a^{m+n}$

Example 2

Simplify (a) $a^5 \div a^2$; (b) $a^7 \div a^3$.

Solution

(a) $a^5 \div a^2$

$= \dfrac{a \times a \times a \times a \times a}{a \times a}$ $\quad \dfrac{\text{5 factors}}{\text{2 factors}}$

$= a \times a \times a$ $\quad (5 - 2) = 3$ factors

$= a^3$

(b) $a^7 \div a^3$

$= \dfrac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a}$ $\quad \dfrac{\text{7 factors}}{\text{3 factors}}$

$= a \times a \times a \times a$ $\quad (7 - 3) = 4$ factors

$= a^4$

In general, $a^m \div a^n = \dfrac{\overbrace{a \times a \times a \times \ldots \times a}^{m \text{ times}}}{\underbrace{a \times a \times a \times \ldots \times a}_{n \text{ times}}}$ $\quad \dfrac{m \text{ factors}}{n \text{ factors}}$

$= a \times a \times a \ldots \times a$ $\quad (m - n)$ factors

$= a^{m-n}$ where $m > n$

Example 3

Simplify (a) $(a^3)^2$; (b) $(a^2)^4$.

Solution

(a) $(a^3)^2$

$= \underbrace{(a \times a \times a)}_{3 \text{ factors}} \times \underbrace{(a \times a \times a)}_{3 \text{ factors}}$ — 3×2 factors

$= a^{3 \times 2}$

$= a^6$

(b) $(a^2)^4$

$= \underbrace{(a \times a)}_{2 \text{ factors}} \times \underbrace{(a \times a)}_{2 \text{ factors}} \times \underbrace{(a \times a)}_{2 \text{ factors}} \times \underbrace{(a \times a)}_{2 \text{ factors}}$ — 2×4 factors

$= a^{2 \times 4}$

$= a^8$

In general, $(a^m)^n = a^m \times a^m \times a^m \times \ldots \times a^m$ to n factors

$\underbrace{m \text{ factors} + m \text{ factors} + \ldots + m \text{ factors}}_{(m \times n) \text{ factors}}$

$= a^{m \times n}$

$= a^{mn}$

Example 4

Which is bigger, 2^{140} *or* 3^{105}*?*

Express each of the following as a power of a single number.

(a) $2^3 \times 3^3$ (b) $9^2 \times 8^2$

Solution

(a) $2^3 \times 3^3$

$= 2 \times 2 \times 2 \times 3 \times 3 \times 3$

$= (2 \times 3) \times (2 \times 3) \times (2 \times 3)$

$= (2 \times 3)^3$

$= 6^3$

(b) $9^2 \times 8^2$

$= 9 \times 9 \times 8 \times 8$

$= (9 \times 8) \times (9 \times 8)$

$= (9 \times 8)^2$

$= 72^2$

In general, $a^m \times b^m = \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \times \underbrace{(b \times b \times \ldots \times b)}_{m \text{ factors}}$

$= \underbrace{(a \times b) \times (a \times b) \times \ldots \times (a \times b)}_{m \text{ factors}}$

$= (a \times b)^m$

Example 5

Express each of the following as a power of a single number.

(a) $3^2 \div 5^2$ *(b)* $4^3 \div 7^3$

Solution

(a) $3^2 \div 5^2 = \dfrac{3 \times 3}{5 \times 5}$

$= \left(\dfrac{3}{5}\right) \times \left(\dfrac{3}{5}\right)$

$= \left(\dfrac{3}{5}\right)^2$

(b) $4^3 \div 7^3 = \dfrac{4 \times 4 \times 4}{7 \times 7 \times 7}$

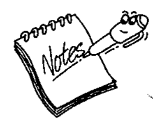
$= \left(\dfrac{4}{7}\right) \times \left(\dfrac{4}{7}\right) \times \left(\dfrac{4}{7}\right)$

$= \left(\dfrac{4}{7}\right)^3$

In general, $a^m \div b^m = \dfrac{a \times a \times \ldots \times a}{b \times b \times \ldots \times b}$ $\dfrac{m \text{ factors}}{m \text{ factors}}$

$= \left(\dfrac{a}{b}\right) \times \left(\dfrac{a}{b}\right) \times \ldots \times \left(\dfrac{a}{b}\right)$ m factors

$= \left(\dfrac{a}{b}\right)^m$



*The **index** is also known as the **exponent**.*

Below is a summary of the Laws of Indices:

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $a^m \times b^m = (a \times b)^m$
5. $a^m \div b^m = \left(\dfrac{a}{b}\right)^m$

where m and n are positive integers, $m > n$, $a \neq 0$, $b \neq 0$.

Example 6

Simplify the following:

(a) $(5^3 \times 7^4) \times (7^5 \times 5^2)$ *(b)* $(3^5 \times 8^4) \div (8^3 \times 3^2)$ *(c)* $(2a^2b^3) \times (3ab^2)$
(d) $(4a^3b)^3$ *(e)* $a^2 \times b^2 \times c^2$ *(f)* $a^7 \div b^7 \times c^7$
(g) $(-2xy^2)^2$ *(h)* $(-3x^2y)^5 \div (-9x^3y^2)^2$

Solution

(a) $(5^3 \times 7^4) \times (7^5 \times 5^2)$
$= 5^3 \times 5^2 \times 7^4 \times 7^5$
$= 5^{3+2} \times 7^{4+5}$
$= 5^5 \times 7^9$

(b) $(3^5 \times 8^4) \div (8^3 \times 3^2)$
$= 3^{5-2} \times 8^{4-3}$
$= 3^3 \times 8$

(c) $(2a^2b^3) \times (3ab^2)$
$= (2 \times 3) \times a^{2+1} \times b^{3+2}$
$= 6a^3b^5$

(d) $(4a^3b)^3$
$= 4^3(a^3)^3(b)^3$
$= 64a^9b^3$

(e) $a^2 \times b^2 \times c^2$
$= (a \times b)^2 \times c^2$
$= (a \times b \times c)^2$
$= (abc)^2$

(f) $a^7 \div b^7 \times c^7$
$= \left(\frac{a}{b}\right)^7 \times c^7$
$= \left(\frac{a}{b} \times c\right)^7$
$= \left(\frac{ac}{b}\right)^7$

(g) $(-2xy^2)^2$
$= (-2)^2x^2(y^2)^2$
$= 4x^2y^4$

(h) $(-3x^2y)^5 \div (-9x^3y^2)^2$
$= \frac{(-3)^5(x^2)^5y^5}{(-9)^2(x^3)^2(y^2)^2}$
$= \frac{-3^5x^{10}y^5}{9^2x^6y^4}$
$= \frac{-3^5}{(3^2)^2}x^{10-6}y^{5-4}$
$= -3x^4y$

Exercise 2a

1. Simplify the following, giving your answers in index form:

(a) $3^5 \times 3^7$ **(b)** $7^{10} \times 7^{15}$
(c) $8^9 \div 8^3$ **(d)** $15^6 \div 15^2$
(e) $(9^2)^3$ **(f)** $(4^6)^5$
(g) $m^4 \times m^3$ **(h)** $c^3 \times c^5$
(i) $y^5 \times y^2$ **(j)** $b^5 \div b^3$
(k) $x^7 \div x^4$ **(l)** $n^6 \div n$
(m) $(c^3)^4$ **(n)** $(p^4)^4$
(o) $(x^7)^2$ **(p)** $a^5 \times a^7 \times a^4$
(q) $a^3 \times a^6 \div a^2$ **(r)** $(a^2 \times a^4)^3$
(s) $(x^7 \div x^3)^3$ **(t)** $(k^5 \times k^2)^7$
(u) $(h^9 \div h^3)^{12}$

2. Simplify the following, giving your answers in index form:

(a) $5^2 \times 3^7 \times 5^7 \times 3^6$
(b) $6^7 \times 19^3 \times 6^{14} \times 19^6$
(c) $7^9 \times 5^6 \times 5^9 \times 7^3$
(d) $\frac{7^8 \times 7^3}{5^6\ 5^4}$ **(e)** $\frac{3^{14}}{2^7 \times 3^4 \times 2^5}$
(f) $\frac{11^{15}}{7^9 \times 11^3 \times 7^3}$ **(g)** $\frac{5^7 \times 3^9}{5^{11} \times 3^4 \div 5^9}$
(h) $\frac{13^9 \div 13}{7^{10} \times 13 \div 7^3}$ **(i)** $a^5b^6 \times a^3b^2$
(j) $x^7y^4 \times x^6y^9$ **(k)** $m^{12}n^7 \div m^4n^3$
(l) $h^{14}k^{12} \div h^5k^5$

3. Simplify the following:

(a) $(2x^4)^2$ **(b)** $(3a^5)^3$
(c) $(5b^7)^2$ **(d)** $3a^2b \times 4a^3b^5$
(e) $2abc \times 3ab^2c$ **(f)** $7a^3b^2 \times 5a^4b^7$
(g) $(-7a^2)^2 \times (2a^3)^3$ **(h)** $(2x^4)^5 \times (-3x^5)^2$
(i) $(8x^4)^2 \times (2x^6)^3$ **(j)** $8a^5b^4 \div 2a^2b^3$
(k) $12x^7y^6 \div 4x^2y^2$ **(l)** $35a^3b^4 \div 7a^2b$
(m) $(-2ab)^3 \div 4a^2b$ **(n)** $(-12y^3)^4 \div (4x^2)^5$
(o) $(32ab^3)^2 \div 4ab^5$ **(p)** $(a^2b^4)^5 \div (2a^2b^3)^2$
(q) $(ab)^5 \div (2ab)^2$ **(r)** $(a+b)^2 \times (a+b)^3$
(s) $2(a-b)^9 \times (a-b)^6$
(t) $15(2a+b)^9 \div 3(2a+b)^3$
(u) $(6a^3b^2)^2 \div 4ab^3$
(v) $(5a^2b^2)(7ab^3)(2a^5b)$
(w) $(3a^2b^3)(2a^4b^2)^3$
(x) $(5ab^6)(8a^2b)^2 \div (4ab^2)^2$

4. Simplify the following:

(a) $\frac{(2x^3)^5}{3xy} \div \frac{x}{2y^2}$ **(b)** $\frac{12x^5}{x^2y} \div \frac{3y^2}{2xy}$
(c) $\frac{5b^3}{ab} \div \frac{2ab^3}{3b^2}$ **(d)** $\frac{2x^5}{y^2} \div \frac{4x^2}{x^3y}$
(e) $\frac{(2x^2y)^3}{(10xy^3)^2} \times \frac{(5xy^4)^3}{4xy}$
(f) $\frac{8x^8y^4}{(2xy^2)^2} \times \frac{(4x^2y^2)^2}{(3xy)^2}$
(g) $\frac{8a^2b^4}{(2ab)^2(4ab^2)}$ **(h)** $\frac{(2xy^2)^5}{(4x^2y)^2(xy^3)}$
(i) $\frac{(8a^5b^4)(3a^2b^5)^3}{(9ab^2)(2ab)^3}$ **(j)** $\frac{(2xy^3)^4(6x^2y)^2}{(4xy)(16x^3y^2)}$
(k) $\frac{(3x^2)^3(4y^2)^4(3z^3)^2}{(2x)^2(9y^2)^3(z)^4}$

Zero and Negative Indices

The laws for positive integral indices can be extended so that we can give meanings to zero and negative integral indices.

Example 7

Simplify (a) $5^3 \div 5^3$; (b) $8^2 \div 8^2$

Solution

(a) By definition, $5^3 \div 5^3 = \frac{5 \times 5 \times 5}{5 \times 5 \times 5} = 1$

But if we extend Law 2 to situations where $m = n$, then

$$5^3 \div 5^3 = 5^{3-3} = 5^0$$

Therefore it is natural to define $5^0 = 1$.

(b) By definition, $8^2 \div 8^2 = \frac{8 \times 8}{8 \times 8} = 1$

Again extending Law 2, we have

$$8^2 \div 8^2 = 8^{2-2} = 8^0$$

Therefore it is natural to define $8^0 = 1$.

In general, $a^4 \div a^4 = a^{4-4} = a^0$ (Law 2)

But, $a^4 \div a^4 = \frac{a \times a \times a \times a}{a \times a \times a \times a} = 1$

$\therefore$ we have

6. $a^0 = 1$, where a is any real number and $a \neq 0$.

Example 8

Simplify (a) $a^2 \div a^4$; (b) $a^4 \div a^7$.

Solution

(a) $a^2 \div a^4 = a^{2-4}$
$= a^{-2}$

Also, by definition

$$a^2 \div a^4 = \frac{a \times a}{a \times a \times a \times a} = \frac{1}{a^2}$$

Extending Law 2, we have

$$a^{-2} = \frac{1}{a^2}$$

(b) $a^4 \div a^7 = a^{4-7}$
$= a^{-3}$

Also, by definition

$$a^4 \div a^7 = \frac{a \times a \times a \times a}{a \times a \times a \times a \times a \times a \times a} = \frac{1}{a^3}$$

Extending Law 2, we have

$$a^{-3} = \frac{1}{a^3}$$

In general, we have

7. $a^{-n} = \dfrac{1}{a^n}$**, where** n **is a positive integer and** $a \neq 0$**.**

Example 9

Simplify the following, giving your answers in positive indices only.

(a) $a^2 \times a^{-5}$ (b) $a^{-2} \times a^{-5} \div a^2$ (c) $\dfrac{(a^2)^{-5} \times (ab)^4}{(a^{-1}b)^2}$

Solution

(a) $a^2 \times a^{-5} = a^{2+(-5)}$
$= a^{-3}$
$= \dfrac{1}{a^3}$

(b) $a^{-2} \times a^{-5} \div a^2 = a^{-2+(-5)-2}$
$= a^{-9}$
$= \dfrac{1}{a^9}$

(c) $\dfrac{(a^2)^{-5} \times (ab)^4}{(a^{-1}b)^2} = \dfrac{a^{-10} \times a^4b^4}{a^{-2}b^2}$
$= a^{-10+4-(-2)} \times b^{4-2}$
$= a^{-4}b^2$
$= \dfrac{b^2}{a^4}$

Example 10

Simplify the following, giving your answers in negative indices only.

(a) $\dfrac{a^3}{a^5}$ (b) $\dfrac{(a^2b)^3}{a^7b^5}$ (c) $\dfrac{4a^{-3}b^2}{(2ab^2)^3}$

Solution

(a) $\dfrac{a^3}{a^5} = a^{3-5}$
$= a^{-2}$

(b) $\dfrac{(a^2b)^3}{a^7b^5} = \dfrac{(a^2)^3b^3}{a^7b^5}$
$= \dfrac{a^6b^3}{a^7b^5}$
$= \dfrac{1}{ab^2}$
$= a^{-1}b^{-2}$

(c) $\dfrac{4a^{-3}b^2}{(2ab^2)^3} = \dfrac{4a^{-3}b^2}{8a^3b^6}$
$= \dfrac{1}{2}a^{-3-3}b^{2-6}$
$= \dfrac{1}{2}a^{-6}b^{-4}$

Just for fun

1. What is the smallest number you can form using only three odd digits?
2. What is the largest number you can form using only three digits?

Just for fun

Add suitable mathematical symbols +, −, ×, ÷ and √ to the following equations to make them correct.
1992 = 1
1992 = 2
1992 = 3
1992 = 4
1992 = 5
1992 = 6
1992 = 7
1992 = 8
1992 = 9
1992 = 10
1992 = 11
1992 = 12
(eg. −1 + 9 − 9 + 2 = 1)

Example 11

Evaluate each of the following:

(a) 2^{-3}　　*(b)* $2^3 \times 3^{-2}$　　*(c)* $\left(\frac{2}{3}\right)^{-2} \times \left(\frac{5}{4}\right)^0$　　*(d)* $(3^{-2})^3 \times (9^{-3})^{-2}$

Solution

(a) $2^{-3} = \frac{1}{2^3}$

$= \frac{1}{8}$

(b) $2^3 \times 3^{-2} = 2^3 \times \frac{1}{3^2}$

$= \frac{8}{9}$

(c) $\left(\frac{2}{3}\right)^{-2} \times \left(\frac{5}{4}\right)^0 = \frac{1}{\left(\frac{2}{3}\right)^2} \times 1$

$= \frac{1}{\frac{4}{9}}$

$= \frac{9}{4}$

$= 2\frac{1}{4}$

(d) $(3^{-2})^3 \times (9^{-3})^{-2} = 3^{-6} \times 9^6$

$= 3^{-6} \times (3^2)^6$

$= 3^{-6} \times 3^{12}$

$= 3^{-6+12}$

$= 3^6$

$= 729$

Note: $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$

In general, $\left(\frac{a}{b}\right)^{-c} = \left(\frac{b}{a}\right)^c$ where $a \neq 0$, and $b \neq 0$.

Exercise 2b

1. Simplify the following, giving your answers in positive indices only:

(a) $3^7 \times 3^{-12}$　**(b)** $5^4 \times 5^{-7}$
(c) $8^9 \div 8^{15}$　**(d)** $9^8 \div 9^{-4}$
(e) $6^0 \div 6^5$　**(f)** $7^{-4} \times 7^{-5}$
(g) $(5^{-4})^3$　**(h)** $(9^{-2})^{-4}$
(i) $a^2 \times a^{-7}$　**(j)** $a^4 \div a^7$
(k) $(a^2)^{-3}$　**(l)** $(a^{-2})^4 \div a^3$
(m) $2a^{-5} \div 7b^{-5}$
(n) $\frac{1}{a^3b^{-4}}$　**(o)** $\frac{5}{a^{-2}b^{-3}}$
(p) $\frac{1}{(a^3b^{-2})^2}$　**(q)** $\frac{3}{(a^{-4}b^3)^{-3}}$
(r) $\frac{(ab^3)^{-4}}{a^{-3}b^{-7}}$　**(s)** $\frac{(a^3)^{-5} \times (ab)^{17}}{(a^{-1}b)^2}$
(t) $\frac{(a^4)^3(a^{-1}b)^{10}}{a^2b^7}$　**(u)** $\frac{(x^3y)^3(2xy)^{-2}}{4x^{-4}y^{-5}}$
(v) $(a^{-2}b)^{-1} \times (b^{-3}a)^{-1}$
(w) $(2a^2)^3 \times (4ab^{-1})^2$
(x) $(a^{-4}b^{-1})^{-3} \div (b^2a^{-4})^2$
(y) $\frac{(a^2b^{-1})^2}{a^3b^{-5}}$　**(z)** $\frac{(ab^{-2})^{-2}}{(a^2b^{-3})^{-1}}$

2. Simplify the following, giving your answers in negative indices only:

(a) $6^4 \times 6^{-2}$　**(b)** $7^8 \div 7^4$
(c) $5^0 \times 5^{-4}$　**(d)** $9^6 \div 9^{-2}$
(e) $(3^{-2})^{-4}$　**(f)** $(8^{-4})^6$
(g) $a^2 \times b^3$　**(h)** $ab^2 \div ab^3$
(i) $(a^2b)^3$　**(j)** $ab \div ab^5$

(k) $a^2 \div a^5$ **(l)** $a^5 \div b^5$

(m) $abc \div a^5b^4c^2$ **(n)** $a^{-2} \div b^{-2}$

(o) $\frac{1}{a^5b^{-2}}$ **(p)** $\frac{1}{(ab^2)^3}$

(q) $\frac{a^3b^2}{(ab)^{-2}}$ **(r)** $\frac{(ab)^{-4}(ab)^5}{(ab^2)^2(abc^2)^{-3}}$

(s) $\frac{a^3b^{-4}}{ab^2}$ **(t)** $(a^3b^{-4})^{-1}$

(u) $\left(\frac{a}{b^{-4}}\right)^{-2}$ **(v)** $\frac{(a^{-1}b)^3}{a^4b^5}$

(w) $(2^{-3}a^4b)^{-1} \times (4^{-2}b^{-5})$

(x) $\left(\frac{-b^2}{a}\right)^2 \times a^{-4}$

(m) $3^2 \times 4^{-3}$ **(n)** $3^{-2} \times 2^{-1}$

(o) $\left(\frac{3}{4}\right)^{-1} \times \left(\frac{3}{7}\right)^0$ **(p)** $\left(\frac{2}{7}\right)^{-3} \times 49^{-1}$

(q) $\frac{2^3 \times 6^{-5}}{3^{-3} \times 4^{-4}}$ **(r)** $\frac{2^5 \times 9^{-2}}{27^{-3} \times 8^{-4}}$

(s) $2^5 \times 2^3 \div 4^4$ **(t)** $(3^2)^5 \div 9^3 \times 27^{-1}$

(u) $2^{-2} \times 4^3 \div 8^0$

(v) $\left(\frac{3}{4}\right)^{-2} \div \left(\frac{4}{9}\right)^3 \times \left(\frac{27}{16}\right)^{-1}$

(w) $\left(\frac{2}{3}\right)^{-1} \div \left(\frac{4}{9}\right)^{-2} \times 27^0$

(x) $\frac{5^4}{3^7} \times \left(\frac{9}{15}\right)^3 \div \frac{27}{25}$

3. Evaluate each of the following:

(a) 4^{-2} **(b)** 3^{-4}

(c) $(43)^0$ **(d)** 7^{-2}

(e) 2^{-3} **(f)** 5^{-2}

(g) $\left(\frac{-1}{4}\right)^2$ **(h)** $\left(\frac{2}{3}\right)^{-3}$

(i) $\left(\frac{2}{9}\right)^0$ **(j)** $\left(\frac{-2}{5}\right)^{-3}$

(k) $\left(\frac{16}{9}\right)^{-2}$ **(l)** $\left(\frac{3}{4}\right)^{-3}$

4. Simplify the following:

(a) $a^3 \times a^0$ **(b)** $a^2 \times a^{-5}$

(c) $a^{-3} \times a^{-2}$ **(d)** $a^3 \times a^{-3}$

(e) $a^2 \div a^{-2}$ **(f)** $x^7 \div x^{-5}$

(g) $x^{-2} \div x^{-5}$ **(h)** $x^{-7} \div x^{-9}$

(i) $x^{-3} \div x^2$ **(j)** $x^{-4} \div x^{-7} \div x^2$

(k) $a^{500} \div a^{-600}$ **(l)** $a^4 \div a^{-5}$

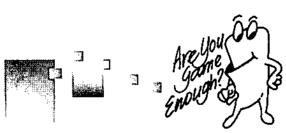
(m) $(x^0)^{-7}$ **(n)** $(x^2yz)^4 \div (xyz)^7$

(o) $(xyz)^5 \div x^2y^3z^4$

(p) $(pqr^2)^{-2} \div (p^2r^2q)^{-5}$

(q) $(a^2 \times a^{-5})^{-3}$

(r) $(pqr)^{-2} \times (p^2q^3)^2 \div (p^4r^3s)^{-7}$



Can you recall what is a

(a) rational number?
(b) positive number?

Fractional Indices

Earlier, we have seen that the rules of indices hold true for integral indices. We shall now find a meaning for a^n, where n is a fraction or rational number, and a is positive.

Consider $a^{\frac{1}{2}}$.

Law 1 holds true for fractional indices:

i.e. $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$

By definition, $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = (a^{\frac{1}{2}})^2$

Therefore, $(a^{\frac{1}{2}})^2 = a$.

Taking the square roots on both sides, we obtain

$$a^{\frac{1}{2}} = \sqrt{a}$$

For any positive integer n, $0^{\frac{1}{n}} = 0$; i.e. $\sqrt[n]{0} = 0$

Consider $a^{\frac{1}{3}}$.

Law 1 holds true for fractional indices:

i.e. $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$

By definition, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = (a^{\frac{1}{3}})^3$

Therefore, $(a^{\frac{1}{3}})^3 = a$.

Taking the cube roots on both sides, we obtain

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

NB: Law 1 which holds true for integral indices *also* holds true for fractional indices defined above.

In general, we define

8. $a^{\frac{1}{n}} = \sqrt[n]{a}$, where $a > 0$ and n is a positive integer.

Let us consider a^b, where b is a ratio of two integers, e.g. $a^{\frac{2}{3}}$.

Law 3 holds true for fractional indices:

i.e. $(a^{\frac{2}{3}})^3 = a^2$

Taking the cube roots on both sides, we obtain

$$a^{\frac{2}{3}} = \sqrt[3]{a^2}$$

Law 3 holds true for fractional indices:

i.e. $a^{\frac{2}{3}} = (a^{\frac{1}{3}})^2$

$$a^{\frac{2}{3}} = (\sqrt[3]{a})^2$$

In general, we define

9. $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$, where m and n are integers and $a > 0$.

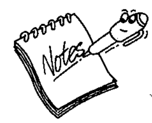
NB: 1. When $m = 1$, $a^{\frac{m}{n}} = a^{\frac{1}{n}} = \sqrt[n]{a}$ which was given above.
2. Because a rational number can be expressed as a fraction, the above definition can be used for rational indices.
3. Any expression involving the radical sign $\sqrt[n]{\ }$, where n is a positive integer is called a **radical** expression.

By the above definition, all the laws of indices we learned earlier for integral indices hold true for fractional indices.

Example 12

Rewrite each of the following in the radical form and evaluate the results.

(a) $4^{\frac{1}{2}}$ *(b)* $27^{-\frac{1}{3}}$ *(c)* $8^{\frac{2}{3}}$ *(d)* $100^{1.5}$



For any positive integer n,

$0^{\frac{m}{n}} = 0$ if m is positive.

$0^{\frac{m}{n}}$ is meaningless if m is 0 or negative.

Solution

(a) $4^{\frac{1}{2}} = \sqrt{4} = 2$ (b) $27^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}$

(c) **Method 1:**

$8^{\frac{2}{3}} = (\sqrt[3]{8})^2$
$= 2^2$
$= 4$

Method 2

$8^{\frac{2}{3}} = \sqrt[3]{8^2}$
$= \sqrt[3]{64}$
$= 4$

Method 3

$8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}$
$= 2^2$
$= 4$

(d) **Method 1:**

$100^{1.5} = 100^{\frac{3}{2}}$
$= (\sqrt{100})^3$
$= 10^3 = 1\ 000$

Method 2:

$100^{\frac{3}{2}} = \sqrt{100^3}$
$= \sqrt{1\ 000\ 000}$
$= 1\ 000$

Method 3

$100^{\frac{3}{2}} = (10^2)^{\frac{3}{2}}$
$= 10^3$
$= 1\ 000$

In (c) and (d) above, which of the three methods do you prefer? Why?

Example 13

Express the following using rational indices:

(a) $\sqrt{p^5}$ *(b)* $\sqrt[5]{a^3}$ *(c)* $\frac{1}{\sqrt[r]{x^k}}$ *(d)* $\frac{1}{\sqrt[x]{x^n}}$

Solution

(a) $\sqrt{p^5} = (p^5)^{\frac{1}{2}} = p^{\frac{5}{2}}$ (b) $\sqrt[5]{a^3} = (a^3)^{\frac{1}{5}} = a^{\frac{3}{5}}$

(c) $\frac{1}{\sqrt[r]{x^k}} = \frac{1}{(x^k)^{\frac{1}{r}}} = \frac{1}{x^{\frac{k}{r}}} = x^{-\frac{k}{r}}$ (d) $\frac{1}{\sqrt[x]{x^n}} = \frac{1}{(x^n)^{\frac{1}{x}}} = \frac{1}{x^{\frac{n}{x}}} = x^{-\frac{n}{x}}$

Example 14

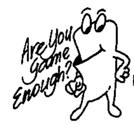
Simplify the following:

(a) $a^{\frac{2}{3}} \times a^{\frac{1}{4}}$ (b) $3a^{\frac{2}{3}} \div a^{\frac{1}{2}}$ (c) $(a^{\frac{3}{4}})^{\frac{8}{15}}$ (d) $(a^{\frac{1}{3}}b^{\frac{2}{5}})^{\frac{3}{4}}$

Solution

(a) $a^{\frac{2}{3}} \times a^{\frac{1}{4}} = a^{\frac{2}{3}+\frac{1}{4}} = a^{\frac{11}{12}}$ (b) $3a^{\frac{2}{3}} \div a^{\frac{1}{2}} = 3a^{\frac{2}{3}-\frac{1}{2}} = 3a^{\frac{1}{6}}$

(c) $(a^{\frac{3}{4}})^{\frac{8}{15}} = a^{\frac{3}{4}\times\frac{8}{15}} = a^{\frac{2}{5}}$ (d) $(a^{\frac{1}{3}}b^{\frac{2}{5}})^{\frac{3}{4}} = a^{\frac{1}{3}\times\frac{3}{4}}b^{\frac{2}{5}\times\frac{3}{4}} = a^{\frac{1}{4}}b^{\frac{3}{10}}$



Find a if
$(2^{15} + 4)^2 - (2^{15} - 4)^2 = 2^a$

Example 15

Simplifying the following, giving your answers in the radical form.

(a) $\sqrt{2} \times \sqrt{3}$ (b) $\sqrt[3]{2} \times \sqrt[3]{7}$ (c) $\sqrt[4]{4} \times \sqrt[5]{8}$ (d) $\sqrt{18} \times \sqrt[5]{64}$

Solution

(a) $\sqrt{2} \times \sqrt{3} = 2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = (2 \times 3)^{\frac{1}{2}} = \sqrt{6}$

(b) $\sqrt[3]{2} \times \sqrt[3]{7} = 2^{\frac{1}{3}} \times 7^{\frac{1}{3}} = (2 \times 7)^{\frac{1}{3}} = \sqrt[3]{14}$

(c) $\sqrt[4]{4} \times \sqrt[5]{8} = 4^{\frac{1}{4}} \times 8^{\frac{1}{5}} = (2^2)^{\frac{1}{4}} \times (2^3)^{\frac{1}{5}} = 2^{\frac{1}{2}} \times 2^{\frac{3}{5}} = 2^{\frac{11}{10}} = \sqrt[10]{2^{11}}$

(d) $\sqrt{18} \times \sqrt[5]{64} = 18^{\frac{1}{2}} \times 64^{\frac{1}{5}} = (9 \times 2)^{\frac{1}{2}} \times (2^6)^{\frac{1}{5}}$

$= 9^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{6}{5}} = 9^{\frac{1}{2}} \times 2^{\frac{17}{10}}$

$= 3 \times \sqrt[10]{2^{17}}$

We have learnt that
$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ *and* $\sqrt{a} \times \sqrt{a} = a$.

Also,
$-1 = \frac{1}{-1} = \frac{-1}{1}$

Thus,
$\sqrt{\frac{1}{-1}} = \sqrt{\frac{-1}{1}}$

$\therefore \quad \frac{\sqrt{1}}{\sqrt{-1}} = \frac{\sqrt{-1}}{\sqrt{1}}$

Multiply both sides by $\sqrt{1}\sqrt{-1}$, *we have*

$\frac{\sqrt{1}}{\sqrt{-1}}\sqrt{1}\sqrt{-1} = \frac{\sqrt{-1}}{\sqrt{1}}\sqrt{1}\sqrt{-1}$

i.e. $\sqrt{1}\sqrt{1} = \sqrt{-1}\sqrt{-1}$

Hence,
$1 = -1$

Is 1 = −1? Can you spot the wrong reasoning?

Example 16

Simplify the following and express your answers in the radical form.

(a) $a^{\frac{1}{2}} \times a^{\frac{2}{3}} \div a^{\frac{3}{4}}$ (b) $(a^{\frac{1}{2}}b^{\frac{2}{3}})^{\frac{3}{4}} \div (a^{\frac{2}{5}}b^{\frac{1}{3}})^{\frac{5}{6}}$

Solution

(a) $a^{\frac{1}{2}} \times a^{\frac{2}{3}} \div a^{\frac{3}{4}} = a^{\frac{1}{2}+\frac{2}{3}-\frac{3}{4}}$

$= a^{\frac{5}{12}} = \sqrt[12]{a^5}$

(b) $(a^{\frac{1}{2}}b^{\frac{2}{3}})^{\frac{3}{4}} \div (a^{\frac{2}{5}}b^{\frac{1}{3}})^{\frac{5}{6}} = (a^{\frac{1}{2}\times\frac{3}{4}}b^{\frac{2}{3}\times\frac{3}{4}}) \div (a^{\frac{2}{5}\times\frac{5}{6}}b^{\frac{1}{3}\times\frac{5}{6}})$

$= (a^{\frac{3}{8}}b^{\frac{1}{2}}) \div (a^{\frac{1}{3}}b^{\frac{5}{18}})$

$= a^{\frac{3}{8}-\frac{1}{3}}b^{\frac{1}{2}-\frac{5}{18}} = a^{\frac{1}{24}}b^{\frac{2}{9}}$

$= \sqrt[24]{a} \times \sqrt[9]{b^2}$

Exercise 2c

1. Rewrite each of the following in the radical form and evaluate the result.

(a) $49^{\frac{1}{2}}$ **(b)** $27^{\frac{1}{3}}$ **(c)** $16^{\frac{1}{4}}$ **(d)** $8^{\frac{1}{3}}$ **(e)** $100^{\frac{1}{2}}$ **(f)** $9^{1.5}$

(g) $16^{\frac{3}{4}}$ **(h)** $125^{\frac{2}{3}}$ **(i)** $32^{\frac{4}{5}}$ **(j)** $36^{\frac{1}{2}}$ **(k)** $27^{\frac{2}{3}}$ **(l)** $81^{\frac{3}{4}}$

(m) $1\,000^{\frac{2}{3}}$ **(n)** $32^{0.6}$ **(o)** $4^{1.5}$ **(p)** $4^{2.5}$ **(q)** $81^{0.25}$ **(r)** $8^{-\frac{1}{3}}$

(s) $9^{-0.5}$ **(t)** $16^{-0.25}$ **(u)** $32^{-0.8}$ **(v)** $16^{-1.5}$ **(w)** $\left(\frac{1}{4}\right)^{\frac{1}{2}}$ **(x)** $\left(\frac{1}{8}\right)^{-\frac{5}{3}}$

(y) $25^{-2.5}$ **(z)** $\left(\frac{1}{16}\right)^{-\frac{1}{4}}$

2. Express each of the following in the index form.

(a) $\sqrt{a^{16}}$ **(b)** $\sqrt[3]{a^{15}}$ **(c)** $\sqrt[3]{27a^9}$ **(d)** $\sqrt[3]{8a^9}$ **(e)** $\sqrt[4]{x^{32}}$

(f) $\sqrt[4]{81x^{20}}$ **(g)** $\sqrt[3]{125p^9q^{15}}$ **(h)** $\sqrt{(a+b)^5}$ **(i)** $\sqrt[4]{16a^2b^6}$ **(j)** $\sqrt[3]{\frac{a^5b^6}{c^4}}$

(k) $\sqrt[3]{\frac{8a}{(a+b)^2}}$ **(l)** $\sqrt[p]{\frac{b^n}{a^m}}$

3. Simplify the following, giving your answers in the radical form.

(a) $\sqrt{3} \times \sqrt{7}$ **(b)** $\sqrt[3]{5} \times \sqrt[3]{3}$ **(c)** $\sqrt{27} \times \sqrt{6}$ **(d)** $\sqrt[4]{8} \times \sqrt[7]{16}$

(e) $\sqrt[5]{4} \times \sqrt[5]{128}$ **(f)** $\sqrt[6]{81} \times \sqrt[5]{27}$ **(g)** $\sqrt[3]{81} \times \sqrt[4]{27}$ **(h)** $\sqrt[4]{125} \times \sqrt[6]{5}$

(i) $\sqrt[3]{25} \div \sqrt[5]{5}$ **(j)** $\sqrt{2} \div \sqrt[7]{32}$ **(k)** $\sqrt[3]{128} \div \sqrt[5]{2}$ **(l)** $\sqrt{27} \div \sqrt[12]{81}$

4. Simplify the following:

(a) $a^{\frac{1}{3}} \times a^{\frac{2}{5}}$ **(b)** $a^{\frac{2}{5}} \times a^{-\frac{1}{3}}$ **(c)** $a^{\frac{4}{5}} \times a^{\frac{5}{6}}$ **(d)** $a^{\frac{2}{11}} \div a^{\frac{4}{9}}$

(e) $a^{\frac{6}{7}} \div a^{\frac{3}{14}}$ **(f)** $12a^{\frac{7}{8}} \div 3a^{\frac{1}{4}}$ **(g)** $(a^{\frac{1}{3}})^{\frac{3}{5}}$ **(h)** $(a^{\frac{1}{3}})^2 \times (a^{\frac{2}{3}})^3$

(i) $a^{\frac{2}{3}} \times a^{\frac{1}{2}} \div a^{\frac{1}{4}}$ **(j)** $a^2b^{\frac{1}{3}} \div a^{\frac{2}{3}}b^{\frac{1}{6}}$ **(k)** $a^{\frac{1}{2}}b^{\frac{2}{3}} \times a^{\frac{2}{3}}b^{\frac{1}{4}}$ **(l)** $(a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}})^6$

(m) $a^{\frac{2}{3}}b^{\frac{5}{6}} \times a^{\frac{1}{2}}b \div (ab)^{\frac{1}{3}}$ **(n)** $(a^{\frac{1}{2}}b^{\frac{1}{3}})^{\frac{4}{3}} \div (a^{\frac{1}{3}}b^{\frac{1}{4}})^{\frac{1}{2}}$

5. Simplify the following and express your answers in the radical form.

(a) $a^{\frac{1}{2}} \times a^{\frac{2}{3}}$ **(b)** $a^{\frac{6}{7}} \times a^{\frac{1}{14}}$ **(c)** $x^{\frac{1}{4}} \div x^{-\frac{2}{3}}$

(d) $x^{\frac{3}{2}} \div x^{-\frac{1}{2}}$ **(e)** $(x^{\frac{1}{2}}y^{\frac{1}{3}})^5$ **(f)** $(x^{\frac{3}{4}}y^{\frac{1}{3}})^6$

(g) $(a^3b^2)^{\frac{1}{2}} \times (b^3a^4)^{-\frac{1}{3}}$ **(h)** $(a^3b^2)^{\frac{1}{4}} \times (a^{\frac{1}{3}}b)^{\frac{3}{4}}$ **(i)** $x^{\frac{1}{5}}y^{\frac{1}{2}} \div x^{-\frac{1}{10}}y^{-1}$

(j) $(x^{\frac{1}{4}}y^3)^{-\frac{1}{2}} \div (x^{\frac{1}{3}}y^{-\frac{1}{4}})^{-5}$ **(k)** $(x^2y^3)^{\frac{1}{5}} \times (x^{\frac{1}{3}}y^2)^{\frac{3}{4}}$ **(l)** $x^{\frac{1}{2}}y^{\frac{1}{3}} \times x^{\frac{2}{5}} \div y^{\frac{1}{4}}$

Equations Involving Indices

This section shows examples of how to solve simple equations involving indices.

Example 17

Solve the following equations:

(a) $2^x = 32$ *(b)* $3^x = \frac{1}{9}$ *(c)* $9^x = 27$ *(d)* $4^x = 32$

(e) $x^4 = 16$ *(f)* $2^x \times 4^{x+1} = 8^{2x-3}$

Solution

(a)
$$2^x = 32$$
$$= 2^5$$
$$\therefore \quad x = 5$$

(b)
$$3^x = \frac{1}{9}$$
$$= \frac{1}{3^2}$$
$$= 3^{-2}$$
$$\therefore \quad x = -2$$

(c)
$$9^x = 27$$
$$(3^2)^x = 3^3$$
$$3^{2x} = 3^3$$
$$\therefore \quad 2x = 3$$
$$x = \frac{3}{2} = 1\frac{1}{2}$$

(d)
$$4^x = 32$$
$$(2^2)^x = 2^5$$
$$2^{2x} = 2^5$$
$$\therefore \quad 2x = 5$$
$$x = \frac{5}{2} = 2\frac{1}{2}$$

(e)
$$x^4 = 16$$
$$x^4 = 2^4$$
$$\therefore \quad x = 2$$

(f)
$$2^x \times 4^{x+1} = 8^{2x-3}$$
$$2^x \times (2^2)^{x+1} = (2^3)^{2x-3}$$
$$2^x \times 2^{2x+2} = 2^{6x-9}$$
$$2^{x+2x+2} = 2^{6x-9}$$
$$\therefore \quad 3x + 2 = 6x - 9$$
$$11 = 3x$$
$$x = \frac{11}{3}$$
$$= 3\frac{2}{3}$$

Notes

An alternative method that may be used for Example 18(c) is:

$$x^{-3} = 8$$
$$(x^{-3})^{-\frac{1}{3}} = 8^{-\frac{1}{3}}$$
$$x = \frac{1}{\sqrt[3]{8}}$$
$$\therefore \quad x = \frac{1}{2}$$

Example 18

Solve the following equations, where x is positive.

(a) $x^{-1} = 4$ *(b)* $2^3 \times 16 = 2^x$ *(c)* $x^{-3} = 8$ *(d)* $x^{\frac{2}{3}} = 64$

Solution

(a)
$$x^{-1} = 4$$
$$\frac{1}{x} = 4$$
$$x = \frac{1}{4}$$

or

$$x^{-1} = 4$$
$$(x^{-1})^{-1} = (4)^{-1}$$
$$\therefore \quad x = 4^{-1}$$
$$= \frac{1}{4}$$

(b)
$$2^3 \times 16 = 2^x$$
$$2^3 \times 2^4 = 2^x$$
$$2^{3+4} = 2^x$$
$$\therefore \quad x = 3 + 4$$
$$= 7$$

(c) $x^{-3} = 8$

$\Rightarrow \frac{1}{x^3} = 8$

i.e. $x^3 = \frac{1}{8}$

$x = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

(d) $x^{\frac{2}{3}} = 64$

$(x^{\frac{2}{3}})^{\frac{3}{2}} = 64^{\frac{3}{2}}$

$x = (\sqrt{64})^3 = 8^3$

$\therefore \quad x = 512$

Exercise 2d

1. Solve the following equations:
   - **(a)** $3^x = 81$ **(b)** $4^x = 64$
   - **(c)** $5^x = 125$ **(d)** $7^x = 49$
   - **(e)** $3^x = 243$ **(f)** $2^x = 64$
   - **(g)** $x^3 = 27$ **(h)** $x^3 = 8$
   - **(i)** $x^5 = 32$ **(j)** $5x^2 = 45$
   - **(k)** $3x^2 = 48$ **(l)** $4x^4 = 324$
   - **(m)** $5x^3 = 320$ **(n)** $3^x = 1$
   - **(o)** $5^{-x} = 1$ **(p)** $2^x \times 8 = 64$
   - **(q)** $2^5 \div 8 = 2^x$ ✲**(r)** $4^x = 2^8$
   - ✲**(s)** $(4^5)^2 = 2^x$ ✲**(t)** $3^7 \div 3^{2x} = 27$
   - ✲**(u)** $9^3 \times 27 = 3^x$ ✲**(v)** $(25^3)^2 \times 125 = 5^x$
   - ✲**(w)** $8^3 \div 2^x = 4$ ✲**(x)** $16 \times (4^x)^3 = 2^{2x}$
   - ✲**(y)** $8^{x+1} = 16^{x-3}$ ✲**(z)** $9^{2x-1} = 27^{2x-5}$

2. Solve the following equations where x is positive:
   - **(a)** $x^{\frac{1}{2}} = 3$ **(b)** $x^{-4} = 256$
   - **(c)** $x^{-\frac{1}{2}} = 5$ **(d)** $3^{-x} = 1$

Problem Solving Involving Indices

In this section, we introduce the concept of indices to solve problems.

Example 19

What is the last digit of the number (a) $9^{1\,997}$ (b) $7^{1\,997}$?

Solution

A calculator will not be of much help because the number involved is far too big. We shall approach this problem using two strategies.

Strategy 1: Look for a pattern

Consider the last digits of $9^1, 9^2, 9^3, 9^4, 9^5, 9^6 \ldots$ and $7^1, 7^2, 7^3, 7^4, 7^5, 7^6 \ldots$ by respectively listing them in tables as shown below.

	9^1	9^2	9^3	9^4	9^5	9^6
Value	9	81	729	6 561	59 049	531 441
Last digit	9	1	9	1	9	1

	7^1	7^2	7^3	7^4	7^5	7^6
Value	7	49	343	2 401	16 807	117 649
Last digit	7	9	3	1	7	9

From the first table, we see that the last digit of odd indices of 9 is 9 and that of even indices is 1. Thus the last digit of $9^{1\,997}$ is 9.

The last digit of the powers of 7 recurs in the pattern 7, 9, 3, 1, 7, 9, ... i.e., it recurs after every 4th power. Thus the last digit of $7^{1\,997}$ is $(7^4)^{499} \times 7^1$ or 7.

Strategy 2: Use logical deduction

We know that the product of 2 or more numbers whose last digits are 1 will give an answer with 1 as its last digit. For example, the last digits of 81^2, 81^3, 71^2, 71^5, 91^4, $1\,331^7$, 121^{15} will be 1.

Since the last digit of 9^2 is 1, it follows that 9^4 (or $9^2 \times 9^2$), 9^6, 9^8, 9^{100}, $9^{1\,000}$ ... will each have 1 as its last digit. Thus $9^{1\,997} = (9^2)^{998} \times 9^1$ and hence its last digit will be 1×9 or 9.

Since the last digit of 7^4 is 1, it follows that the last digit of $(7^4)^n$, where n is an integer, will be 1. Thus $7^{1\,997} = (7^4)^{499} \times 7^1$ and hence its last digit will be 1×7 or 7.

Can you write down the last digits of $7^{1\,001}$, $7^{1\,234}$, $9^{1\,999}$, $9^{2\,020}$?

Exercise 2e

1. Find the last digit of each of the following without the use of a calculator.

(a) 2^{10} **(b)** 2^{11} **(c)** 2^{15} **(d)** 3^{25} **(e)** 3^{153}
(f) 3^{495} **(g)** 4^{517} **(h)** 4^{629} **(i)** 4^{666} **(j)** 5^{701}
(k) 5^{713} **(l)** 5^{788} **(m)** 6^{801} **(n)** 6^{805} **(o)** 6^{888}
(p) 7^{912} **(q)** 7^{957} **(r)** 7^{999} **(s)** $8^{1\,001}$ **(t)** $8^{1\,094}$
(u) $8^{1\,099}$ **(v)** $9^{1\,419}$ **(w)** $9^{1\,534}$ **(x)** $9^{2\,000}$

2. Find the last digit of each of the following:

(a) $3^{1\,991} \times 2^{1\,991}$ **(b)** $7^{211} - 6^{211}$ **c)** $5^{555} + 4^{555}$ **(d)** $9^{188} + 8^{188}$
(e) $3^5 \times 5^{411}$ **(f)** $7^{963} \times 6^{830}$

Further Examples on Changing the Subject of a Formula

In Book 2, we learnt how to change the subject of a formula. We shall now learn how to change the subject of a formula involving radical expressions and exponents.

Example 20

(a) If $\sqrt{a + l} = 2b$, make a the subject of the formula.

(b) If $c + 7 = \frac{x^2}{3}$, make x the subject of the fomula.

Solution

(a) $\sqrt{a + l} = 2b$

$a + l = (2b)^2$

$\therefore \quad a = 4b^2 - l$

(b) $c + 7 = \frac{x^2}{3}$

$x^2 = 3(c + 7)$

$x^2 = 3c + 21$

$\therefore \quad x = \pm\sqrt{3c + 21}$

When solving problems that involve square roots such as illustrated in Example 20(a), square both sides of the equation.

For example 20(b), multiply both sides of the equation by 3 (to get rid of the denominator) and then take square roots on both sides of the equation to obtain a value of x.

Example 21

(a) Make s the subject of the fomula $v = \sqrt{u^2 + 2as}$.

(b) Make x the subject of the formula $\sqrt[3]{ax + b} = k$.

Solution

(a) $v = \sqrt{u^2 + 2as}$

$v^2 = u^2 + 2as$

$2as = v^2 - u^2$

$\therefore \quad s = \frac{v^2 - u^2}{2a}$

(b) $\sqrt[3]{ax + b} = k$

$ax + b = k^3$ (______)

$ax = k^3 - b$

$\therefore \quad x = \frac{k^3 - b}{a}$

For Example 21(a), square both sides of the equation. The value of s can be derived when both sides of the equation are divided by 2a.

For Example 21(b), write out the method used to get rid of the cube root inside the bracket provided.

Exercise 2f

1. Make a the subject of the formula for each of the following:

(a) $\sqrt{a} = b$ **(b)** $\sqrt{2a} = b$ **(c)** $\sqrt{m + a} = b$

(d) $e = \sqrt{5a - 8}$ **(e)** $\sqrt{\frac{a}{2}} = b$ **(f)** $l = \sqrt{\frac{k}{ma}}$

(g) $x = \sqrt{\frac{2a}{5c}}$ **(h)** $\sqrt{3a - 2} = \sqrt{\frac{a}{b}}$ **(i)** $\sqrt{3a - 2k} = z$

(j) $2a^2 = b - 3$ **(k)** $3a^2 - 2 = 3c$ **(l)** $k = ba^2 + z$

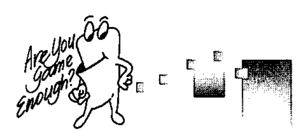
(m) $b = \sqrt{\frac{a^2}{5c}}$ **(n)** $m = n + \frac{na^2}{b}$ **(o)** $A = 4\pi a^2$

(p) $\sqrt[3]{a - b} = c$ **(q)** $V = \frac{1}{3}\pi a^2 h$ **(r)** $V = \frac{4}{3}\pi a^3$

(s) $K = \frac{0.5\,ma^2}{nb}$ **(t)** $ax = \sqrt[3]{a^3 b + 2c}$ **(u)** $y = \frac{xa^2}{R - r}$

2. Make the letter in the brackets the subject of the given formula:

(a) $a = \sqrt{a + 2b}$ *(b)* **(b)** $a^2 + b^2 = c^2$ *(b)*



Find the possible values of X, Y and Z if XY × XY = ZZY.

(c) $(x + y)^2 = x$ (y)

(d) $e = \sqrt{3c - 7a}$ (c)

(e) $x = 2w^2 + b$ (w)

(f) $\sqrt[3]{y - 1} = z$ (y)

(g) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ (b)

(h) $\sqrt[3]{2x^2 - 7} = \frac{y}{z}$ (x)

(i) $t = \sqrt{\frac{4x^2}{m - 3}}$ (x)

(j) $t^2 = \sqrt{\frac{m + 2}{m - 5}}$ (m)

(k) $\frac{1}{a} - \frac{1}{b} = \frac{1}{c - 2}$ (c)

(l) $y = \frac{nx}{a(4x - 3)}$ (x)

(m) $t = \frac{3x - 8mg}{12kg}$ (g)

(n) $r - q = \frac{pq + ps}{q - s}$ (s)

(o) $D = \sqrt{b^2 - 4ac}$ (b)

*(p) $2ps^2 = 3qs^2 + 5p$ (s)

(q) $px = \sqrt{3q - r^2}$ (r)

*(r) $\frac{p}{q} = \frac{1}{3n}\sqrt{\frac{h + 2k}{3h + k}}$ (h)

(s) $\frac{a + x}{a - x} = \frac{b - y}{b + y}$ (x)

(t) $\frac{x}{y} = \sqrt{1 - \frac{v^2}{u^2}}$ (v)

1. Find three distinct positive integers x, y and z such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

2. Make each of the following equations correct by filling in +, −, ×, ÷, ( ) or { } in the appropriate places.

1. 1 2 3 = 1
2. 1 2 3 4 = 1
3. 1 2 3 4 5 = 1
4. 1 2 3 4 5 6 = 1
5. 1 2 3 4 5 6 7 = 1
6. 1 2 3 4 5 6 7 8 = 1
7. 1 2 3 4 5 6 7 8 9 = 1

Further Examples on Algebraic Fractions

Example 22

Simplify (a) $\frac{2d^2}{2d - e} \times \frac{2de - e^2}{6de}$ (b) $\frac{mc(mx + ca)}{m^2x^2 + a^2c^2 + 2mxca}$

Solution

(a) $\frac{2d^2}{2d - e} \times \frac{2de - e^2}{6de} = \frac{{}^1\not{2}d^{\not{2}1}}{\cancel{2d - e}} \times \frac{\cancel{e(2d - e)}}{{}_3\not{6}\not{d}\not{e}}$

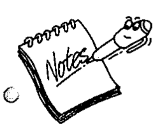
$= \frac{d}{3}$

(b) $\frac{mc(mx + ca)}{m^2x^2 + a^2c^2 + 2mxca} = \frac{mc\cancel{(mx + ca)}}{(mx + ca)^{\not{2}\,1}}$

$= \frac{mc}{mx + ca}$

Example 23

Simplify (a) $\frac{a^2 - 2ab}{a + b} \div \frac{a^2 - 4b^2}{a^2 + 2ab + b^2}$ (b) $\frac{3b - c}{c + b} \div \frac{c - 3b}{b + c}$



The following relations are useful when simplifying algebraic expressions:
(a) $a - b = -(b - a)$
(b) $b - a = -(a - b)$

Solution

(a) $\frac{a^2 - 2ab}{a + b} \div \frac{a^2 - 4b^2}{a^2 + 2ab + b^2} = \frac{a(a - 2b)}{a + b} \div \frac{(a + 2b)(a - 2b)}{(a + b)(a + b)}$

$= \frac{a\cancel{(a - 2b)}}{\cancel{a + b}} \times \frac{\cancel{(a + b)}(a + b)}{(a + 2b)\cancel{(a - 2b)}}$

$= \frac{a(a + b)}{(a + 2b)}$

(b) $\frac{3b-c}{c+b} \div \frac{c-3b}{b+c} = \frac{3b-c}{c+b} \times \frac{b+c}{c-3b}$

$= \frac{^1\cancel{3b-c}}{\cancel{c+b}} \times \frac{\cancel{b+c}}{-\cancel{(3b-c)}_{-1}} = \frac{1}{-1} = -1$

Exercise 2g

Simplify the following fractions:

1. $\frac{a^2+ab}{a^2-ab} \times \frac{ab^2+b^3}{a^3+a^2b}$
2. $\frac{a+5b}{a^2+6ab} \times \frac{a^3+6a^2b}{ab+5b^2}$
3. $\frac{x-y}{y+x} \div \frac{y-x}{x+y}$
4. $\frac{p-q}{p-r} \times \frac{q-r}{q-p} \times \frac{r-p}{r-q}$
5. $\frac{c^2-d^2}{c^2-2cd+d^2} \times \frac{1}{cd+d^2}$
6. $\frac{e}{e^2-f^2} \div \frac{e^2-2ef}{e^2+2ef+f^2}$
7. $(a^2-4b^2) \div \frac{a^2+2ab}{ab}$
8. $\frac{x^2-xy}{x^2-y^2} \div \frac{xy}{2x+2y}$
9. $\frac{d^2-4}{d^2-3d+2} \div \frac{d}{d-1}$
10. $\frac{b^2-bc}{2c} \times \frac{6c}{b^2-c^2}$
11. $\frac{m^2-m-6}{m^2-9} \times \frac{m^2}{m^2+2m}$
12. $\frac{a^2-b^2}{(a+b)^2} \div \frac{(a-b)^2}{5a}$
13. $\frac{b^2-4bc+4c^2}{b^2-4c^2} \div \frac{c}{b-2c}$
14. $\frac{x+3xz}{x^2} \times \frac{3z^2+xz}{z^2}$
15. $\frac{a^2}{a^2-4} \div \frac{3a-a^2}{a^2-5a+6}$
16. $\frac{x^2-y^2}{x^2-9y^2} \div \frac{x+y}{x+3y}$
17. $\frac{3(b^2-4)}{9p^2} \times \frac{6p^3}{4b+8}$
18. $\frac{m^2-mp}{p^2-pq} \div \frac{p^2-mp}{mp-mq}$
19. $\frac{y^2-4y+4}{2-6y} \times \frac{2y+4}{3y^2-12}$
20. $\frac{a^2-36}{x^2-y^2} \div \frac{a+6}{y-x}$
21. $\frac{x^2-3x-4}{x^2-4x} \div \frac{x^2-4x+4}{x^2-4}$
22. $\frac{10def}{30f+30d-30e} \times \frac{12d-12e+12f}{8ef}$

☆23. $\left(y+\frac{12}{y}-7\right) \div \left(y-\frac{4}{y}-3\right)$

☆24. $\frac{x^2-5x}{x-1} \div \frac{x^2-25}{x^2+x-20} \times \frac{x^2+x-2}{x^2-2x-8}$

☆25. $\frac{2x^2-13x+15}{2x-1} \times \frac{2x+1}{4x^2-9} \div \frac{x-5}{2x-1}$

☆26. $\frac{6x^2-x-2}{9x^2+6x-8} \times \frac{2x^2-5x-12}{4x^2+4x-3} \div \frac{2x^2-7x-4}{6x^2+5x-4}$

☆27. $\frac{x^2+18x+80}{x^2+15x+56} \div \frac{x^2+5x-50}{x^2+6x-7} \div \frac{x-1}{x-5}$

☆28. $\frac{x^2-1}{x+3} \div \frac{x^2+x}{x^2-4x+3} \div \frac{x^2-2x+1}{7}$

☆29. $\frac{x(2x-y)^2}{2x^2-xy} \times \frac{y}{2x+y} \div \frac{y(4x^2-y^2)}{4x^2+4xy+y^2}$

☆30. $\frac{x^2+4xy+3y^2}{x+3y} \times \frac{x^2-2xy+y^2}{x^2-y^2} \times \frac{x+y}{x-y}$

Example 24

Simplify each of the following:

(a) $\frac{x+2y}{3} - \frac{2x-3y}{6}$ *(b)* $\frac{4}{x-y} - \frac{1}{y-x}$

Solution

(a) $\frac{x+2y}{3} - \frac{2x-3y}{6} = \frac{2(x+2y)-(2x-3y)}{6}$

$= \frac{2x+4y-2x+3y}{6} = \frac{7y}{6}$

(b) $\frac{4}{x-y} - \frac{1}{y-x} = \frac{4}{x-y} + \frac{1}{x-y}$

$= \frac{5}{x-y}$

NB: The expression $y-x$ can also be written as $-(x-y)$.

Example 25

Simplify (a) $\dfrac{5m-2p}{4m^2+mp-3p^2}-\dfrac{1}{4m-3p}$

(b) $\dfrac{1}{x^2-3x+2}+\dfrac{2}{x^2-4x+3}-\dfrac{3}{x^2-5x+6}$

Solution

(a) $$\frac{5m-2p}{4m^2+mp-3p^2}-\frac{1}{4m-3p}=\frac{5m-2p}{(m+p)(4m-3p)}-\frac{1}{4m-3p}$$
$$=\frac{5m-2p-(m+p)}{(m+p)(4m-3p)}$$
$$=\frac{4m-3p}{(m+p)(4m-3p)}=\frac{1}{m+p}$$

(b) $$\frac{1}{x^2-3x+2}+\frac{2}{x^2-4x+3}-\frac{3}{x^2-5x+6}$$
$$=\frac{1}{(x-1)(x-2)}+\frac{2}{(x-1)(x-3)}-\frac{3}{(x-2)(x-3)}$$
$$=\frac{(x-3)+2(x-2)-3(x-1)}{(x-1)(x-2)(x-3)}=\frac{x-3+2x-4-3x+3}{(x-1)(x-2)(x-3)}$$
$$=\frac{-4}{(x-1)(x-2)(x-3)}$$

For Example 25(a), the expression $4m^2+mp-3p^2$ is factorised before simplification.

The common denominator, i.e. the LCM for the expression is $(m+p)(4m-3p)$.

For Example 25(b), the LCM of
$(x-1)(x-2)$,
$(x-1)(x-3)$ and
$(x-2)(x-3)$
is $(x-1)(x-2)(x-3)$.

Exercise 2h

Express the following as fractions with a single denominator.

1. $\dfrac{1+x}{3}+\dfrac{x-5}{6}$
2. $\dfrac{1}{3x}-\dfrac{2}{5y}$
3. $\dfrac{x+1}{2}-\dfrac{x-3}{3}$
4. $\dfrac{2(x+y)}{x}-\dfrac{3(x-3y)}{5x}$
5. $\dfrac{5}{x-y}-\dfrac{7}{y-x}$
6. $\dfrac{1}{a}-\dfrac{2}{a+b}$
7. $\dfrac{x}{a-b}-\dfrac{1}{b-a}$
8. $3-\dfrac{x-2}{3x}$
9. $\dfrac{4x-3}{8x^2}-\dfrac{2}{3x}$
10. $\dfrac{5}{a}+\dfrac{3}{a+2}$
11. $\dfrac{x+5}{x^2-6x}-\dfrac{3}{x-6}$
12. $\dfrac{5a}{a^2-ab}-\dfrac{3b}{b^2-ab}$
13. $\dfrac{3x}{x^2-y^2}-\dfrac{3y}{y^2-x^2}$
14. $\dfrac{3}{p+3r}-\dfrac{p-15r}{p^2-9r^2}$
15. $\dfrac{(m+n)^2}{m^2-n^2}+\dfrac{m^2+mn}{n^2-mn}$
16. $\dfrac{r(3-r)}{r^2+3r-10}+\dfrac{r-1}{r+5}$
17. $\dfrac{a+b}{a-b}+\dfrac{a^2-4b^2}{a^2-b^2}-\dfrac{a-3b}{a+b}$
18. $\dfrac{5a}{a-9}+\dfrac{a^2-2a+1}{a^2-12a+27}-\dfrac{6a}{a-3}$

*The use of letters and symbols in Mathematics was introduced by Francois Viéte (1540–1603), a French lawyer, in the 16th century. Since then, algebra has become a generalized arithmetic. The letters, known as **variable**, are used in placed of numbers. This helps greatly in solving complicated problems which involve tedious numerical calculations.*

Most games that involve skills require the usage of mathematics and logical manipulation.

The following is a simple game that requires mental skill and dexterity, so prepare yourself.

This game requires 2 ***players*** *only.)*

There are two piles of toothpicks, known simply as ***pile A and pile B****.*

The rules of the game are:

(1) Each player is to remove 1 or more or all the toothpicks from either pile A or pile B.

(2) The loser is the player who takes the last toothpick.

Students should pair up and play the game a few times before coming up with a winning strategy.

Students can start off with a table as shown below:

No of toothpicks in piles A	B	First move by P	Winner P or Q
1	1	1 from A	P
1	1	1 from B	P
2	1	1 from A	:
1	2	1 from B	:
2	1	2 from B	:
:	:	:	:

With this, they may be able to come up with a winning strategy.

19. $\dfrac{8x^2+18y^2}{4x^2-9y^2} - \dfrac{2x+3y}{2x-3y} + \dfrac{2x-3y}{2x+3y}$

20. $\dfrac{x}{x^2+xy} - \dfrac{y}{x^2-y^2} + \dfrac{x+y}{xy-y^2}$ **21.** $\dfrac{2}{m-4} + \dfrac{1}{m} - \dfrac{3}{m-3}$

22. $\dfrac{m-4}{2m-1} + \dfrac{5m^2+9m+14}{2m^2+3m-2} - \dfrac{3m-5}{m+2}$

23. $\dfrac{2}{a-1} + \dfrac{1}{(a-1)^2} + \dfrac{2}{(a-1)^3}$ **24.** $\dfrac{4}{a-1} - \dfrac{3}{a-2} - \dfrac{1}{a-3}$

25. $\dfrac{2}{a^2-1} - \dfrac{1}{a+1} - \dfrac{1}{a-1}$ **26.** $\dfrac{1}{2a-3} - \dfrac{2}{3-2a} + \dfrac{18}{9-4a^2}$

27. $\dfrac{1-\dfrac{1}{x^2}}{1-\dfrac{1}{x}}$ **28.** $\dfrac{\left(\dfrac{1}{1-x^2}\right)}{\left(\dfrac{1+x}{1-x}\right)}$

29. $\dfrac{x^2+2x}{x^2+x-2} - \dfrac{3x}{x+1}$ **30.** $\dfrac{6}{x^2+2x-8} - \dfrac{5}{x^2+x-12}$

31. $\dfrac{2}{3x-2} - \dfrac{5}{9x^2-4} - \dfrac{x}{3x+2}$ **32.** $\dfrac{x+3}{x^2-4x-21} - \dfrac{x}{x^2+x-42}$

33. $\dfrac{x+2}{x^2+3x+2} + \dfrac{x-5}{x^2-x-6}$ **34.** $\dfrac{x}{x^2-9} - \dfrac{3}{x-3} + \dfrac{x}{x+3}$

Equations Involving Algebraic Fractions

Example 26

Solve the following equations:

(a) $b + 2 = \dfrac{9}{b+2}$ *where* $b \neq -2$.

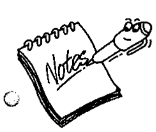
(b) $\dfrac{4}{m^2-m-6} = \dfrac{3}{m^2-2m-3}$ *where* $m \neq 3, -1, \text{or} -2$.

(c) $\dfrac{5}{x-1} - 3 = \dfrac{4}{x}$ *where* $x \neq 0$ *or 1.*

Solution

(a)
$$b + 2 = \frac{9}{b+2}$$
$$(b+2)(b+2) = 9$$
$$b^2 + 4b + 4 = 9$$
$$b^2 + 4b - 5 = 0$$
$$(b-1)(b+5) = 0$$
$$\therefore \quad b = 1 \text{ or } -5$$

(b)
$$\frac{4}{m^2-m-6} = \frac{3}{m^2-2m-3}$$
$$\frac{4}{(m-3)(m+2)} = \frac{3}{(m-3)(m+1)}$$
$$4(m+1) = 3(m+2)$$
$$4m + 4 = 3m + 6$$
$$\therefore \quad m = 2$$



Multiply the equation in Example 26(b) by the LCM of the denominators, i.e. $(m-3)(m+2)(m+1)$.

Multiply by $x(x-1)$ for Example 26(c), as this is the LCM of x and $x-1$.

(c)
$$\frac{5}{x-1} - 3 = \frac{4}{x}$$
$$5x - 3x(x-1) = 4(x-1)$$
$$5x - 3x^2 + 3x = 4x - 4$$
$$3x^2 - 8x + 4x - 4 = 0$$
$$3x^2 - 4x - 4 = 0$$
$$(3x+2)(x-2) = 0$$
$$\therefore \quad 3x + 2 = 0 \quad \text{or} \quad x - 2 = 0$$
$$\text{i.e. } x = -\frac{2}{3} \quad \text{or} \quad x = 2$$

Exercise 2i

Solve the following equations:

1. $\frac{10}{c} = c - 3$
2. $15 - m = \frac{14}{m}$
3. $a - 6 = -\frac{8}{a}$
4. $1 = \frac{7}{3} + \frac{5}{e}$
5. $x + 3 = \frac{6}{x+4}$
6. $\frac{4}{b-3} = b - 3$
7. $\frac{m-2}{m+4} = m$
8. $\frac{2}{d+3} - \frac{d-6}{d^2-9} = 0$
9. $\frac{c-1}{c+3} = \frac{c-1}{c+2}$
10. $\frac{a}{a-3} - \frac{8}{a} = 2$
11. $\frac{n}{n^2-2n-8} + \frac{1}{n^2+n-2} = 0$
12. $\frac{4}{m^2+3m+2} - \frac{3}{m^2+5m+6} = 0$
13. $\frac{5}{c+2} - \frac{5}{c^2-4} = 0$
14. $\frac{3}{m-4} - \frac{m+2}{m^2-3m-4} = \frac{1}{2m+2}$
15. $\frac{4}{a^2-a-2} + \frac{3}{a^2-4} = \frac{2}{a^2+3a+2}$
16. $\frac{d+3}{d} = \frac{3}{4} - \frac{d+9}{2d}$

The use of symbols in sentences was first introduced by George Boole, an English genius, in the late 19th century. His basic idea was that if simple sentences in a logical argument were represented by precise symbols, then the logical relation between two sentences could be read as an algebraic equation. Boole used most of his spare time learning about Mathematics and created a new branch of Mathematics now known as Boolean Algebra, in his honour. Boolean Algebra has many practical uses in the design of electronic computers.

Summary

1. If m and n are integers, then
   - **(a)** $a^m \times a^n = a^{m+n}$ e.g. $3^5 \times 3^8 = 3^{13}$
   - **(b)** $a^m \div a^n = a^{m-n}$ e.g. $5^{14} \div 5^3 = 5^{11}$
   - **(c)** $(a^m)^n = a^{mn}$ e.g. $(5^2)^5 = 5^{10}$
   - **(d)** $(a^m \times b^m) = (a \times b)^m$ e.g. $3^5 \times 2^5 = 6^5$
   - **(e)** $a^m \div b^m = \left(\frac{a}{b}\right)^m$ e.g. $5^4 \div 3^4 = \left(\frac{5}{3}\right)^4$
   - **(f)** $a^0 = 1$ e.g. $5^0 = 1$
   - **(g)** $a^{-n} = \left(\frac{1}{a^n}\right)$ e.g. $5^{-3} = \frac{1}{5^3}$
2. $a^{\frac{1}{n}} = \sqrt[n]{a}$ where $a > 0$ and n is a positive integer.
3. $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$ where m and n ($m \neq n$) are integers and $a > 0$, $n > 0$.
4. To make x the subject of a formula, it is necessary to rearrange the formula so that only x appears on the LHS of the formula.

Review Questions 2

Do not use a calculator for this exercise.

1. Simplify the following:

(a) $(a^4)^2$ **(b)** $(-a^3)^2$ **(c)** $(2x^5)^2$

(d) $(-x^4)^3$ **(e)** $-3(x^2)^2$ **(f)** $(-2x^3)^6$

(g) $(-p^3q)^2$ **(h)** $(-a^2b^4)^3$ **(i)** $a^5 \times a^0$

(j) $a^3 \times a^{-3}$ **(k)** $(a^0)^3$ **(l)** $(a^{-2})^0$

(m) $a^{-7} \times a^2$ **(n)** $a^{-2} \times a^{-4}$ **(o)** $a^{-9} \div a^{-3}$

(p) $a^8 \times a^{-4}$ **(q)** $(a^{-3})^{-1}$ **(r)** $(a^3)^{-5}$

(s) $(2a^{-1})^4$ **(t)** $(ab^3c^2)^2$ **(u)** $(-x^4y^2z)^4$

2. Evaluate the following:

(a) 34^0 **(b)** $(-7)^0$ **(c)** 8^{-2}

(d) $\left(\frac{-1}{4}\right)^{-1}$ **(e)** $\left(\frac{4}{5}\right)^{-2}$ **(f)** $\left(\frac{4}{7}\right)^{-2}$

(g) $5^{-2} \times 4^3$ **(h)** $2^{-3} \div 3^{-3}$ **(i)** $78^{-1} \times 13^3$

(j) $\left(\frac{-1}{7}\right)^{-2} \div 49^2$ **(k)** $3^4 \div \left(\frac{1}{3}\right)^{-4}$ **(l)** $\left(\frac{4}{9}\right)^{-2} \times \left(\frac{27}{8}\right)^{-3}$

(m) $8^{\frac{2}{3}}$ **(n)** $343^{\frac{1}{3}}$ **(o)** $9^{2.5}$

(p) $0.04^{\frac{1}{2}}$ **(q)** $\left(3\frac{3}{8}\right)^{-\frac{2}{3}}$ **(r)** $32^{-1.2}$

(s) $4^{1\frac{1}{2}} \times 4^{-1}$ **(t)** $16^{-\frac{3}{4}} \times 4$ **(u)** $2^{\frac{1}{3}} \times 12^{\frac{2}{3}} \times 6^{\frac{1}{3}}$

(v) $2^{-3} \times 4^{\frac{1}{2}} \times 8^{\frac{1}{3}}$ **(w)** $8^{-\frac{4}{3}} \times 32^{1.2}$ **(x)** $8^{-\frac{2}{3}} \times 16^{-\frac{5}{4}}$

3. Simplify the following:

(a) $20x^6y^7 \div 5x^{-1}y$ **(b)** $3a^5b^{-2} \times 4a^{-2}b^5$ **(c)** $(-x^2y^3z^4)^4$

(d) $\dfrac{(-p^4)^5}{-(p^5)^4}$ **(e)** $\dfrac{2^{10} \times 3^9}{(2^5)^2 \times (3^3)^3}$ **(f)** $\dfrac{(a^5)^2 \times (-a^2)^3}{(-a^3)^5}$

(g) $\dfrac{(a^4)^6}{-a^3 \times (-a^2)^9}$ **(h)** $\dfrac{p^7 \times q^9 r^{11}}{p^{-1} \times (q^3)^3 \times (r^3)^4}$ **(i)** $\dfrac{(r^7)^2 \times (s^2)^6}{(r^2)^6 \times (s^3)^4}$

(j) $\dfrac{4x^2y^4 \times 8x^4y^2}{(4x^2y^2)^2}$ **(k)** $\dfrac{(a^4)^{-6} \times (b^4)^4}{(a^{-3})^8 \times (b^8)^2}$ **(l)** $\dfrac{(x^{-3})^{-6} \times (y^2)^4 \times 2^{-5}}{(x^9)^2 \times y^8 \times (z^2)^{-3}}$

(m) $2a^{\frac{1}{2}} \times 3a^{\frac{1}{3}}$ **(n)** $2x^3 \div 4x^{-2}$ **(o)** $5x^{\frac{1}{2}} \times 3x^{-\frac{1}{3}}$

(p) $(2x^2y)^3 \div (4xy)^{-2}$ **(q)** $(27a)^{\frac{1}{3}} \times (64a^2)^{\frac{1}{3}}$ **(r)** $(x^{\frac{1}{2}} + 1)^2$

(s) $(2a^{\frac{1}{2}} + 1)(a^{\frac{1}{2}} - 2)$ **(t)** $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$

For Your Information

Find the values of x and y such that $2^{x^y} = 256^2$.

Problem Solving

1. Show that the sum of the squares of any four consecutive odd numbers is divisible by 4.

2. Show that the sum of the squares of any three consecutive even numbers is divisible by 4.

3. Solve the equation $4^x - 9(2^x) + 8 = 0$.

4. Solve the simultaneous equations $9^x \times 27^{2y} = 1$ and $2^{3x} \times 8^{4y} = \frac{1}{8}$.

5. Given that $t = 2\pi\sqrt{\frac{a^2 + b^2}{10}}$, express a in terms of b, t and π.

6. If $$M$ is just sufficient to pay the wages of one clerk for x days or the wages of one secretary for y days, how many days will $$M$ be just sufficient to pay the wages of one clerk and one secretary together? Give your answer in terms of x and y.

7. Find the remainder when $3^{1\,993} + 11$ is divided by 9.

8. If $x : y = 3 : 4$, find the numerical value of $\frac{x}{x - y} - \frac{x^2}{x^2 + y^2}$.

9. If $\sqrt{a\sqrt{a\sqrt{a}}} = a^k$, find k.

10. Given that today is Sunday, what day of the week will it be 14^{101} days from today?

11. Given that today is Monday, what day of the week will it be 8^{101} days from today?

12. Given that today is Tuesday, what day of the week will it be 50^{101} days from today?

13. Evaluate $\frac{2^{x+1} - 2^{x-2}}{2^{x-1} - 2^{x+2}}$.

14. If $10^x = 3$, find the value of 10^{2x+1}.

For Your Information

It is a known fact that

$4^2 = 16$

$\therefore \quad 4 = \sqrt{16}$.

Hence, it is impossible to have

$4^2 = -16$

$\therefore \quad 4 \neq \sqrt{-16}$

In this case, there is no real n^{th} root of a negative number when n is even.

In other words, $\sqrt{-16}$ is a complex root.

However, when n is odd,

$-27 = (-3)^3$

$\therefore \quad \sqrt[3]{-27} = -3$

or

$-243 = (-3)^5$

$\therefore \quad \sqrt[5]{-27} = -3,$

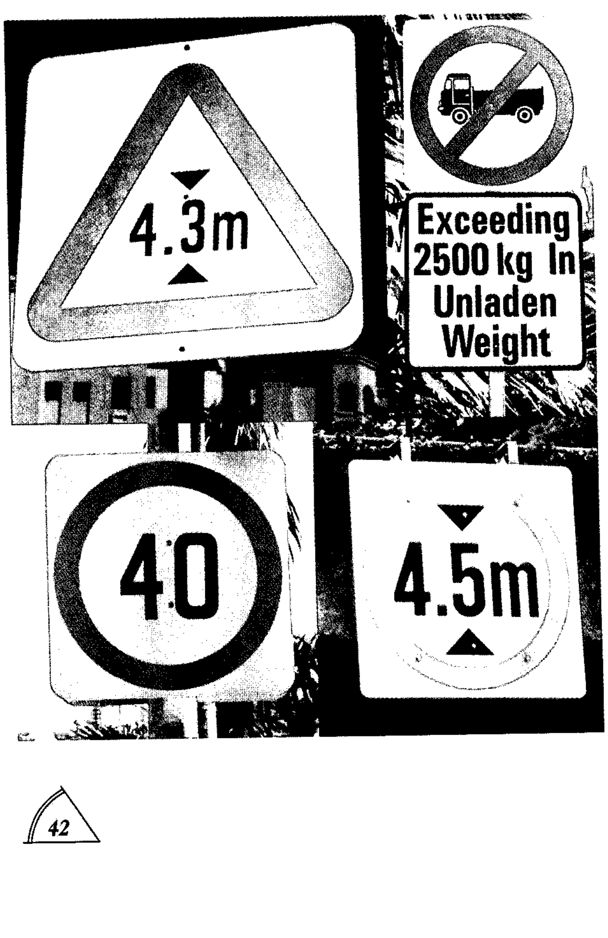
thus it is possible for the existence of a real n^{th} root for a negative number when n is odd.

CHAPTER 3

Linear Inequalities

In this chapter, you will learn more about

△ simple laws of linear inequalities;
△ simple methods for solving linear inequalities.



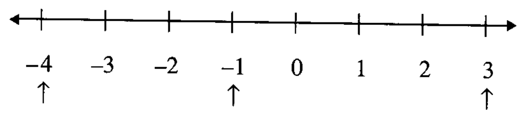
Preliminary Problem

Have you ever noticed the different traffic signs on Singapore roads? Each of these signs indicates an idea of inequality. Do you know the meaning of each of the traffic signs?

Inequalities

In Book 1, you learnt about ordering of numbers on the number line. A number on the number line is **greater** than any number on its left and **smaller** than any number on its right. We use the symbol '>' to represent 'is greater than' and the symbol '<' to represent 'is smaller than'.

For example,



3 lies on the right of −1. Hence 3 is greater than −1 and we write $3 > -1$.

Note: A positive number is always greater than a negative number, e.g. $2 > -4$.

−4 lies on the left of −1. Hence −4 is smaller than −1 and we write $-4 < -1$.

Consider any two numbers, x and y, on the number line. One and only one of the following statements must be true:

(a) $x > y$ (b) $x = y$ (c) $x < y$

This is known as the **Law of Trichotomy**.

Equations and Inequalities

Find the number of points, (x, y), where x and y are positive integers lying on the line $3x + 4y = 29$.

Let us think of x students attending a lecture.

(a) If there are 200 students, we write $x = 200$. This is an **equation**. It has only one answer, i.e., x has only one value.
(b) If there are less than 200 students, we write $x < 200$.
This is an **inequality**. It has many answers, i.e., x can have any numerical value ranging from 0 to 199 inclusive. In this example, negative values of x have no real meaning.

For an inequality with one unknown, x, all values of x that satisfy the inequality are called the solutions of the inequality. For example,

(a) some of the solutions of the inequality $x > -3$ are −2, −1, 0, 1, 2, 3, etc.;
(b) some of the solutions of the inequality $x < 3$ are 2, 1, 0, −1, −2, −3, etc.

Notice that there are infinitely many possible solutions to each of the above inequalities.

Exercise 3a

1. Fill in the blanks with '>', '=' or '<' to make the following statements true:

   (a) 5 ____ 8 (b) -2 ____ 0
   (c) 3 ____ -3 (d) 0.5 ____ 0.4
   (e) 2^2 ____ 4 (f) 2 ____ $\sqrt{2}$
   (g) $\sqrt{0.8}$ ____ $\sqrt[3]{0.8}$
   (h) -5 ____ -8
   (i) $(-4)^2$ ____ $(-3)^2$
   (j) $\frac{5}{6}$ ____ $\frac{6}{5}$
   (k) $-\frac{5}{6}$ ____ $-\frac{6}{5}$
   (l) 1.4 ____ $\sqrt[3]{1.4}$

2. Fill in the blanks with '>', '=' or '<' to make the following statements true.

   (a) 0.9 ____ $\sqrt[3]{0.9}$
   (b) $5 + x$ ____ $7 + x$
   (c) $n - 2$ ____ n
   (d) $m + 3$ ____ m
   (e) $10 - k$ ____ $8 - k$
   (f) $x - 2$ ____ $x - 5$
   (g) $2x + 3$ ____ $2x - 3$
   (h) $\sqrt{0.4}$ ____ 0.4
   (i) 0.5^2 ____ 0.5
   (j) 0.5^2 ____ 0.25
   (k) 5×0 ____ 4×0
   (l) -5×0 ____ -6×0

Properties of Inequalities

1. For any three numbers x, y and z, if $x > y$ and $y > z$, then $x > z$. This is known as the **transitive** property of inequalities.

   e.g. if $x = 10$, $y = 5$, $z = 2$, then $10 > 5$, $5 > 2$ and $10 > 2$;
   if $x = 1$, $y = 0$, $z = -4$, then $1 > 0$, $0 > -4$ and $1 > -4$.

2. We can add or subtract a number from both sides of an inequality without changing the inequality sign. For any two numbers x and y, and a positive number a,

   if $x > y$ (e.g. $5 > 3$ and $2 > 0$)
   then $x + a > y + a$ (e.g. $5 + 2 > 3 + 2$)
   $x - a > y - a$ (e.g. $5 - 2 > 3 - 2$)

   This is also true for a negative number b;

   if $x > y$ (e.g. $5 > 3$ and $-2 < 0$)
   then $x + b > y + b$ (e.g. $5 + 2 > 3 + 2$)
   $x - b > y - b$ (e.g. $5 - 2 > 3 - 2$)

3. We can multiply and divide both sides of an inequality by a **positive** number without changing the inequality sign. For any two numbers x and y, and a third number $a > 0$,

   if $x > y$ (e.g. $5 > 3$ and $2 > 0$)
   then $ax > ay$ (e.g. $2 \times 5 > 2 \times 3$)
   $\frac{x}{a} > \frac{y}{a}$ $\left(\text{e.g. } \frac{5}{2} > \frac{3}{2}\right)$

Different digits are used to represent different letters. For example, A = 1, 2, 3, … . No zeros should appear at the beginning of a number. Find out what the numbers are in each of the following cases:

1.
```
   THREE
 + FOUR
   SEVEN
```

2. (a) ABCDEF × 1 = ABCDEF
(b) ABCDEF × 3 = BCDEFA
(c) ABCDEF × 2 = CDEFAB
(d) ABCDEF × 6 = DEFABC
(e) ABCDEF × 4 = EFABCD
(f) ABCDEF × 5 = FABCDE

This is not true for a **negative** number b;

if $\quad x > y$ and $b < 0 \quad (5 > 3$ and $-2 < 0)$

then $\quad bx < by \quad [(-2) \times 5 < (-2) \times 3]$

$\quad \frac{x}{b} < \frac{y}{b} \quad \left(\frac{5}{-2} < \frac{3}{-2}\right)$

Thus we have to change the inequality sign when we multiply or divide both sides of an inequality by a negative number.

Solving Inequalities

Inequalities are solved in almost the same way as equations.

Example 7

Solve the following inequalities:

(a) $x + 3 < 7$ *(b)* $2x - 1 > 5$ *(c)* $6 - x > 4$

Solution

(a) $x + 3 < 7$

$x + 3 - 3 < 7 - 3$ (Subtract 3 from both sides.)

$\therefore \quad x < 4$

(b) $2x - 1 > 5$

$2x - 1 + 1 > 5 + 1$ (Add 1 to both sides.)

$2x > 6$

$\therefore \quad x > 3$ (Divide both sides by 2.)

(c) $6 - x > 4$

$6 - x - 6 > 4 - 6$ (Subtract 6 from both sides.)

$-x > -2$

$\therefore \quad x < 2$ (Multiply both sides by –1; at the same time change > to <).

or $6 - x > 4$

$6 - x + x > 4 + x$ (Add x to both sides.)

$6 > 4 + x$

$2 > x$

$\therefore \quad x < 2$

It is often convenient to represent the solution to an inequality using the number line. For example, to represent $x < 3$, use the number line as shown in Fig. 3.1. The small circle shows that the value 3 is not included as a possible answer, whereas any value to the left of 3 is.

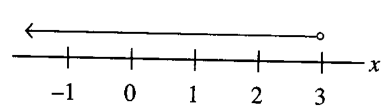


Fig. 3.1

Example 2

Solve the inequality $2x - 3 > 2(1 - x)$.

Solution

Without using a calculator or a table, determine which of the following is larger: $(\sqrt{10} + \sqrt{29})$ or $\sqrt{73}$.

$$2x - 3 > 2(1 - x)$$
$$2x - 3 > 2 - 2x$$
$$2x - 3 + 3 + 2x > 2 - 2x + 3 + 2x \quad \text{(add } 3 + 2x \text{ to both sides)}$$
$$4x > 5$$
$$x > \frac{5}{4}$$

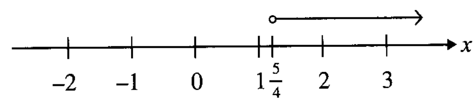


Fig. 3.2

This solution is represented by the number line as shown in Fig. 3.2.

Example 3

Solve the inequality $\frac{1}{3}x > \frac{1}{4}(x - 1)$.

Solution

$$\frac{1}{3}x > \frac{1}{4}(x - 1)$$
$$12 \times \frac{1}{3}x > 12 \times \frac{1}{4}(x - 1)$$
$$4x > 3(x - 1)$$
$$4x > 3x - 3$$
$$4x - 3x > -3$$
$$\therefore \quad x > -3$$

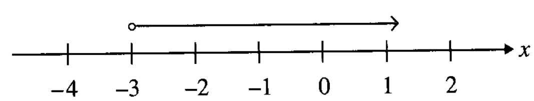


Fig. 3.3

The solution is shown by the number line in Fig. 3.3.

Exercise 3b

1. Fill in the blanks with '>', '=' or '<' to make each of the following statements correct.

(a) If $15 > 10$ and $10 > p$, then 15 ___ p.

(b) If $-3 > x$ and $x > y$, then -3 ___ y.

(c) If $a < 60$ and $60 < b$, then a ___ b.

(d) If $x + 1 = y$, then x ___ y.

(e) If $m - 2 = n$, then m ___ n.

(f) If $x > y$, then $4x$ ___ $4y$.

(g) If $x > y$, then $\frac{x}{100}$ ___ $\frac{y}{100}$.

(h) If $x > y$, then $(-2)x$ ___ $(-2)y$.

(i) If $x > y$, then $\frac{x}{-2}$ ___ $\frac{y}{-2}$.

(j) If $p < q$ and $q < 0$, then p ___ 0.

(k) If $u > 0$, then $(-3)u$ ___ 0.

(l) If $u > 0$, then $\frac{u}{6}$ ___ 0.

2. Solve the following inequalities, illustrating each solution with a number line.

(a) $x - 1 < 1$ **(b)** $x + 2 > 6$ **(c)** $4 - x < 3$ **(d)** $5 - x > -1$
(e) $7 > 3x$ **(f)** $2x + 4 < 0$ **(g)** $2x + 7 > 3$ **(h)** $7 - 2x > 6$
(i) $3x - 2 > 18 - 2x$ **(j)** $7 - 2x < 11 + x$ **(k)** $5x + 3 < 3(2 + x)$ **(l)** $3(x - 2) < 2x + 1$

3. Solve the following inequalities, illustrating each solution with a number line.

(a) $2(3x + 5) > 4x$ **(b)** $3(x + 5) > 2(x + 2) + 8$ **(c)** $\frac{1}{3}(2x - 1) > \frac{3}{5}x$

(d) $\frac{1}{4}(x + 4) > \frac{1}{3}(x + 1)$ **(e)** $\frac{1}{2}(2 - x) > \frac{1}{4}(3 - x) + \frac{1}{2}$ **(f)** $\frac{x-2}{4} + \frac{2}{3} < \frac{x-4}{6}$

(g) $\frac{x+1}{2} + \frac{3x+1}{4} < \frac{3x-1}{4} + 2$ **(h)** $\frac{3x+4}{5} - \frac{x+1}{3} > 1 - \frac{x+5}{3}$

(i) $\frac{x+1}{2} - \frac{x+3}{3} > \frac{x+1}{4} + 1$ **(j)** $\frac{x+3}{4} - \frac{x+2}{5} < 1 + \frac{x+5}{6}$

Difference Between < and ⩽

It is important to know the difference between these two statements:

(a) x is smaller than 3 (b) x is not greater than 3.

In (a), x can take any value smaller than 3, but not including 3. This is represented by Fig. 3.4(a): the small empty circle means that 3 is not included in the set of possible answers.

In (b), x can take any value smaller than or equal to 3. This is represented by Fig. 3.4(b): the small circle-dot means that 3 is included in the set of possible solutions.

The inequalities $x \leqslant 3$ and $x \ngtr 3$ are said to be equivalent: the symbol $\leqslant$ represents 'is smaller than or equal to'; and the symbol $\ngtr$ represents 'not greater than'. Similarly, the inequalities $x \geqslant 3$ and $x \nless 3$ are equivalent: the symbol $\nless$ represents 'not smaller than'.

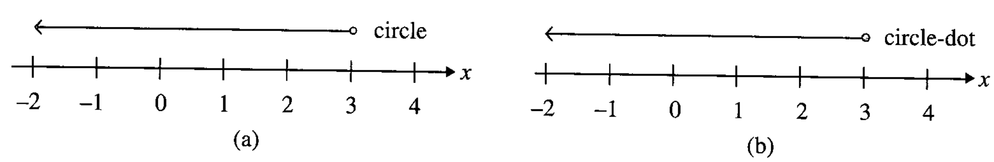


Fig. 3.4

Example 4

Solve the inequality $x - 7 \leqslant 5 - 2x$.

Solution

$x - 7 \leqslant 5 - 2x$
$x + 2x \leqslant 5 + 7$
$3x \leqslant 12$
$x \leqslant 4$

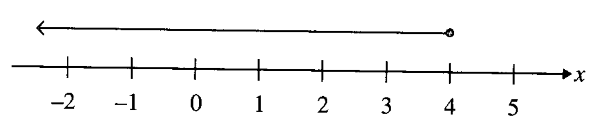


Fig. 3.5

The solution is shown by the number line in Fig. 3.5.

Example 5

Solve the inequality $\frac{2}{3}(x-2) \geqslant \frac{1}{2}(x-3)$.

Solution

$$\frac{2}{3}(x-2) \geqslant \frac{1}{2}(x-3)$$
$$6 \times \frac{2}{3}(x-2) \geqslant 6 \times \frac{1}{2}(x-3)$$
$$4(x-2) \geqslant 3(x-3)$$
$$4x-8 \geqslant 3x-9$$
$$4x-3x \geqslant -9+8$$
$$\therefore \quad x \geqslant -1$$

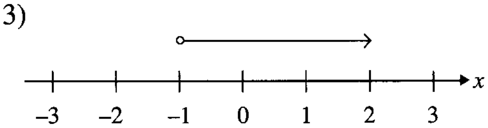


Fig. 3.6

The solution is shown by the number line in Fig. 3.6

Two travellers, one carrying 5 buns and the other, 3 buns, met an Arab in a desert. The Arab was very hungry and, as he had no food with him, the two men shared the buns with him, each having an equal share of the 8 buns. In return for their kindness, the Arab gave them 8 gold coins and told them to share the money fairly. The second traveller who had 3 buns said that he should receive 3 gold coins and that the other 5 gold coins should go to the first traveller. But the latter said that he should get more than 5 gold coins as he had given the Arab more buns. They could not agree and so a fight ensued. Can you help them solve their problem?

Exercise 3c

Solve the following inequalities, illustrating each solution using the number line.

1. $x-2 \leqslant 3$

2. $x+2 \geqslant 3$

3. $4-x \leqslant 4$

4. $x \geqslant \frac{1}{2}x-1$

5. $x-7 \geqslant 1-x$

6. $2x+1 \leqslant 5-4x$

7. $4-5x \leqslant x-5$

8. $\frac{1}{2}x \geqslant 1+\frac{1}{3}x$

9. $2(x-3) \geqslant 1$

10. $2(x-5) \leqslant 2-x$

11. $5(x-4) \leqslant 2x$

12. $3-4x \geqslant 3x-4$

13. $3(1-4x) \leqslant 8-7x$

14. $\frac{1}{4}(2x+3) \leqslant (7-4x)$

15. $\frac{1}{6}(2-x)-3 \geqslant \frac{x}{10}$

16. $\frac{1}{2}(3x+2) \geqslant \frac{1}{3}(2x-7)$

17. $\frac{1}{3}(x+2) \geqslant \frac{2}{3}+\frac{1}{4}(x-1)$

18. $\frac{4}{3}(2x+3) \geqslant 10-\frac{4x}{3}$

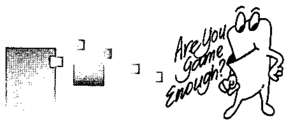
19. $4\left(\frac{x}{3}+\frac{3}{4}\right) \geqslant 3\left(\frac{x}{2}-5\right)$

20. $\frac{x-2}{4}-\frac{x-5}{6} \geqslant \frac{1}{3}$

21. $\frac{1}{4}\left(\frac{10x-5}{7}+6\right) < \frac{2}{3}(x-4)$

Problem Solving Involving Inequalities

In this section, we shall look at how inequalities can be used to solve problems.



Consider

$1 + 5 = 2 + 4$ ___ (1)

(a) Add 3 to each term in (1) and we get:

$(1 + 3) + (5 + 3)$
$= (2 + 3) + (4 + 3)$
$4 + 8 = 5 + 7$ ___ (2)

(b) Add (1 + 5) to the RHS of (2) and we get
$(1 + 5) + 5 + 7$
Add (2 + 4) to the LHS of (2) and we get
$(2 + 4) + 4 + 8$

Complete the following by filling in >, = or < in the boxes provided.

$1 + 5 + 5 + 7 \square 2 + 4 + 4 + 8$ ___ (3)

$1^2 + 5^2 + 5^2 + 7^2 \square 2^2 + 4^2 + 4^2 + 8^2$ ___ (4)

Add 6 to each term in (3) and we get:

$7 + 11 + 11 + 13 \square 8 + 10 + 10 + 14$ ___ (5)

Complete the following by filling in >, = or < in the boxes provided.

$1 + 5 + 5 + 7 + 8 + 10 + 10 + 14 \square 2 + 4 + 4 + 8 + 7 + 11 + 11 + 13$ ___ (6)

$1^2 + 5^2 + 5^2 + 7^2 + 8^2 + 10^2 + 10^2 + 14^2 \square 2^2 + 4^2 + 4^2 + 8^2 + 7^2 + 11^2 + 11^2 + 13^2$ ___ (7)

Check whether the following is correct:

$1^3 + 5^3 + 5^3 + 7^3 + 8^3 + 10^3 + 10^3 + 14^3$
$= 2^3 + 4^3 + 4^3 + 8^3 + 7^3 + 11^3 + 11^3 + 13^3$

Example 6

In 1998, 48 665 students from all races in Singapore received Edusave Merit Bursary Awards ranging from \$150 for a Primary pupil to \$400 for a Pre-university pupil. These awards are given to the top 25% of pupils from each school. 8 of Mr Ong's grandchildren received the Edusave Merit Bursary award in 1998. Find the value of the largest and smallest possible amount that could be paid out to Mr Ong's grandchildren.

Solution

Each of Mr Ong's grandchild should receive \$150 at least and \$400 at most.
Therefore, the largest possible amount to be paid out = 8 × \$400 = \$3 200.
The smallest possible amount to be paid out = 8 × \$150 = \$1 200.

Example 7

Kumar scored 66 and 72 marks respectively for his two class tests. What is the lowest mark he must score for the third test if an average score of at least 75 is required, to qualify for a bonus prize.

Solution

Let x be the marks scored by Kumar in the third test.

$$\frac{66 + 72 + x}{3} \geqslant 75$$

$$\therefore \quad 66 + 72 + x \geqslant 75 \times 3$$
$$x \geqslant 225 - 66 - 72$$
$$x \geqslant 87$$

i.e., Kumar must score at least 87 marks to qualify for the bonus prize.

Example 8

A Mathematics test consists of 20 multiple-choice questions. A correct answer is awarded 3 points and 1 point is deducted for a wrong answer while no points are awarded or deducted for an unanswered question. A boy attempted a total of 19 questions and his total score for the test was above 32. Find the minimum number of correct answers he obtained.

Solution

Let x be the number of correct answers and y the number of incorrect answers.

$$x + y = 19 \text{ ——— } (1)$$
$$3x - y > 32 \text{ ——— } (2)$$

From (1) we have $y = 19 - x$ ________ (3)

Substitute (3) into (2):

$$3x - (19 - x) > 32$$
$$3x - 19 + x > 32$$
$$4x > 51$$
$$x > 12.75$$

Since x must be a whole number, the minimum number of questions attempted correctly was 13.

Method:

Step 1: Write the equation using variables a, b, c and d.

$a + b = c + d$ ___ (1)

Step 2: Add 3 to each term in (1):

$a + 3 + b + 3 = c + 3 + d + 3$ ___ (2)

Step 3: Add the LHS of (1) to the RHS of (2) and vice versa.

Step 4: Generate two comparisons such as the following:

(a) $a + b + c + 3 + d + 3 \square c + d + a + 3 + b + 3$ ___ (3)

(b) $a^2 + b^2 + c^2 + 3^2 + d^2 + 3^2 \square c^2 + d^2 + a^2 + 3^2 + b^2 + 3^2$ ___ (4)

Step 5: Add 6 to each term in (3).

Step 6: Do as in step 3 but use equations (3) and (5) from page 49 instead.

You can proceed further and generate some more comparisons.

Exercise 3d

1. In 1998, 24 754 students in Singapore received Edusave Merit Scholarship Award ranging from \$250 for a Primary pupil to \$500 for a Secondary pupil. These awards are given to the top 10% of the pupils from each school. 6 of Mr Muthu's grandchildren received the Edusave Merit Scholarship Award in 1998. What is the greatest and smallest possible amount that could be paid out to Mr Muthu's grandchildren?

2. In December 1998, The Singapore Buddhist Lodge donated \$15 000 and 50 bursaries worth between \$250 and \$350 each to Jamiyah, the Muslim Missionary Society of Singapore. What is the maximum and minimum possible amount that the Lodge donated?

3. The youngest member of the extended Lim family is 5 years old and the eldest is 78. What are the possible ages of the other members of the Lim family?

4. The perimeter of a square is not more than 80 cm. What is the largest possible area of the square?

5. Joanne and Elvin intend to buy a birthday present for Carol. They decide that the cost of the present should not be more than \$20 and that Joanne will pay \$2 more than Elvin. What is the maximum amount that Elvin has to pay?

6. Ali scored 74, 82 and 60 for three of his Mathematics tests. What is the lowest mark he must score for his fourth test if he aims to achieve an average of at least 78 for the four tests?

7. A Mathematics competition consists of 30 multiple-choice questions. A correct answer is awarded 4 marks while 1 mark is deducted for a wrong answer. No marks will be awarded or deducted for questions not attempted. A student skipped 3 questions and had a score of more than 44. Find the minimum number of correct answers obtained.

8. Ghani and Samy shared a business venture with an initial capital of not more than \$60 000. If Ghani put in \$5 000 more than Samy, what is the maximum amount that Samy invested?

Linear Inequalities in One Variable

Given two or more linear inequalities which are connected by the word 'and', the solution(s) to each inequality must satisfy all the others simultaneously. In other words, only the *common* solutions of the inequalities should be considered.

Example 9

Solve the inequalities $3x + 4 > 13$ and $4x - 13 < 11$.

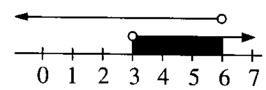
Solution

Solving the two inequalities separately, we get

$$\begin{aligned} 3x + 4 &> 13 \quad &\text{and} \quad 4x - 13 &< 11 \\ 3x &> 13 - 4 & 4x &< 11 + 13 \\ 3x &> 9 & 4x &< 24 \\ x &> 3 & x &< 6 \end{aligned}$$

The solutions satisfying both inequalities lie in the overlapping shaded region, i.e. $3 < x < 6$.

The solutions to the two inequalities are shown by the following number lines:



Example 10

Find the integer values of x for which $3x \leqslant x + 6$ and $2x + 4 < 3x + 6$.

Solution

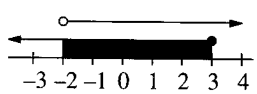
Solving the two linear inequalities separately, we have

$$\begin{aligned} 3x &\leqslant x + 6 \quad &\text{and} \quad 2x + 4 &< 3x + 6 \\ 3x - x &\leqslant 6 & 3x - 2x &> 4 - 6 \\ 2x &\leqslant 6 & x &> -2 \\ x &\leqslant 3 & & \end{aligned}$$

The solutions satisfying both inequalities are x, such that $-2 < x \leqslant 3$. The integer values of x in this range are -1, 0, 1, 2 and 3.

NB: -2 is not a solution to the inequality.

The solutions to the two inequalities are shown by the number lines and the shaded region below:



Example 11

Solve the following linear inequalities: $4x + 9 > x + 15$ and $2x + 17 \leqslant 11$.

Solution

$$\begin{aligned} 4x + 9 &> x + 15 \quad &\text{and} \quad 2x + 17 &\leqslant 11 \\ 4x - x &> 15 - 9 & 2x &\leqslant 11 - 17 \\ 3x &> 6 & 2x &\leqslant -6 \\ x &> 2 & x &\leqslant -3 \end{aligned}$$

There are no solutions.

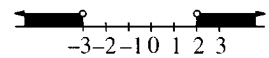
Example 12

Solve the inequalities $3x - 1 < x + 5 \leqslant 4x + 14$.

Solution

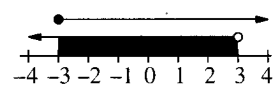
$$3x - 1 < x + 5 \quad \text{and} \quad 4x + 14 \geqslant x + 5$$
$$\text{i.e.} \quad 3x - x < 5 + 1 \qquad 4x - x \geqslant 5 - 14$$
$$2x < 6 \qquad 3x \geqslant -9$$
$$x < 3 \qquad x \geqslant -3$$

$\therefore$ the solution is $-3 \leqslant x < 3$.



For Example 11 the number line shows that the two arrows and the two shaded regions do not overlap, i.e., there is no solution satisfying both inequalities simultaneously. Hence, the linear inequalities $4x + 9 > x + 15$ and $2x + 17 \leqslant 11$ have no solution.

The number line for Example 12 is shown below



Example 13

Given that $3 \leqslant x \leqslant 7$ and $-5 \leqslant y \leqslant -1$, find

(a) the largest possible value of $2x - y$;
(b) the smallest possible value of xy;
(c) the largest possible value of $x^2 + y^2$.

Solution

(a) The largest possible value of $2x - y$ occurs when x is largest and y is smallest.
$\therefore$ the largest possible value of $2x - y = 2(7) - (-5) = 14 + 5 = 19$.

(b) The smallest possible value of xy occurs when x and y are smallest if both x and y are positive. In this case y is negative, thus the smallest possible value of xy occurs when xy is numerically greatest.
$\therefore$ the smallest value of $xy = 7 \times (-5) = -35$.

(c) The largest possible value of $x^2 + y^2$ occurs when both x and y are numerically largest.
$\therefore$ the largest value of $x^2 + y^2 = 7^2 + (-5)^2 = 49 + 25 = 74$.

By saying "numerically greatest", we mean the greatest absolute value.

Exercise 3e

Solve the following inequalities, illustrating each solution using the number line.

1. **(a)** $2x \leqslant 3x + 2$ **(b)** $-3 - x > 2x - 7$ **(c)** $18 - 3x > 5x - 4$
(d) $\frac{1}{2}x - 1 \geqslant \frac{1}{3}x + 4$ **(e)** $\frac{x}{5} - 4 < 2\frac{1}{4}x + 2$ **(f)** $\frac{3}{4}x + 3 \leqslant x - 6$

2. **(a)** $x - 4 \leqslant 3$ and $3x \geqslant -6$ **(b)** $2x + 5 < 15$ and $3x - 2 > -6$
(c) $x - 2 \leqslant 6$ and $4x - 3 > 18$ **(d)** $5x - 1 < 4$ and $3x + 5 \geqslant x + 1$
(e) $2x - 5 \geqslant 1$ and $3x - 1 > 26$ **(f)** $\frac{1}{2}x - 4 > \frac{1}{3}x$ and $\frac{1}{6}x + 1 > \frac{1}{8}x + 3$

3. **(a)** $-4 \leqslant 2x \leqslant 3x - 2$ **(b)** $1 - x < -2 \leqslant 3 - x$ **(c)** $-3 > x - 9 > 2x$
(d) $6 \geqslant 4x > 3x - 1$ **(e)** $2x \leqslant x + 6 < 3x + 5$ **(f)** $3x - 2 \geqslant 10 \geqslant x + 4$

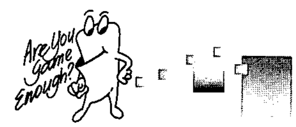
4. (a) $3 - x \leqslant x - 4 \leqslant 2x - 9$ (b) $1 - x < x - 1 < 11 - 2x$
(c) $x - 3 < 2x - 1 < 5 + x$ (d) $20 - 2x > 2x + 4 > 1 - x$
(e) $3x - 5 < x + 1 \leqslant 2x + 1$ (f) $x + 2 \geqslant 1 - 3x > x - 11$

5. (a) $\frac{x}{4} + 3 \leqslant 4 \leqslant \frac{1}{2}x + 6$ (b) $\frac{x}{3} \geqslant \frac{x}{2} + 1 \geqslant x - 1$
(c) $2(1 - x) > x - 1 \geqslant \frac{1}{2}(x - 7)$ (d) $10 \geqslant \frac{3x - 2}{2} \geqslant 4$
(e) $\frac{x - 2}{3} < \frac{2x + 1}{5} \leqslant 3$ (f) $\frac{x}{2} + \frac{1}{5} \geqslant \frac{2x}{5} > x - 5$

List the integer values of x which satisfy the following inequalities:

6. (a) $3x - 5 < 26 \leqslant 4x - 6$ (b) $3x + 2 < 19 < 5x - 4$
(c) $-4 \leqslant 7 - 3x \leqslant 2$ (d) $-10 < 7 - 2x \leqslant -1$
(e) $4x + 5 \leqslant 5x - 2 < 4x + 7$ (f) $7 \leqslant 2x + 1 \leqslant 21$

7. (a) $x + 1 < 27 < x + 4$ (b) $10 \leqslant 2x + 3 \leqslant 15$
(c) $3 < 5x - 1 \leqslant 29$ (d) $2x - 3 < x + 3 < 3x + 8$
(e) $\frac{1}{2}x + 6 < \frac{1}{4}x + 10 < x + 5$ (f) $\frac{1}{3}(x + 7) < \frac{1}{6}(x + 37) < x$



Bag A contains 2 black caps. Bag B contains 2 white caps, and Bag C contains 1 black cap and 1 white cap. However, Matt decides to switch around all the three labels on the bags. As a result, the three bags are now wrongly labelled. If Matt is allowed to draw one cap from any one of the three bags only once, and then look at its colour, which bag should he choose, so that he can determine the colours of the caps in each bag? Why?

8. Given that $-5 < 2x \leqslant 7$, write down
(a) the largest integer value of x;
(b) the smallest integer value of x;
(c) the largest rational value of x.

9. Given that $\frac{1}{2} < x \leqslant 15\frac{1}{2}$, write down
(a) the smallest prime number; (b) the largest prime number; (c) the smallest integer.

10. Given that $3 \leqslant x \leqslant 5$ and $-4 \leqslant y \leqslant -1$, find
(a) the greatest possible value of $x - y$; (b) the smallest possible value of xy;
(c) the largest possible value of $\frac{x}{y}$; (d) the smallest possible value of $(x^2 + y^2)$.

11. If x is an integer such that $200 < 3x^3 < 400$, find x.

12. If x is a positive integer such that $350 < 7x^2 < 450$, find x.

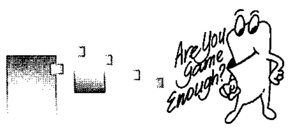
13. An integer x is such that $x + 2 < 5\sqrt{17} < x + 3$. State the value of x.

14. An integer k is such that $k + 1.5 < 6\sqrt{7} < k + 2.5$. State the value of k.

15. The length and breadth of a rectangle are x cm and y cm respectively. If the rectangle has an area of 24 cm^2, state the possible pairs of integer values of x and y, if $x > y$.

16. It is given that $6 \leqslant x \leqslant 8$ and $0.2 \leqslant y \leqslant 0.5$. If $z = \frac{x}{y}$, find the limits in which z must lie.

17. Evelyn intends to give her 12 good friends a treat at the school canteen. Her friends can have either a bowl of noodles or a plate of chicken rice. A bowl of noodles costs \$1.20 while a plate of chicken rice is \$1.50.
(a) What is the maximum and minimum amount of money she may have to spend?
(b) If she intends to spend not more than \$16 on the treat, what is the maximum number of plates of chicken rice her friends can have if two of them insist on having noodles?



1. *Can you find out the largest values for*
   (a) $x^2 + y^2$
   (b) $x^2 - y^2$
   if $-10 \leqslant x \leqslant 10$ *and* $-5 \leqslant y \leqslant 5$?
2. *Can you find out the smallest values for*
   (a) $x^2 + y^2$
   (b) $x^2 - y^2$
   if $-10 \leqslant x \leqslant 10$ *and* $-5 \leqslant y \leqslant 5$?

Summary

1. For any two numbers, x and y, only one of the following is true:

$$x > y, x = y \quad \text{or} \quad x < y$$

2. For any three numbers, x, y and z, if $x > y$ and $y > z$, then $x > z$.

3. We can add, or subtract, a number from both sides of an inequality without changing the inequality sign.

4. We can multiply, or divide, both sides of an inequality by a **positive** number without changing the inequality sign.

5. We have to change the inequality sign when we multiply, or divide, both sides of an inequality by a **negative** number.

Review Questions 3

1. Solve each of the following inequalities, illustrating your answer using the number line.

(a) $9 - 5x < 3$ **(b)** $2x + 1 < 24$ **(c)** $2x - 3 > 0.5x$
(d) $10x - 3 \leqslant 8$ **(e)** $0.4x - 5 \geqslant 0.2x$ **(f)** $2 + 5x > 25$
(g) $25 - 2x \geqslant 4$ **(h)** $-9 \leqslant 5 - 2x$ **(i)** $3 - 4x \leqslant -7$
(j) $3x - 4 < 29$ **(k)** $27 \leqslant 4x - 5$ **(l)** $3x + 2 \leqslant 93$

2. Solve each of the following inequalities:

(a) $3 + \frac{x}{4} > 5 + \frac{x}{3}$ **(b)** $\frac{4x}{9} - 5 < 3 - \frac{2x}{3}$

(c) $\frac{4x}{5} - \frac{3}{4} \geqslant x - \frac{1}{2}$ **(d)** $\frac{x + 4}{4} - \frac{3x - 5}{6} < \frac{x}{2} + \frac{1}{2}$

(e) $\frac{x - 2}{3} < \frac{2x + 3}{5} + \frac{5}{8}$ **(f)** $5 - \frac{2x - 5}{6} \leqslant \frac{x + 3}{2} + \frac{2(x + 1)}{3}$

3. Given that $x \leqslant 14\frac{1}{2}$, state the largest possible value of x if

(a) x is an integer; **(b)** x is a prime number; **(c)** x is a rational number.

4. Given that $27 - 2x \leqslant 8$, find

(a) the least value of x; **(b)** the least integer value of x.

5. Find **(a)** the largest integer x such that $3x < 27$;

(b) the smallest integer x such that $\frac{2x}{3} > 7$;

(c) the smallest prime number x such that $\frac{5x}{6} > 13$;

(d) the largest prime number k such that $3k + 2 < 23$.

6. List the integer values of x for which
   (a) $5x > 69 - 2x$ and $27 - 2x \geqslant 4$;
   (b) $-10 \leqslant x < -4$ and $2 - 5x < 35$.

7. Given that $-3 \leqslant x \leqslant 7$ and $4 \leqslant y \leqslant 10$, calculate
   (a) the smallest possible value of $x - y$;
   (b) the largest possible value of $\frac{x}{y}$;
   (c) the largest possible value of $x^2 - y^2$;
   (d) the smallest possible value of $x^3 + y^3$.

8. A piece of writing paper weighs 3 g and an envelope, 5 g. It costs 60 cents to send a letter weighing 20 g or less by air to Hong Kong. Mary has 60 cents worth of stamps. What is the maximum number of pieces of writing paper that she can use to write to her pen-pal in Hong Kong?

9. An apple costs 35 cents and a pear costs 55 cents. John has $20 and he has to buy at least 5 of each fruit. What is the maximum number of fruit he can buy with $20? How much change can he get back?

10. In 1998, 39 245 students in Singapore of all races received Good Progress Awards ranging from $150 for a Primary school pupil to $250 for a Secondary school pupil. These awards were given to the pupils who have shown great improvement in their grades from each school. 9 of the cousins of Fatimah received the Good Progress Award in 1998. What is the greatest and smallest possible amount that could be paid out to Fatimah's cousins?

Problem Solving

1. Solve the following inequalities:
   (a) $\frac{1}{4} + \frac{1}{3}x > 3x - \frac{1}{2}$
   (b) $\frac{1}{3}(4x - 3) > \frac{1}{2}(x + 5)$
   (c) $(x + 2)(x - 3) < 0$
   (d) $\frac{x + 2}{x - 3} > 0$

2. State whether each of the following statements is true or false. If your answer is 'false', offer an explanation to support your case.
   (a) If $a > b$ and both a and b are negative, then $\frac{a}{b} > 1$.
   (b) If $a > b$ and both a and b are negative, then $a^3 > b^3$.
   (c) If $a > b$ and both a and b are negative, then $\frac{b}{a} - \frac{a}{b}$ is positive.
   (d) If $a > 2$, then $\frac{1}{a} < \frac{1}{2}$.
   (e) If a and b are positive and $a < \frac{1}{b}$, then $b < \frac{1}{a}$.

3. If $-1 \leqslant x \leqslant 5$ and $2 \leqslant y \leqslant 6$, find the greatest and least possible values of
   (a) $y - x$;
   (b) $\frac{x}{y}$;
   (c) $\frac{x^2}{y}$.

4. Solve the inequality $|2x - 7| < 3$.

5. Find the range of values of x for which $(x + 2)(x - 5) > 0$.

6. Find the range of values of x for which $\frac{3x - 5}{(x - 7)^2} > 0, x \neq 7$.

7. Solve the inequality $|x^2 - 5x| < 6$.

8. Mary wants to buy at least 5 ball-point pens and more than twice as many pencils. Each ball-point pen costs 20 cents and each pencil costs 10 cents. If she has only $3, what is the maximum number of pencils and ball-point pens she can buy?

9. A farmer intends to buy more ducks and chickens. He has to pay $4 for each duck and $3 for each chicken. He has to spend $1 on the feed for each duck and $2 for each chicken after which he will be able to sell the ducks and chickens at $6 each. He has only $36 to buy the poultry and $20 to buy the feed. How many ducks and chickens must he buy, in order to obtain maximum profit?

10. A woman on a diet needs at least 10, 12 and 12 units of vitamins A, B and C respectively per day to satisfy her minimum dietary requirements. A liquid product selling for $3 per bottle contains respectively 5, 2 and 1 units of vitamins A, B and C per bottle. Another dry product selling for $2 per box contains respectively 1, 2 and 4 units of vitamins A, B and C per box. How many units of each product can she buy at a minimum cost to meet her dietary needs at the same time?

CHAPTER 4

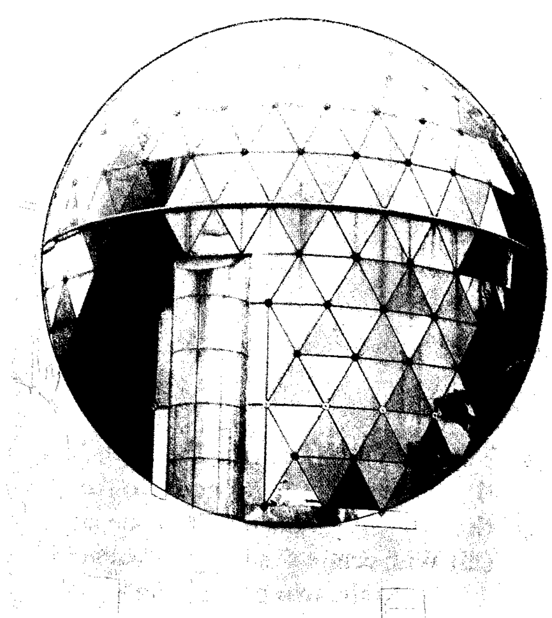
Congruent and Similar Triangles

In this chapter, you will learn how

- △ to test for the similarity/congruency between two triangles;
- △ to solve problems involving similar/congruent triangles.

Preliminary Problem

Look at the spherical structure in the photograph. Do you see any similar and congruent triangles in the design of this structure?



Congruent Triangles

In-Class Activity

You can carry out this activity yourself.

1. For two triangles to be congruent, all six pairs of corresponding measurements (i.e. three pairs of sides and three pairs of angles) must be equal. However, given any two triangles, do we need to consider all the six pairs of measurements in order to determine whether they are congruent? If the answer is no, then what is the minimum number of pairs of measurements to be considered and what are they?

2. Construct a triangle using the three lengths, l, m, n, as given in Fig. 4.1. Compare the triangle you have constructed with those constructed by your classmates by placing one triangle on top of another, or by using a protractor to measure all the angles of the two triangles.

   Are the triangles congruent? If the answer is yes, can we thereby say that any two triangles that have three pairs of equal sides are congruent?

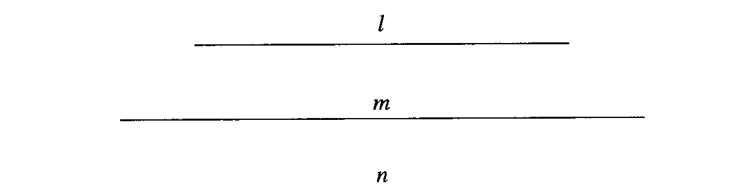


Fig. 4.1

3. Two lengths l and m and one angle $n°$ are as shown in Fig. 4.2 below.

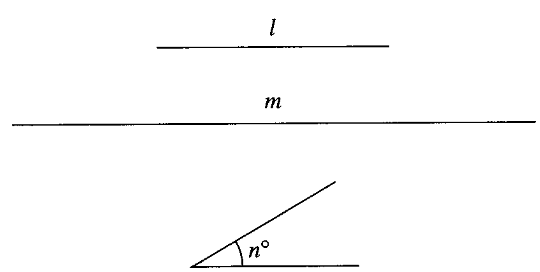


Fig. 4.2

   (a) Using the given lengths and angle, construct a triangle ABC as follows:
   - **(i)** Draw the base BC equal to the length m.
   - **(ii)** Using a protractor, draw the angle, at B, equal to $n°$.
   - **(iii)** With centre C and radius equal to the length l, draw an arc to cut the arm other than BC. Label the two points A and A' as shown in Fig. 4.3 on the next page.
   - **(iv)** Join AC and $A'C$.

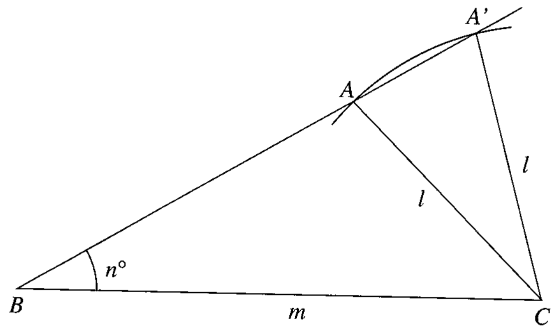


Fig. 4.3

Do you agree that triangles ABC and $A'BC$ have two pairs of sides of equal lengths and a pair of equal angles?

Are $\triangle ABC$ and $\triangle A'BC$ congruent?

Do you agree that two triangles having two pairs of sides of equal lengths and a pair of equal angles are not necessarily congruent?

(b) Using the same measurements given in Fig. 4.2 earlier, construct another triangle PBC as follows:

(i) Repeat steps **(i)** and **(ii)** in **(a)**.

(ii) With centre B and radius equal to the length l, draw an arc to cut the arm other than BC. Label the point P as shown in Fig. 4.4 below.

(iii) Join PC.

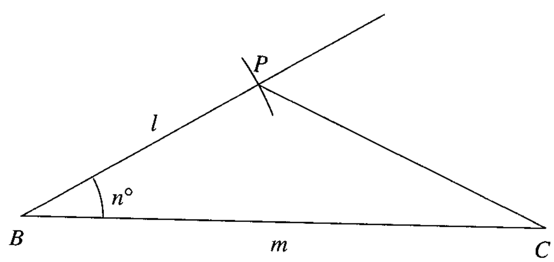


Fig. 4.4

Is the constructed triangle PBC congruent to the triangle in Fig. 4.4 and to the triangles constructed by your classmates?

Do you notice that the construction procedures in **(b)** assure that the given angle $n°$ is included between the two given lengths l and m and thus, produce triangles that are congruent?

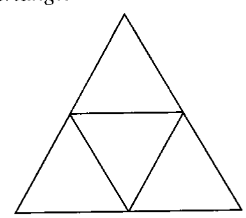
Can we conclude that two triangles having two pairs of sides of equal lengths and a pair of equal **included** angles are congruent?

Congruency Tests

From the discussion earlier, we derive two tests for congruency of two triangles.

Just for Fun

The diagram below shows four equilateral triangles formed by using 9 toothpicks. By removing 3 toothpicks and rearranging the figure, can you form 4 congruent triangles?



1. **SSS Property:** In $\triangle ABC$ and $\triangle PQR$, if $AB = PQ$, $AC = PR$ and $BC = QR$, then $\triangle ABC \equiv \triangle PQR$ (see Fig. 4.5).

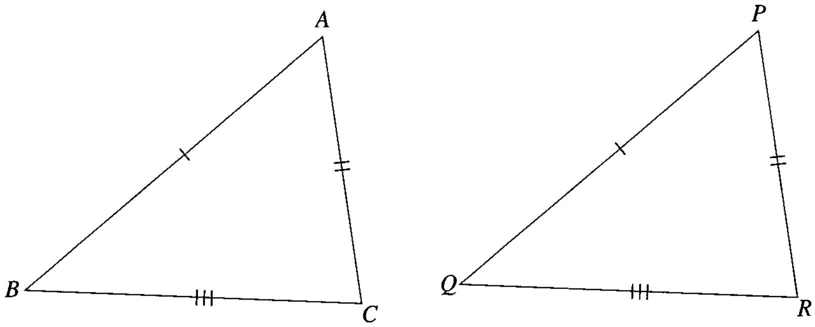


Fig. 4.5

This property is commonly referred to as the SSS property of congruent triangles and is stated as follows:

If the three sides of one triangle are equal to the three sides of the other triangle, then the two triangles are congruent.

Note: When congruent triangles are named, the letter must be written in the correct order so that it becomes clear which letters of the two triangles correspond to each other.

2. **SAS Property:** In $\triangle ABC$ and $\triangle PQR$, if $AB = PQ$, $AC = PR$ and $\hat{A} = \hat{P}$, then $\triangle ABC \equiv \triangle PQR$ (see Fig. 4.6)

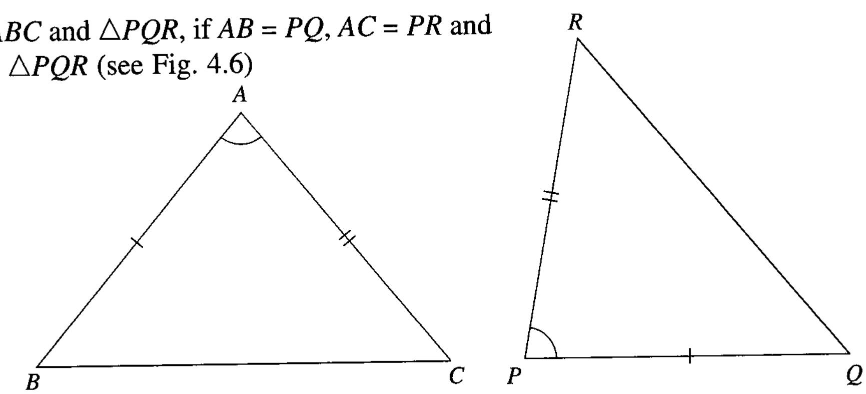


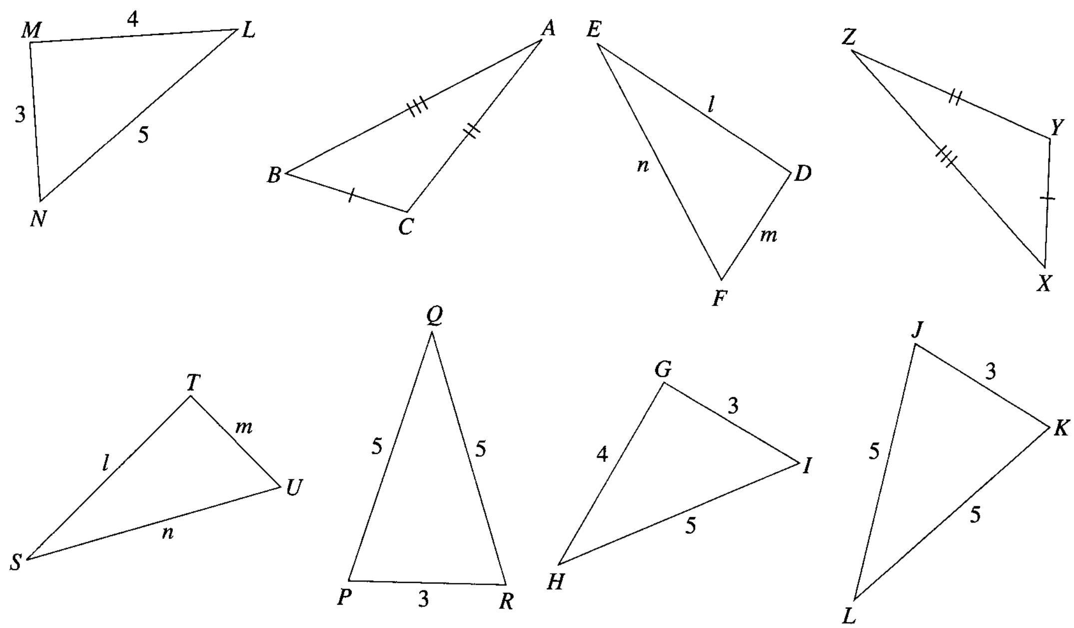
Fig. 4.6

This property is commonly referred to as the SAS property of congruent triangle and can be stated as follows:

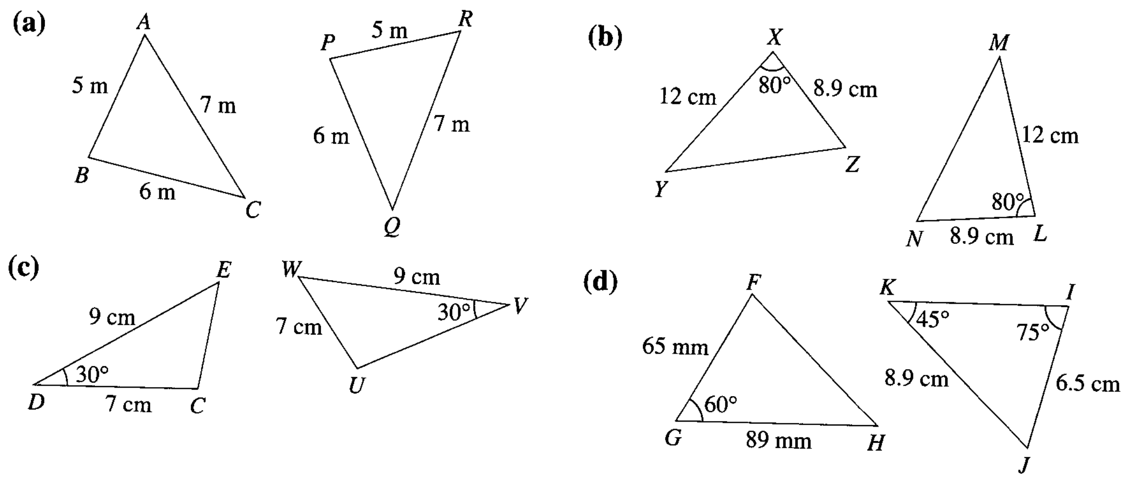
If two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle, then the two triangles are congruent.

Exercise 4a

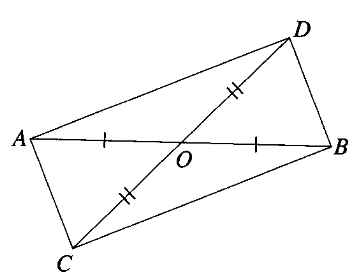
1. State which pairs of triangles are congruent.



2. For each pair of triangles (not drawn to scale), decide whether they are congruent or not. If they are congruent, state the case of congruency.



3. In the diagram, AOB and COD are straight lines, $AO = BO$ and $CO = DO$.

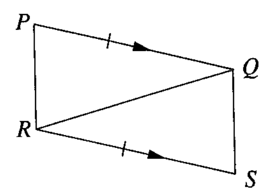


(a) Identify a pair of congruent triangles and state the case of congruency.
(b) Write down two pairs of equal angles.

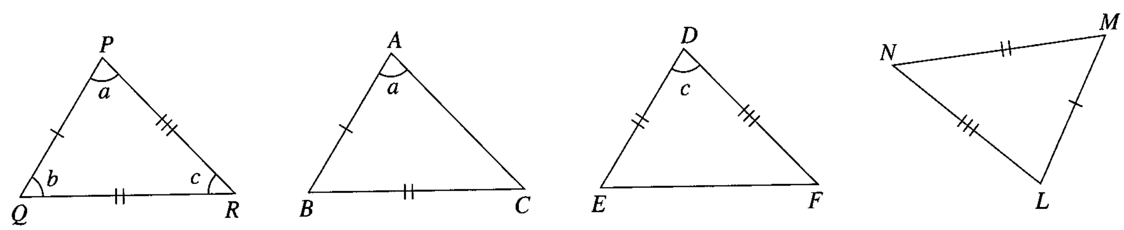
4. In the diagram, PQ is equal and parallel to RS.

(a) Identify two congruent triangles and state the case of congruency.

(b) If $PR = 5$ cm and $Q\hat{S}R = 50°$, find the length QS and $Q\hat{P}R$.



5. Identify the triangles that are congruent to triangle PQR.



6. Copy and complete the following:

In $\triangle ABC$ and $\triangle ADC$,

$AB =$ ______

$BC =$ ______

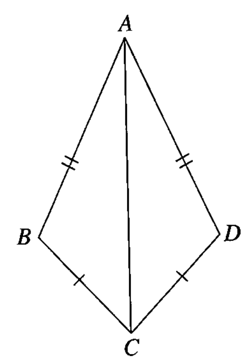
$AC =$ ______

$\therefore \triangle ABC \equiv$ ______ (SSS)

$\therefore B\hat{A}C =$ ______,

$\hat{B} =$ ______ and

$A\hat{C}B =$ ______



7. Copy and complete the following:

In $\triangle PQR$ and $\triangle SPT$,

$PQ =$ ______

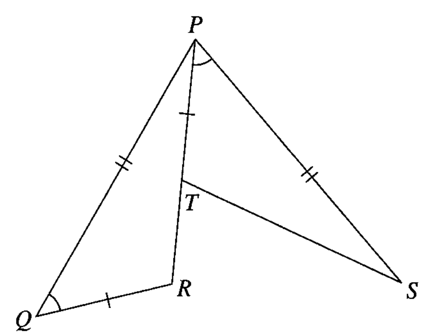
$P\hat{Q}R =$ ______

$QR =$ ______

$\therefore$ ______ $\equiv \triangle SPT$ (______)

$\therefore Q\hat{P}R =$ ______,

$P\hat{R}Q =$ ______ and ______ $= ST$.



In-Class Activity

1. 40°, 60° and 80° are the angle measurements of the three angles of a triangle. Using any two of the measurements and a length of 4 cm, three possible triangles can be constructed. Do they all have the same angles? Are they congruent?

2. Construct the three triangles mentioned in **1**. Compare your triangles with those in Fig. 4.7 below, and with those constructed by your classmates.

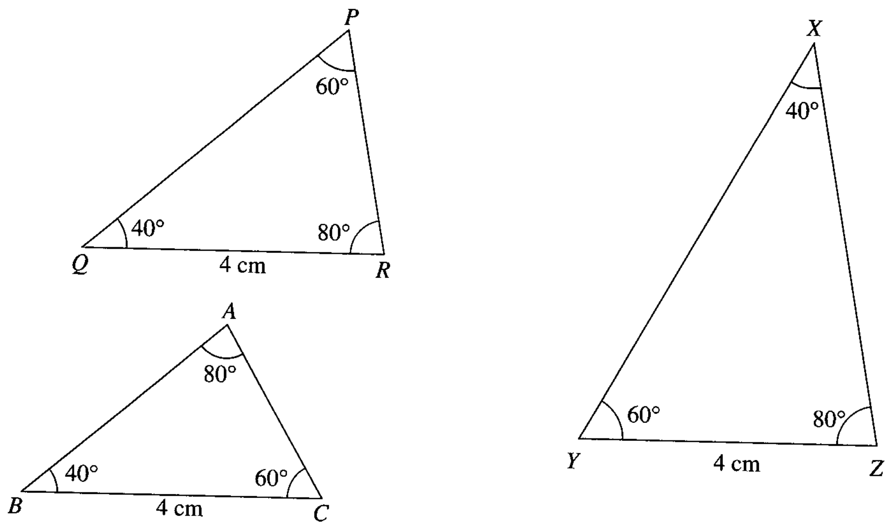


Fig. 4.7

3. Do you notice that the triangles constructed by you and your classmates, having the 4-cm length opposite the same angle, are congruent? The sides opposite the same angle are called **corresponding sides**.

4. Through this investigation, do you agree that two triangles having two pairs of equal angles and a pair of equal corresponding sides are congruent?

5. Fig. 4.8 below shows two right-angled triangles ABC and PQR, in which angle B = angle $Q = 90°$. The sides AC and PR, opposite the right angles B and Q respectively, are **hypotenuses**.

If $AB = PQ$ and $BC = QR$, are the triangles ABC and PQR congruent? If the answer is yes, state the test of congruency.

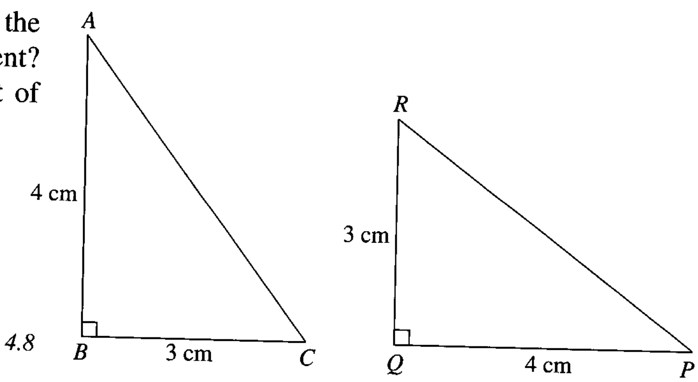


Fig. 4.8

6. If, instead, $AC = PR$ and **either** $AB = PQ$ **or** $BC = QR$, are the triangles ABC and PQR congruent?
   (a) Construct $\triangle LMN$ in which angle $M = 90°$, $MN = 4$ cm and the hypotenuse $LN = 7$ cm.
   (b) Compare your triangle with those constructed by your classmates. Are they congruent?
   (c) Are you surprised at the outcome, recalling that the test of congruency SAS involves two pairs of equal sides and a pair of equal included angles? As a challenge, you may want to refer to Pythagoras' theorem, which was discussed in Book 2, for further investigation.

From the above discussion, we derive two more tests for congruency of two triangles.

7. AAS Property: In $\triangle ABC$ and $\triangle PQR$, if $\hat{A} = \hat{P}$, $\hat{B} = \hat{Q}$ and $BC = QR$, then $\triangle ABC \equiv \triangle PQR$ (see Fig. 4.9(a))

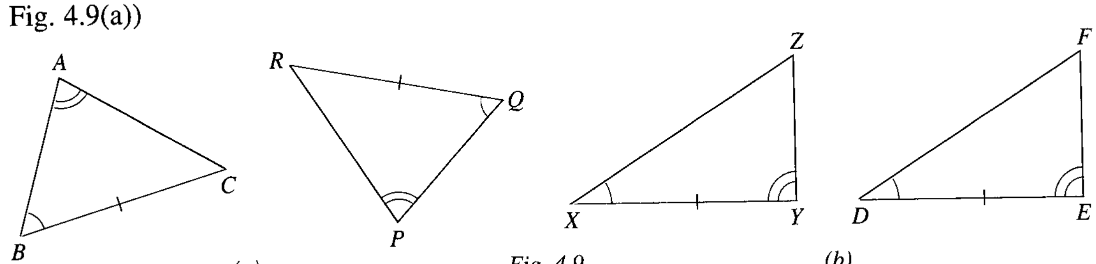


Fig. 4.9

Similarly, in $\triangle XYZ$ and $\triangle DEF$, if $\hat{X} = \hat{D}$, $\hat{Y} = \hat{E}$ and $XY = DE$, then $\triangle XYZ \equiv \triangle DEF$ (see Fig. 4.9(b)).

This property is commonly referred to as the **AAS** property of congruent triangles and can be stated as follows:

If two angles and a side of one triangle are equal to two angles and the corresponding side of the other triangle, then the two triangles are congruent.

8. RHS Property: In $\triangle ABC$ and $\triangle PQR$, if $\hat{B} = \hat{Q} = 90°$, $AC = PR$ and $AB = PQ$, then $\triangle ABC \equiv \triangle PQR$ (see Fig. 4.10).

This property is commonly referred to as the **RHS** property of congruent right-angled triangles and can be stated in this way:

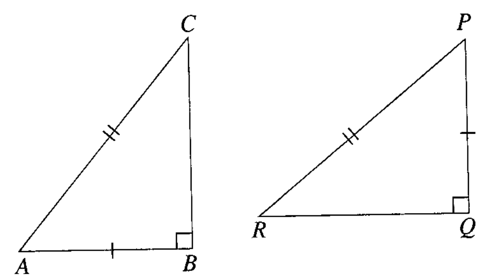
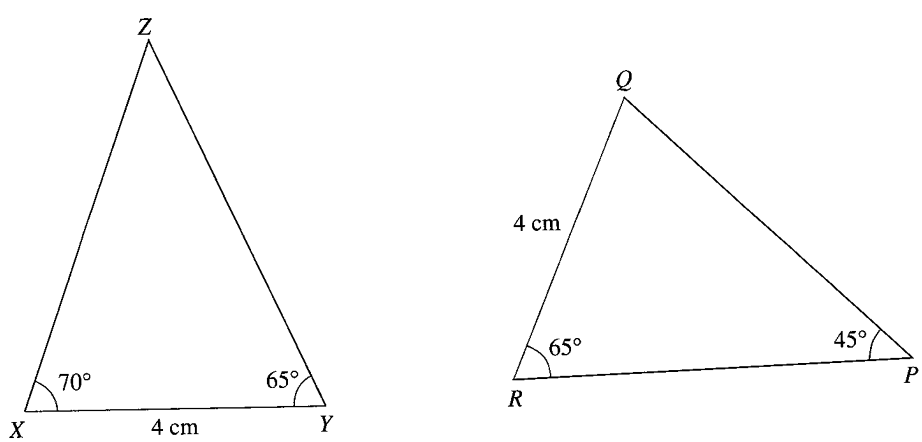


Fig. 4.10

If the hypotenuse and one side of one right-angled triangle are equal to the hypotenuse and one side of the other right-angled triangle, then the two right-angled triangles are congruent.

Exercise 4b

1.

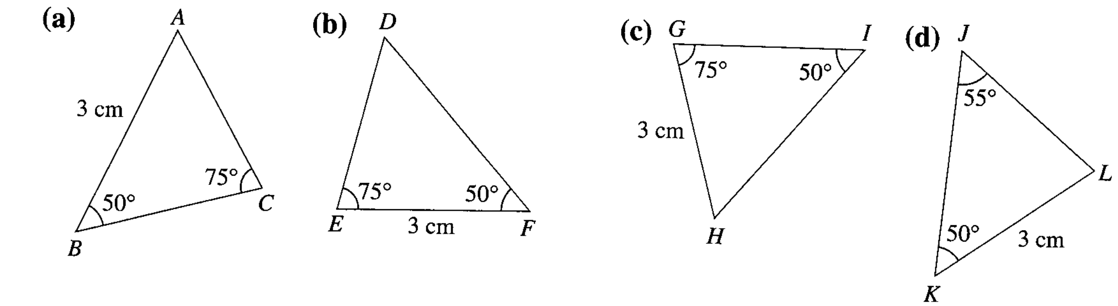


(a) Are XY and QR corresponding sides? Why?

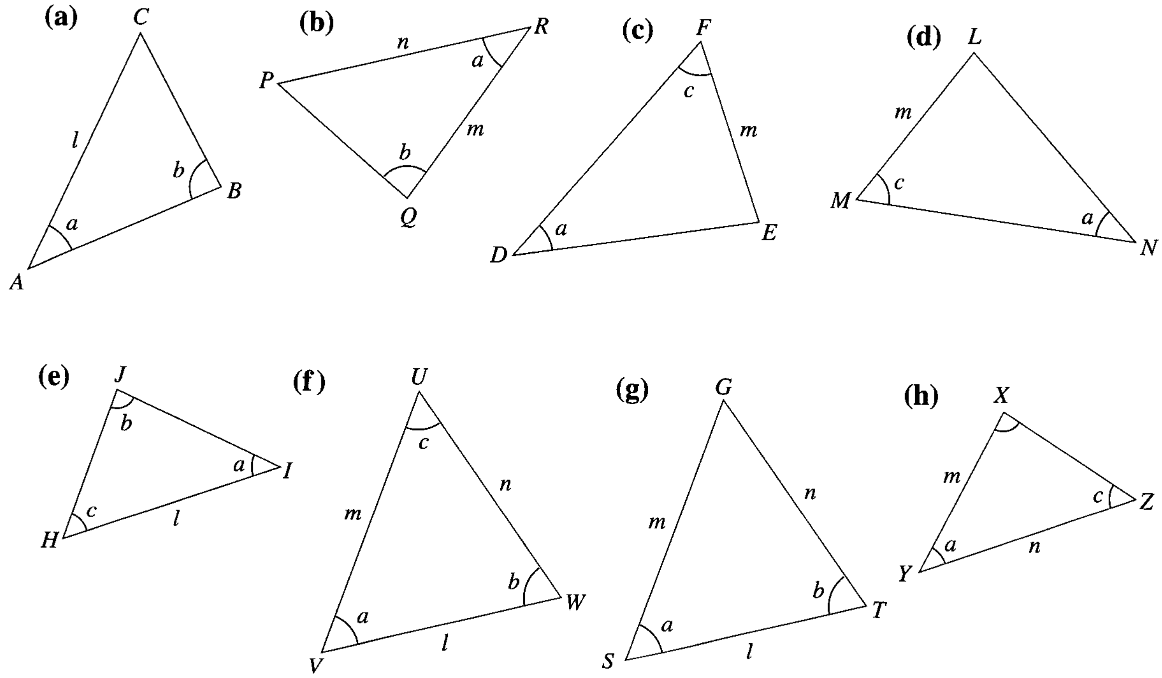
(a) Are XY and QR corresponding sides? Why?

(b) State whether the following statement is true or false: "$\triangle XYZ$ and $\triangle QRP$ are congruent." Give a reason to support your answer.

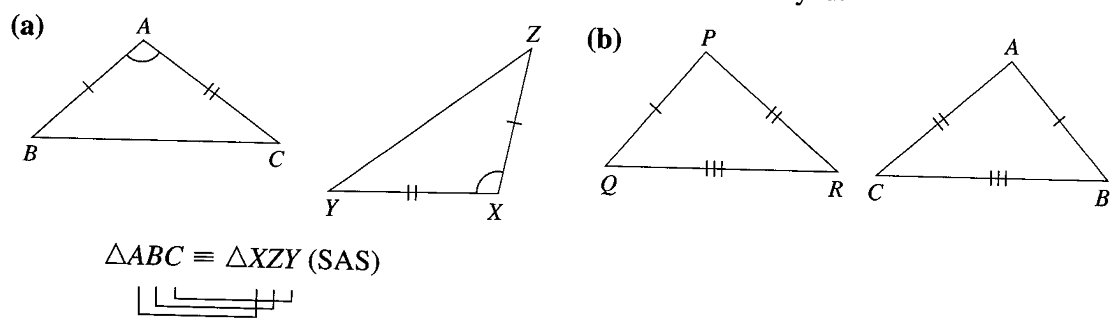
2. Pick out a pair of congruent triangles from those shown below:



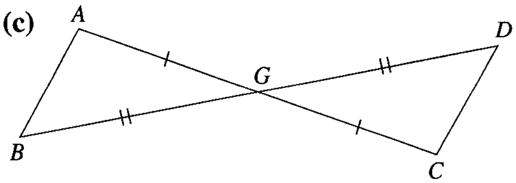
3. State which pairs of triangles are congruent, given that $\hat{a} + \hat{b} + \hat{c} = 180°$.

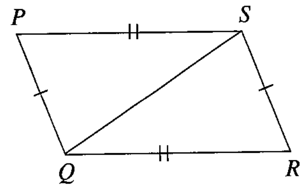


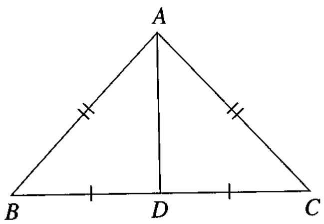
4. Identify the triangles that are congruent. Keep the letters in the correct order and state the case of congruency (i.e., SSS, SAS, AAS or RHS). **(a)** has been done for you.

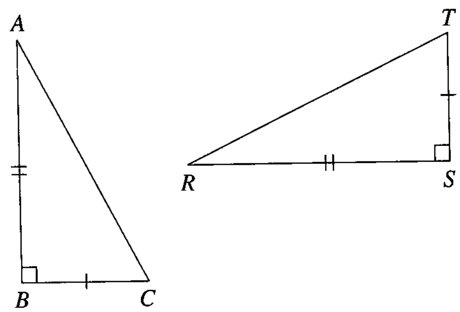


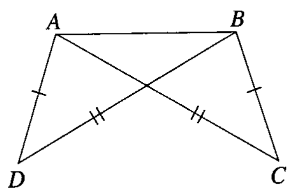
$\triangle ABC \equiv \triangle XZY$ (SAS)

(c)


(d)


(e)


(f)


(g)


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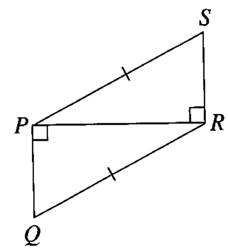
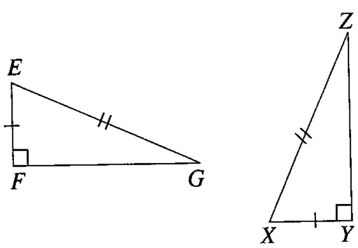
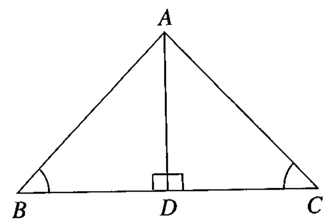
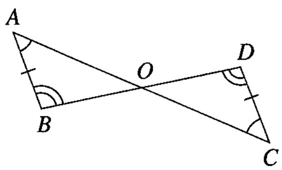
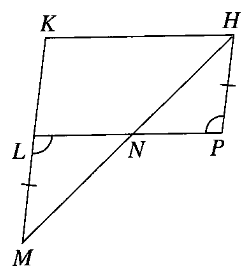


(i)


(j)


(k)


(l)


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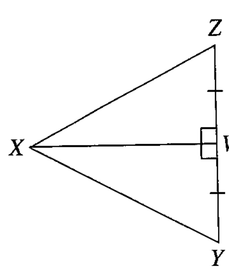

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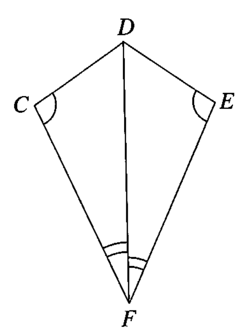
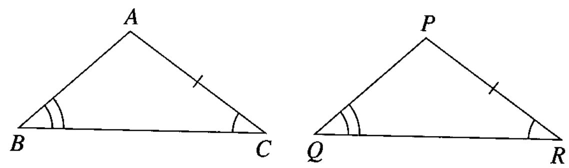


Exercise 4c

1. Copy and complete the following:



In $\triangle ABC$ and $\triangle PQR$,

$A\hat{B}C = P\hat{Q}R$

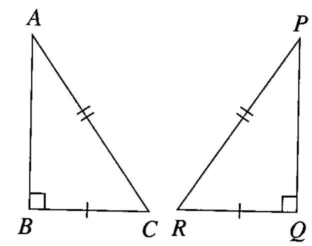
$A\hat{C}B =$ _____

$AC =$ _____

$\therefore \quad \triangle ABC \equiv$ _____ (AAS)

$\therefore \quad B\hat{A}C =$ _____, $AB =$ _____ and $BC =$ _____.

2. Copy and complete the following:



In $\triangle ABC$ and $\triangle PQR$,

$A\hat{B}C =$ _____

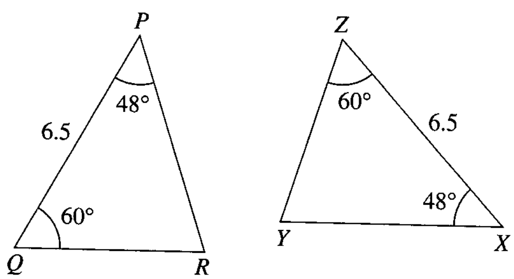
$AC =$ _____

$BC =$ _____

$\therefore \quad \triangle ABC \equiv$ _____ (_____)

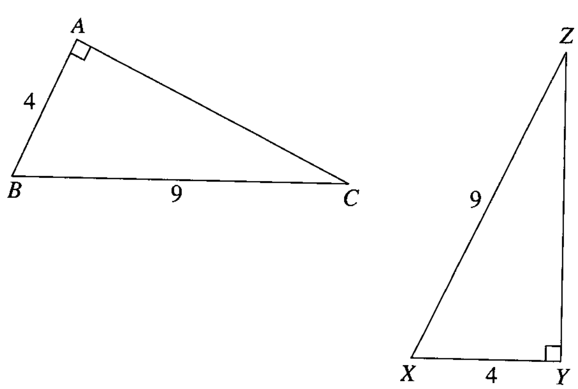
$\therefore \quad B\hat{A}C =$ _____, $A\hat{C}B =$ _____ and $AB =$ _____.

3.



(a) Show that the two triangles are congruent.
(b) State the other equal pairs of angles and sides.
(Use questions **1** and **2** as models.)

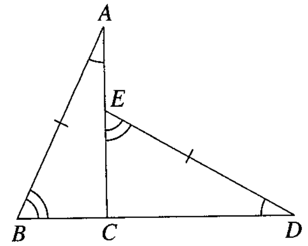
4.



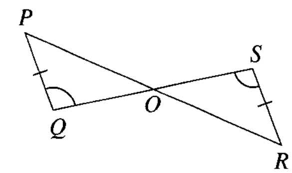
(a) Show that the two triangles are congruent.
(b) State the other equal pairs of angles and sides.

5. Study each of the following pairs of triangles carefully. If the triangles are congruent, state the case of congruency. Also, name the other three pairs of equal measurements.

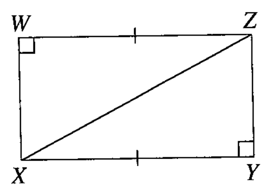
(a)



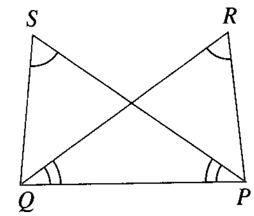
(b)



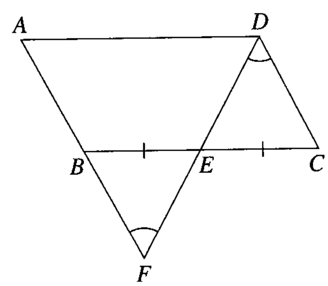
(c)



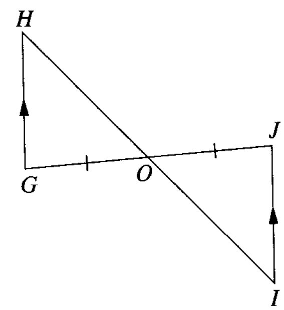
(d)



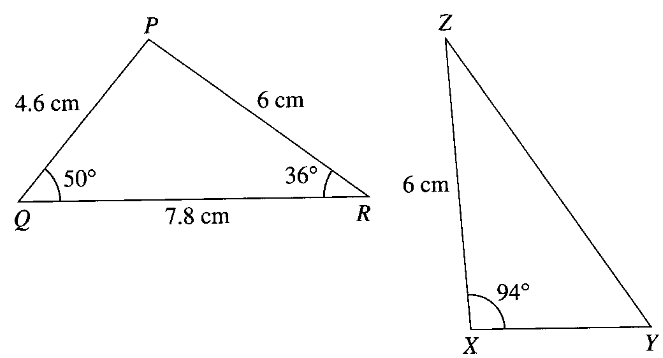
(e)



(f)



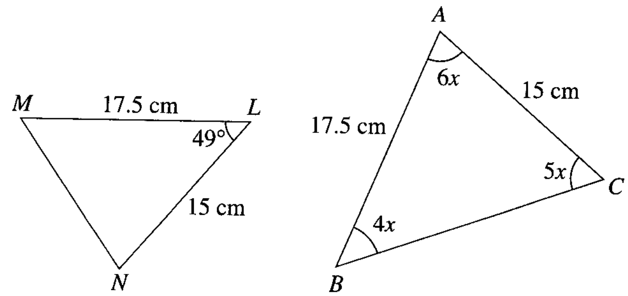
6.



Given $\triangle PQR \equiv \triangle XYZ$, write down

(a) the length of XY;
(b) the length of ZY;
(c) the value of $\hat{P}$;
(d) the value of $\hat{Z}$;
(e) the value of $\hat{Y}$.

7. **(a)** Find the value of x.
(b) What is the value of $\hat{A}$?
(c) Is $\triangle ABC \equiv \triangle LMN$? Give reasons to support your answer.



Simple Applications of Congruent Triangles

The following examples explain how we can use the ideas of congruent triangles to solve different problems.

Just For Fun

How many ways can you cut a square piece of cake into two congruent parts?

Example 1

In Book 1, we learnt how to construct the bisector of a given angle. In Fig. 4.11 below, the bisector of $P\hat{O}Q$ is constructed using the method learnt in Book 1. Here we use congruent triangles to show why the method works.

Solution

In $\triangle ROT$ and $\triangle SOT$,
$OR = OS$ (why?), $RT = ST$ (why?)
and OT is common to both triangles.

$\therefore$ by the SSS property, $\triangle ROT \equiv \triangle SOT$.

Thus, $\hat{a} = \hat{b}$ and OT is the angle bisector of $P\hat{O}Q$.

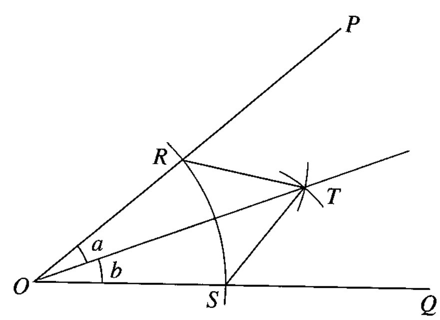


Fig. 4.11

Example 2

To measure the width of the internal trough, AB, of a machine tool which cannot be measured directly, we make use of a device as shown in Fig. 4.12. The device is made up of two parts, AA′ and BB′, hinged halfway at O. By measuring the distance A′B′, we can then obtain the length of AB. Why?

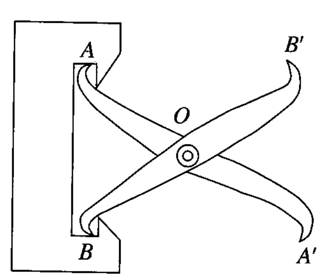


Fig. 4.12

Solution

Fig. 4.13 is obtained by joining AA' and BB'.

In $\triangle OAB$ and $\triangle OA'B'$,

$OA = OA'$, $OB = OB'$

and $\hat{1} = \hat{2}$ (why?)

$\therefore$ by the SAS property, $\triangle OAB \equiv \triangle OA'B'$.

Thus, $AB = A'B'$.

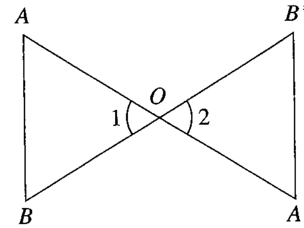


Fig. 4.13

Example 3

In Fig. 4.14, to measure the width, AB, of a river, a man walks from point A along the river bank to point C, such that AC is perpendicular to AB. He plants a pole at D, the mid-point of AC. From C, he walks along a path perpendicular to AC and stops at E, such that points E, D and B lie along a straight line. By measuring the distance CE, he will be able to know the width (AB) of the river. Why?

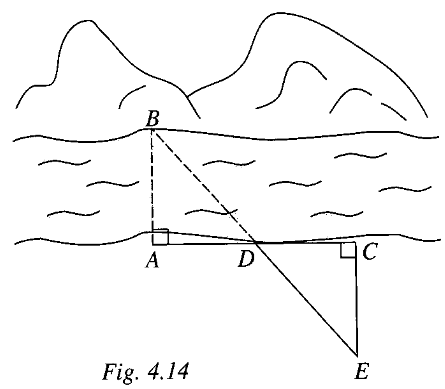


Fig. 4.14

Solution

In $\triangle BAD$ and $\triangle ECD$.

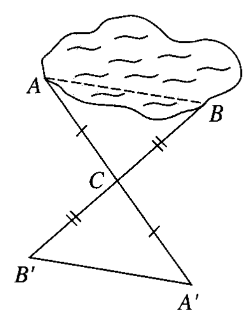
$B\hat{A}D = E\hat{C}D$, $A\hat{D}B = C\hat{D}E$ (why?) and $AD = CD$ (why?)

$\therefore$ by the AAS property, $\triangle BAD \equiv \triangle ECD$.

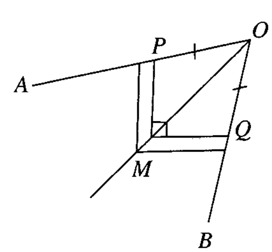
Thus, $AB = CE$.

Exercise 4d

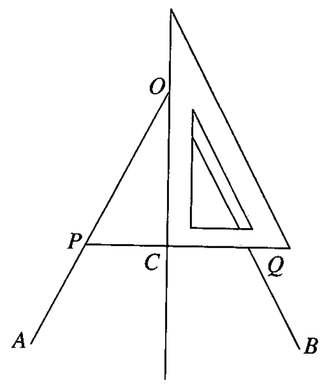
1. The diagram below illustrates how the length AB (which cannot be measured directly) of a pond is measured. Choose a point C and measure the length of AC and the length of BC. Produce AC and BC to A' and B' respectively, so that $CA' = AC$ and $CB' = BC$. By measuring the length of $B'A'$, we will be able to find the length of AB. Why?



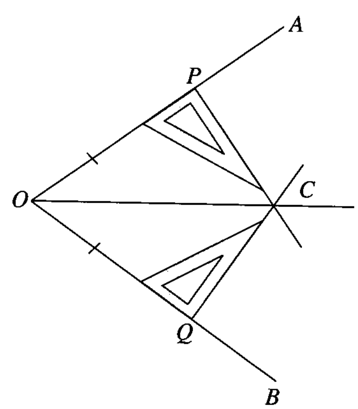
2. Using a set square, we can bisect a given angle. In the diagram, P and Q are marked along the arms, OA and OB, respectively of $A\hat{O}B$, such that $OP = OQ$. Move a 90° – 45° – 45° set square away from O until the 45° edges coincide with P and Q as shown in the diagram. Prove that OM is the angle bisector of $A\hat{O}B$.



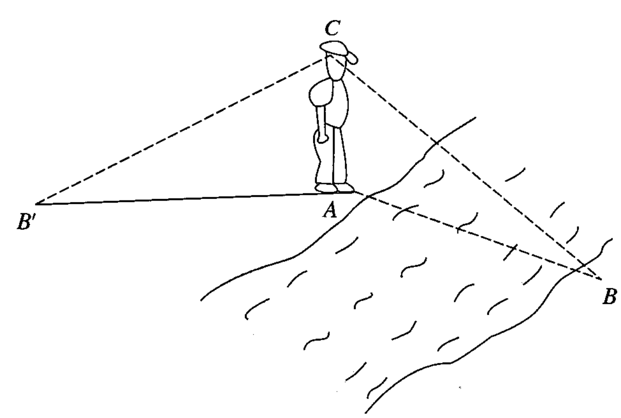
3. In the diagram shown below, P is on OA and Q is on OB, such that $OP = OQ$. Place a set square with one side along PQ and another side passing through O, as shown in the diagram. Prove that OC is the angle bisector of $A\hat{O}B$.



4. In the diagram below, P and Q are points along the arms, OA and OB respectively, of the angle AOB, such that $OP = OQ$. A set square is used to construct perpendiculars to OA and to OB at P and Q respectively. The perpendiculars meet at C. Prove that OC is the angle bisector of $A\hat{O}B$.



5. The diagram shows a man standing at a point, A, along a river bank. He looks directly across to the opposite bank, adjusting his cap so that his line of vision CB passes through the lowest point at the rim of his cap and falls on the point B. He then turns around without moving his head. His new line of vision CB' through the lowest point at the rim of his cap now falls on a point B' on the same side of the river. State which measurement he can make in order to find the width (AB) of the river. Why?



Similar Triangles

In Book 2, we learnt that two polygons are similar if

(1) all the corresponding angles are equal;
(2) all the corresponding sides are proportional.

Hence, two triangles ABC and PQR are similar if, in $\triangle ABC$ and $\triangle PQR$,

(1) $\hat{A} = \hat{P}$, $\hat{B} = \hat{Q}$, $\hat{C} = \hat{R}$ and

(2) $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = k$, where k is a constant (see Fig. 4.15).

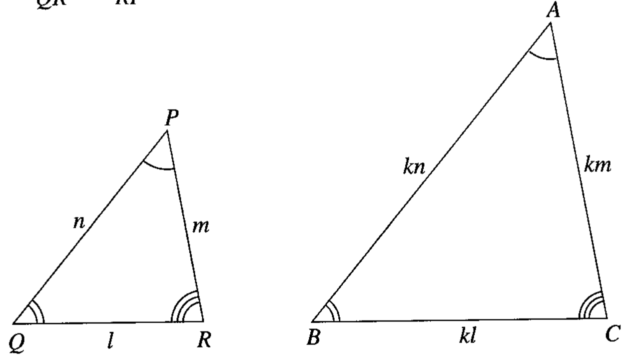


Fig. 4.15

Conditions **(1)** and **(2)** are necessary to ensure similarity for general polygons. However, we do not need both conditions to determine similarity between two triangles. Remember that we only require three pairs of measurements to determine congruency between two triangles although two congruent triangles have altogether six pairs of equal corresponding measurements. In fact, we can use tests for congruency between two triangles to establish the tests for similarity between two triangles.

In-Class Activity

1. (a) Are two congruent triangles necessarily similar?

(b) Recall that if two triangles ABC and PQR are congruent, then

(1) $\hat{A} = \hat{P}$, $\hat{B} = \hat{Q}$, $\hat{C} = \hat{R}$ and

(2) $AB = PQ$, $BC = QR$, $CA = RP$.

(c) Can we say that the statement "$AB = PQ$, $BC = QR$, $CA = RP$" in **(2)** above is equivalent to "$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = 1$"?

(d) Do you agree that two congruent triangles must be similar?

2. (a) Are two similar triangles necessarily congruent?

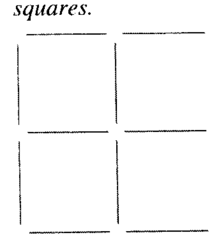
(b) Referring to the statement "$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = k$, where k is a constant", must the value of k be 1 for a pair of similar triangles ABC and PQR?

(c) Do you agree that two similar triangles may not be congruent?

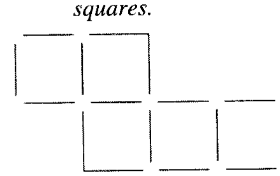
Just for Fun

1. 12 toothpicks are arranged as shown.

By shifting only 2 toothpicks, form a figure with (a) six squares; (b) seven squares.



2. The diagram below shows 16 toothpicks arranged to form five squares.



By shifting only 2 toothpicks, form a figure with four squares.

3. Consider the two congruent triangles ABC and PQR in Fig. 4.16(a). The sides of $\triangle PQR$ are enlarged proportionately in Fig. 4.16(b).

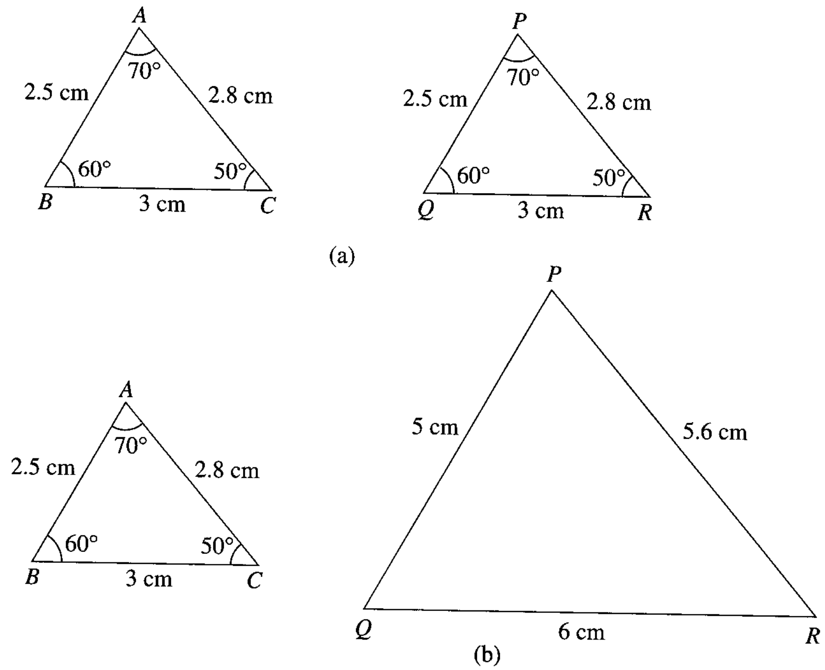


Fig. 4.16

4. Answer the following questions referring to the triangles ABC and PQR in Fig. 4.16(b).
   (a) Measure $\hat{P}$, $\hat{Q}$, and $\hat{R}$ in the enlarged $\triangle PQR$.
   (b) Do the values of the angles remain unchanged?
   (c) What is the value of the ratios $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$?
   (d) Can we conclude that two congruent triangles will remain similar when we enlarge or reduce the sides of one triangle using the same ratio?

5. **(a)** We know that two triangles ABC and PQR are congruent if their corresponding sides are equal, i.e., $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = 1$. **(SSS Property)**
   (b) Based on the discussion in **4**, do you agree that two triangles ABC and PQR are similar if the **ratios** of their corresponding sides are equal, i.e., $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = k$, where k is a constant?
   (c) We know that two triangles ABC and PQR are congruent if their two sides and the included angle are equal, i.e., $\frac{AB}{PQ} = \frac{BC}{QR} = 1$ and $\hat{B} = \hat{Q}$. **(SAS Property)**
   (d) Do you agree that two triangles ABC and PQR are similar if the **ratios** of two sides and the included angle are equal, i.e., $\frac{AB}{PQ} = \frac{BC}{QR} = k$, where k is a constant and $\hat{B} = \hat{Q}$?
   (e) We know that two triangles ABC and PQR are congruent if they have two angles and a corresponding side equal, i.e, $\hat{B} = \hat{Q}$, $\hat{C} = \hat{R}$ and $\frac{AB}{PQ} = 1$. **(AAS Property)**
   (f) Do you agree that two triangles ABC and PQR are similar if $\hat{B} = \hat{Q}$, $\hat{C} = \hat{R}$ and $\frac{AB}{PQ} = k$, where k is a constant?

(g) Did you notice that "$\frac{AB}{PQ} = k$" in **(f)** above is redundant? Do you know why?

(h) Can we simply state that two triangles ABC and PQR are similar if they have two angles equal, i.e. $\hat{B} = \hat{Q}$, $\hat{C} = \hat{R}$?

Tests For Similarity Between Two Triangles

From the above, we derive the following tests for similarity between two triangles.

1. In $\triangle ABC$ and $\triangle PQR$, if $\hat{A} = \hat{P}$ and $\hat{B} = \hat{Q}$, then $\triangle ABC$ and $\triangle PQR$ are similar (see Fig. 4.17).

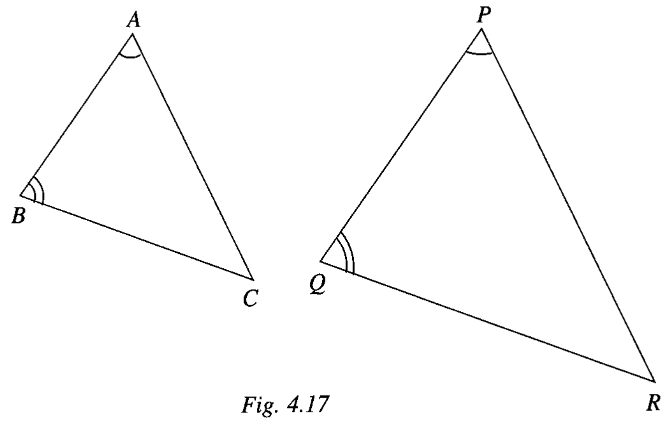


Fig. 4.17

2. In $\triangle ABC$ and $\triangle PQR$, if $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = \frac{1}{k}$, where k is a constant, then $\triangle ABC$ and $\triangle PQR$ are similar (see Fig. 4.18).

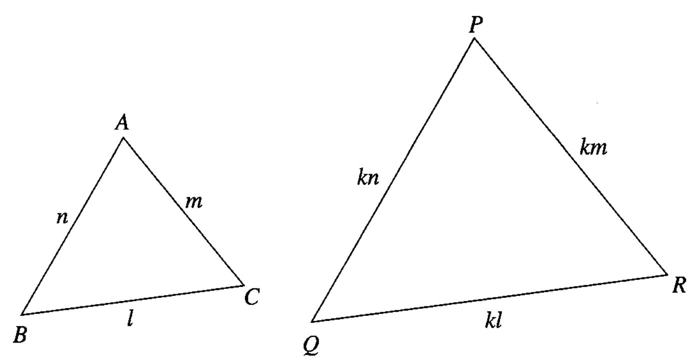


Fig. 4.18

3. In $\triangle ABC$ and $\triangle PQR$, if $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{1}{k}$ and $\hat{B} = \hat{Q}$, then $\triangle ABC$ and $\triangle PQR$ are similar (see Fig. 4.19).

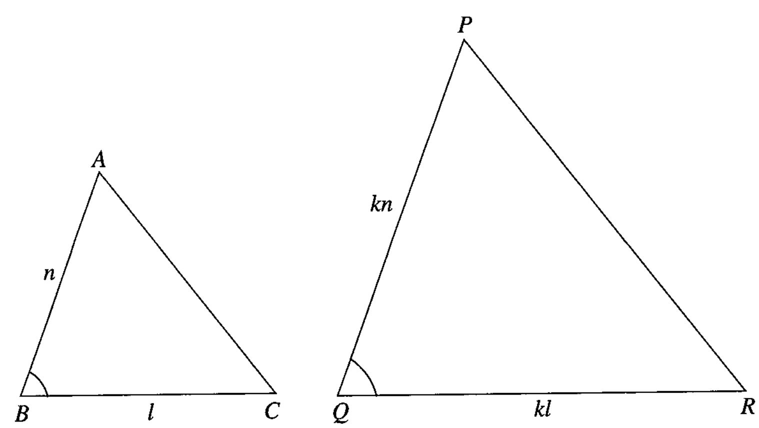
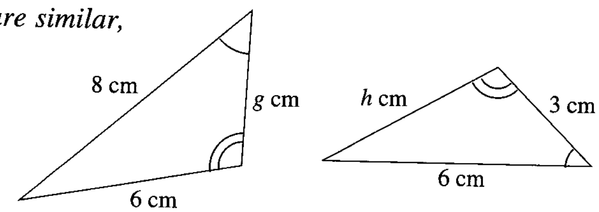


Fig. 4.19

Example 4

Given that the two triangles on the right are similar, find the values of g and h.



Solution

Since the two triangles are similar, we have

$$\frac{g}{3} = \frac{8}{6} = \frac{6}{h} \quad \text{or} \quad \frac{3}{g} = \frac{6}{8} = \frac{h}{6}$$

Using $\frac{g}{3} = \frac{8}{6}$,

$$g = \frac{8}{6} \times 3 = 4$$

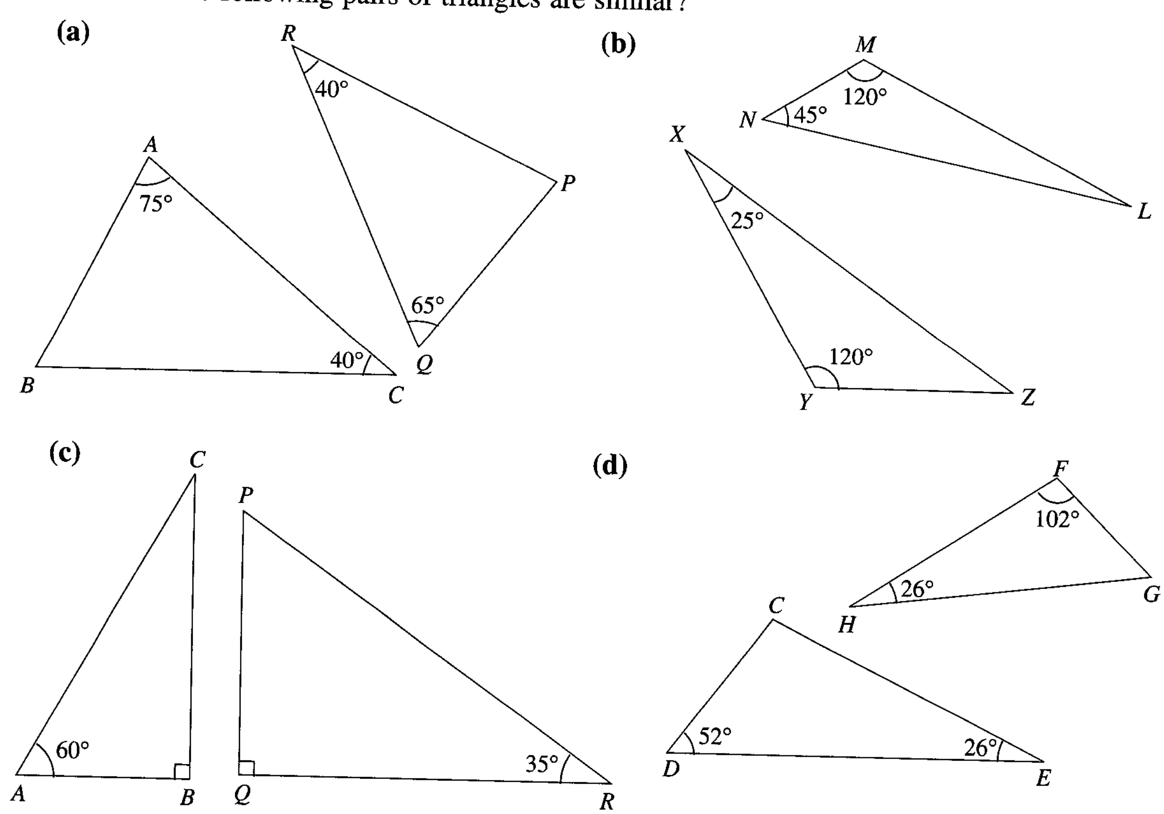
Using $\frac{6}{8} = \frac{h}{6}$,

$$h = \frac{6}{8} \times 6 = 4\frac{1}{2}$$

Exercise 4e

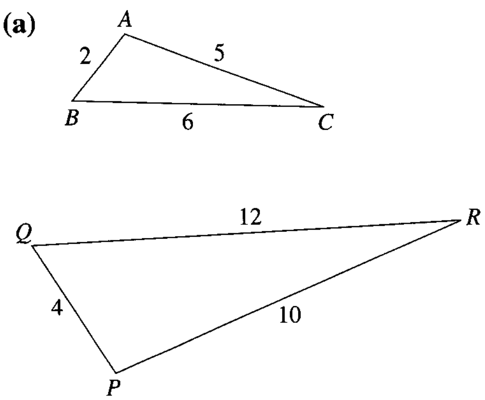
Note: The diagrams in this exercise are not drawn to scale.

1. Which of the following pairs of triangles are similar?

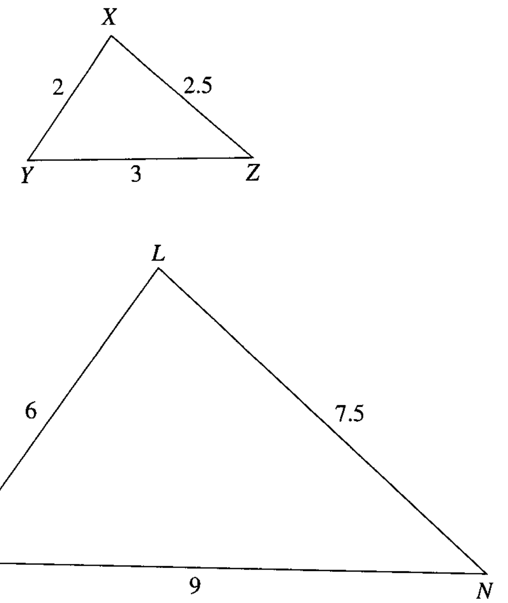


2. which of the following pairs of triangles are similar?

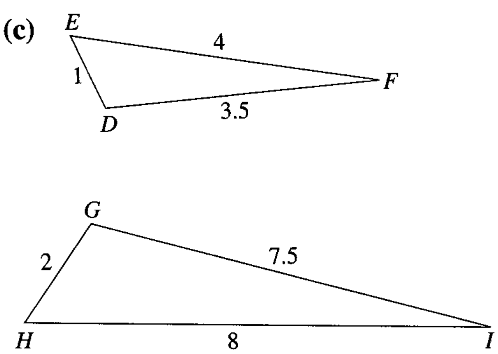
(a)



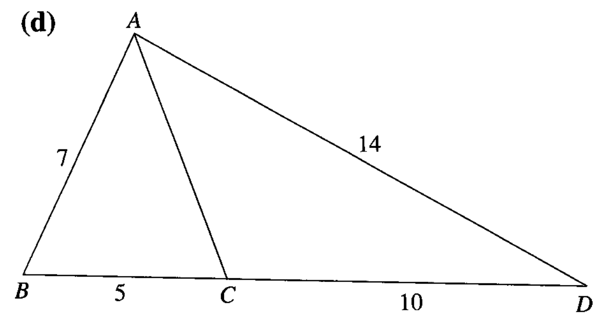
(b)



(c)

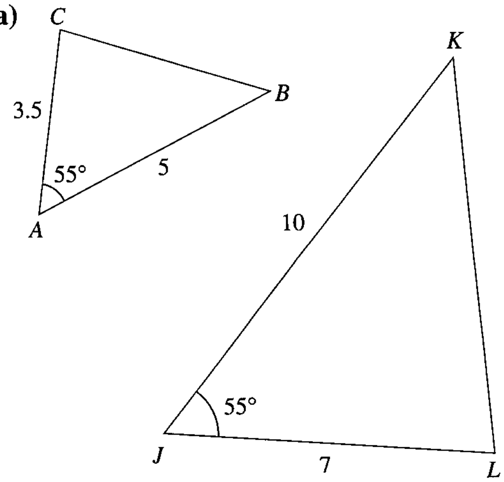


(d)

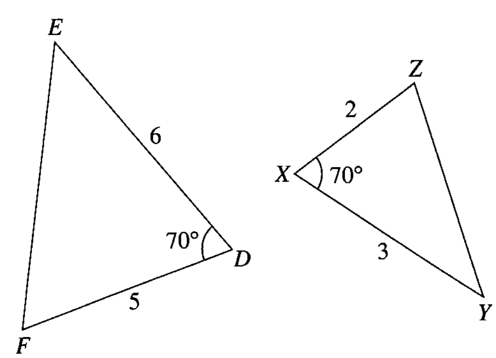


3. Which of the following pairs of triangles are similar?

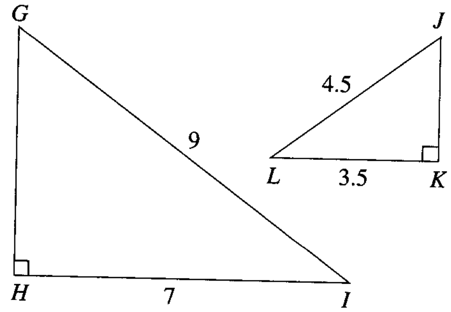
(a)



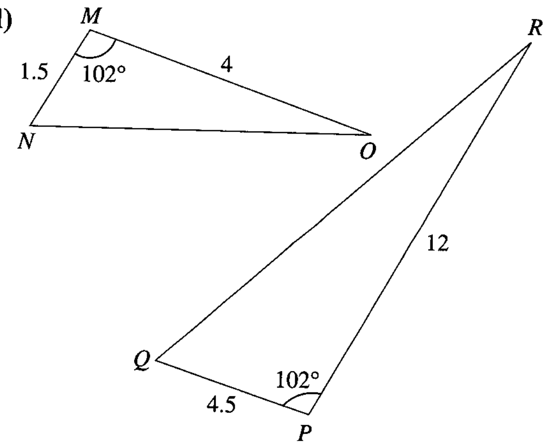
(b)



(c)

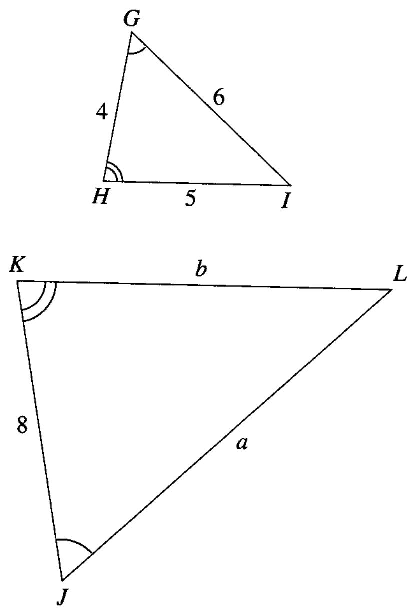


(d)



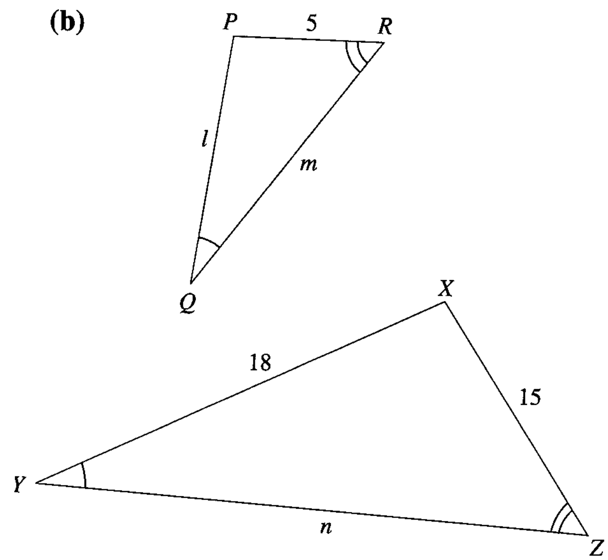
4. Each of the following pairs of triangles are similar. Answer the questions concerning the lengths of the sides.

(a)



What are the measurements of a and b?

(b)

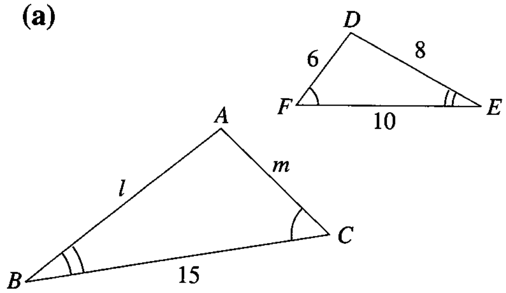


(i) What is the ratio of $\frac{m}{n}$?

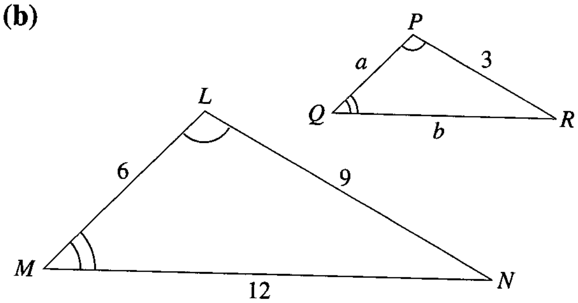
(ii) What is the length of l?

5. Each of the following pairs of triangles are similar. Find the length of each of the unknown sides indicated by letters of the alphabet. (All lengths are in cm.)

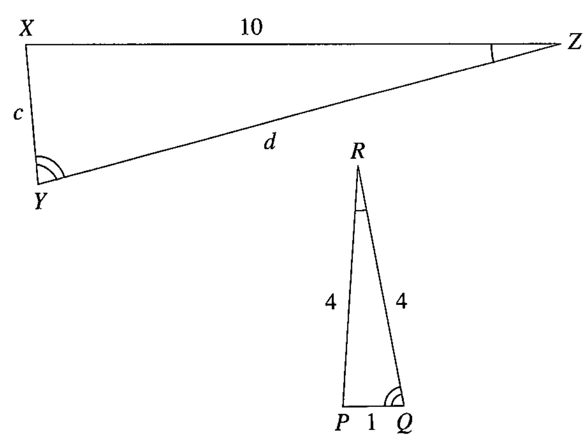
(a)



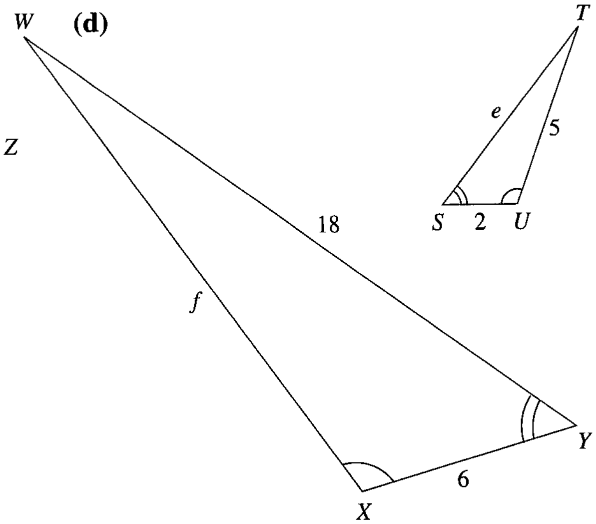
(b)



(c)

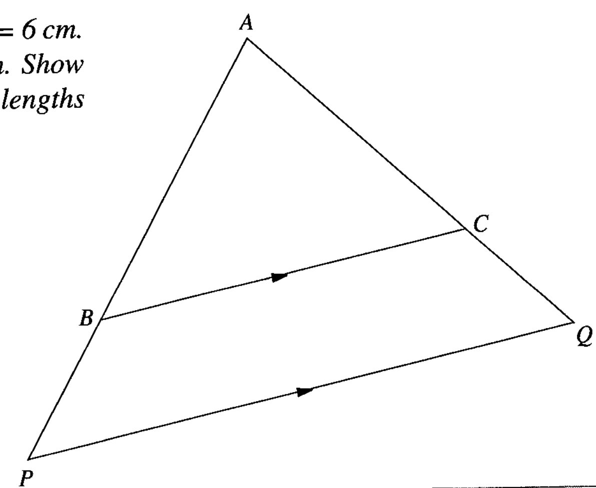


(d)



Example 5

In the given figure, BC is parallel to PQ, AB = 6 cm. AC = 5.6 cm, BC = 7.2 cm and AP = 9 cm. Show that $\triangle ABC$ is similar to $\triangle APQ$ and find the lengths of CQ and PQ.



Solution

In $\triangle ABC$ and $\triangle APQ$, $A\hat{B}C = A\hat{P}Q$ (corr $\angle$s; $BC \parallel PQ$)
$A\hat{C}B = A\hat{Q}P$ (corr $\angle$s; $BC \parallel PQ$)

Since two of the angles of one triangle are equal to two corresponding angles of the other triangle, the two triangles are similar.

Since $\triangle ABC$ is similar to $\triangle APQ$,

$$\frac{AB}{AP} = \frac{BC}{PQ} = \frac{AC}{AQ} \quad \text{or} \quad \frac{AP}{AB} = \frac{PQ}{BC} = \frac{AQ}{AC}$$

$$\frac{6}{9} = \frac{7.2}{PQ} = \frac{5.6}{AQ} \quad \text{or} \quad \frac{9}{6} = \frac{PQ}{7.2} = \frac{AQ}{5.6}$$

Using $\frac{PQ}{7.2} = \frac{9}{6}$,

$$PQ = \frac{9}{6} \times 7.2 = 10.8 \text{ cm}$$

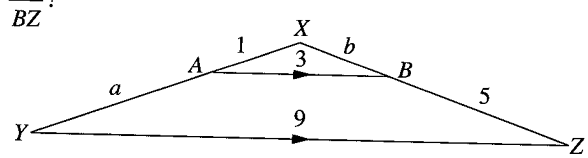
Using $\frac{AQ}{5.6} = \frac{9}{6}$,

$AQ = \frac{9}{6} \times 5.6 = 8.4$ cm

$\therefore \quad CQ = 8.4 - 5.6 = 2.8$ cm

Example 6

In the diagram, AB is parallel to YZ, AB = 3 cm, YZ = 9 cm, XA = 1 cm, AY = a cm, XB = b cm and BZ = 5 cm. Find the values of a and b and hence, find the numerical values of $\frac{XA}{AY}$ *and* $\frac{XB}{BZ}$. *Is* $\frac{XA}{AY} = \frac{XB}{BZ}$?



Find out how you can use eight straight lines of equal lengths to make a square and four congruent equilateral triangles.

Solution

$\triangle XAB$ and $\triangle XYZ$ are similar. (Why?)

$\therefore \quad \frac{XA}{XY} = \frac{AB}{YZ}$

i.e., $\quad \frac{1}{1+a} = \frac{3}{9} = \frac{1}{3}$

$1 + a = 3$

$a = 2$

Similarly, $\quad \frac{XB}{XZ} = \frac{1}{3}$

i.e, $\quad \frac{b}{b+5} = \frac{1}{3}$

$3b = b + 5$

$b = \frac{5}{2} = 2\frac{1}{2}$

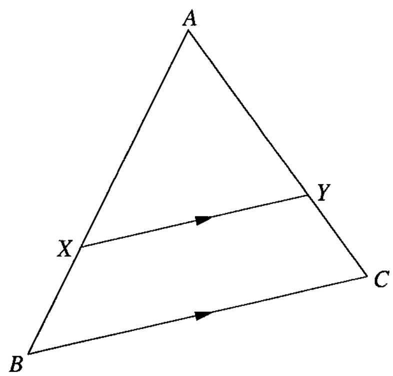
$\therefore \quad \frac{XA}{AY} = \frac{1}{a} = \frac{1}{2}$

$\frac{XB}{BZ} = \frac{b}{5} = \frac{5}{2} \times \frac{1}{5} = \frac{1}{2}$

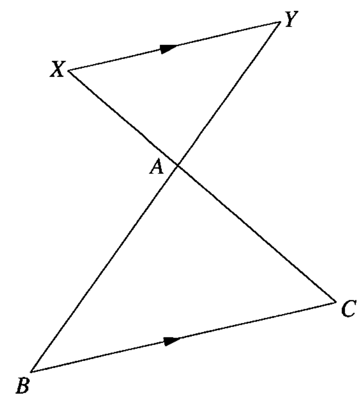
$\therefore \quad \frac{XA}{AY} = \frac{XB}{BZ}$

From this example, we arrive at the following result:

A straight line drawn parallel to one side of a triangle divides the other two sides proportionally. Conversely, if a line divides two sides of a triangle proportionally, then it is parallel to the third side.



(a)



(b)

Fig. 4.20

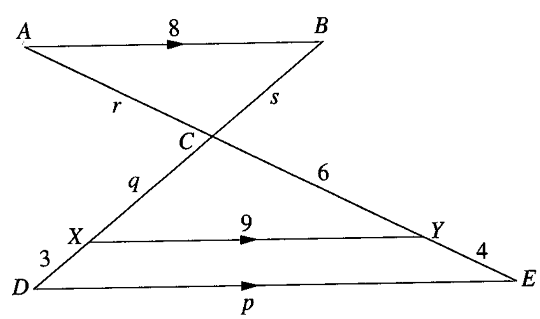
In Fig. 4.20(a) and (b), XY is parallel to BC and similar triangles are formed accordingly.

In Fig. 4.20(a), $\frac{AX}{XB} = \frac{AY}{YC}$

In Fig. 4.20(b), $\frac{XA}{CA} = \frac{YA}{BA}$

Example 7

Find the values of p, q, r and s in the figure shown on the right.



Solution

In the similar triangles CXY and CDE, we have

$$\frac{DE}{XY} = \frac{CE}{CY} \quad \text{i.e.,} \quad \frac{p}{9} = \frac{10}{6}$$

$$p = \frac{10}{6} \times 9 = 15$$

Since XY is parallel to DE, $\frac{CX}{XD} = \frac{CY}{YE}$ i.e., $\frac{q}{3} = \frac{6}{4}$

$$q = \frac{9}{2} = 4\frac{1}{2}$$

In the similar triangles ABC and YXC, $\frac{BC}{XC} = \frac{AB}{YX}$ and $\frac{AC}{YC} = \frac{AB}{YX}$.

$$\text{i.e.,} \quad \frac{s}{q} = \frac{8}{9}$$

$$s = \frac{8}{9} \times \frac{9}{2} = 4$$

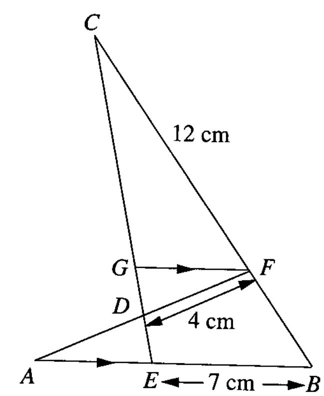
$$\text{and} \quad \frac{r}{6} = \frac{8}{9}$$

$$r = \frac{8}{9} \times 6 = 5\frac{1}{3}$$

Example 8

In the diagram, GF is parallel to AB, AB = CF and BF = FG.

(a) Show that △ABF is congruent to △CFG.
(b) Show that △CDF is similar to △ADE.
(c) Given that CF = 12 cm, DF = 4 cm and EB = 7 cm, calculate the length of DE.



Solution

(a) In $\triangle ABF$ and $\triangle CFG$, $AB = CF$ (given)

$BF = FG$ (given)

$A\hat{B}F = C\hat{F}G$ (corr $\angle$s; $GF \,//\, AB$)

$\therefore$ $\triangle ABF$ and $\triangle CFG$ are congruent (SAS).

(b) In $\triangle CDF$ and $\triangle ADE$, $C\hat{D}F = A\hat{D}E$ (vert opp $\angle$s)

$D\hat{C}F = D\hat{A}E$ (corr $\angle$s; $\triangle ABF \equiv \triangle CFG$)

$\therefore$ $\triangle CDF$ and $\triangle ADE$ are similar.

(c) Since $\triangle CDF$ and $\triangle ADE$ are similar,

$$\frac{DE}{DF} = \frac{AE}{CF}$$

$$AE = AB - EB$$

$$= 12 - 7 = 5 \text{ cm}$$

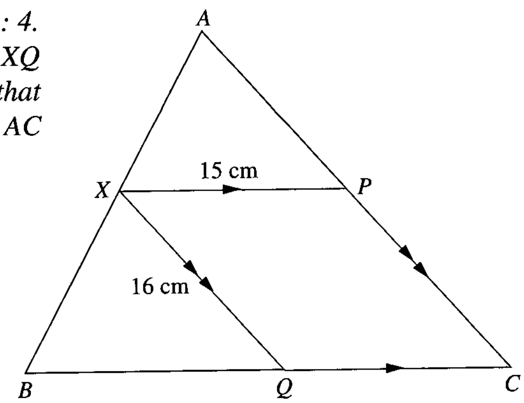
$$\frac{DE}{4} = \frac{5}{12}$$

$$\therefore \quad DE = \frac{5}{12} \times 4$$

$$= 1\frac{2}{3} \text{ cm}$$

Example 9

The side AB of $\triangle ABC$ is divided at X in the ratio 3 : 4. P and Q are points on CA and CB such that XP and XQ are parallel to BC and AC respectively. Given that XP = 15 cm and XQ = 16 cm, find the lengths of AC and BQ.



Solution

$\triangle AXP$ is similar to $\triangle ABC$.

Since $A\hat{X}P = A\hat{B}C$ and $A\hat{P}X = A\hat{C}B$ (corr $\angle$s; $XP \,//\, BC$),

$$\frac{AX}{AB} = \frac{XP}{BC} = \frac{AP}{AC} \quad (XP \,//\, QC \text{ and } XQ \,//\, PC \text{ (given)})$$

$\therefore$ $XPCQ$ is a parallelogram with PC = 16 cm and QC = 15 cm. Since $AX : XB = 3 : 4$, it implies that $AX : AB = 3 : 7$.

Using $\dfrac{AX}{AB} = \dfrac{AP}{AC}$,

we have $\dfrac{3}{7} = \dfrac{AP}{AP + 16}$

$3(AP + 16) = 7(AP)$

$3AP + 48 = 7AP$

$4AP = 48$

$AP = 12$ cm $\quad \therefore \quad AC = AP + PC = 12 + 16 = 28$ cm

Using $\dfrac{AX}{AB} = \dfrac{XP}{BC}$,

we have $\dfrac{3}{7} = \dfrac{15}{BC}$

$3(BC) = 7(15)$

$BC = 35$ cm $\quad \therefore \quad BQ = BC - QC = 35 - 15 = 20$ cm.

Exercise 4f

1. Copy and complete the following:

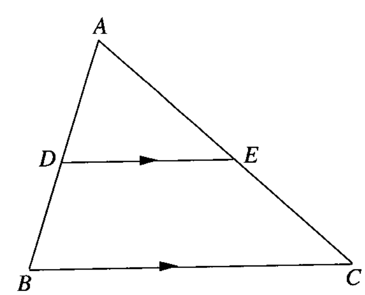
In $\triangle ABC$ and $\triangle ADE$,

$\hat{A} = \hat{A}$ (common angle)

$A\hat{B}C$ = _____ (corr $\angle$s; $DE \,//\, BC$)

$A\hat{C}B$ = _____ (_____)

$\therefore$ $\triangle ABC$ and $\triangle ADE$ are similar.



2. Copy and complete the following:

(a) In $\triangle OAB$ and $\triangle ODC$,

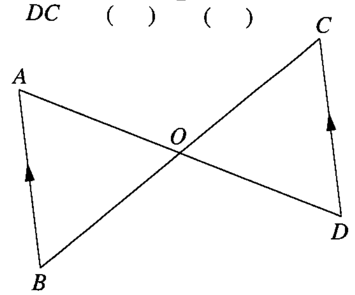
$A\hat{O}B =$ ______ (______)

$O\hat{A}B =$ ______ (______)

______ $= O\hat{C}D$ (______)

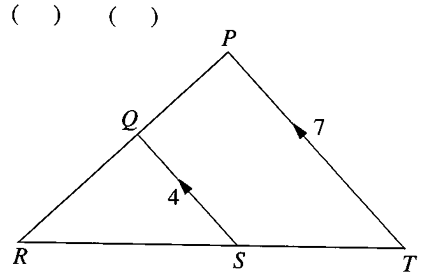
$\therefore$ $\triangle OAB$ and $\triangle ODC$ are similar.

(b) $\frac{AB}{DC} = \frac{(\ \ )}{(\ \ )} = \frac{(\ \ )}{(\ \ )}$



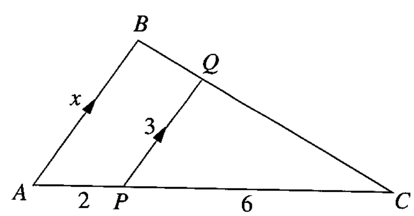
3. Show that $\triangle PRT$ is similar to $\triangle QRS$. Hence, complete the following ratios:

$\frac{7}{4} = \frac{PR}{(\ \ )} = \frac{(\ \ )}{(\ \ )}$

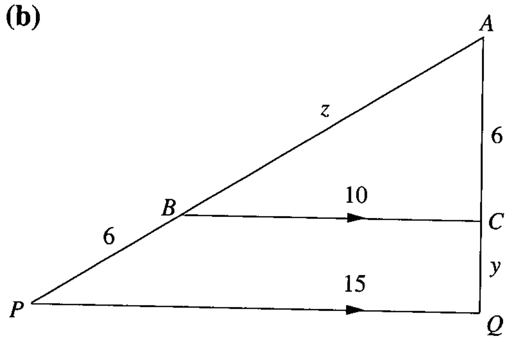


4. Find the lengths of the unknown sides marked with letters in the following figures. All lengths are given in cm.

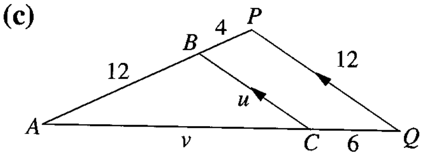
(a)



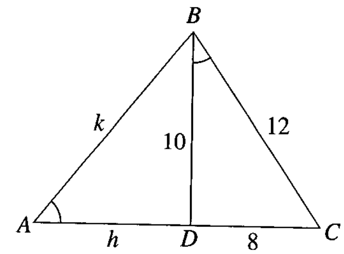
(b)



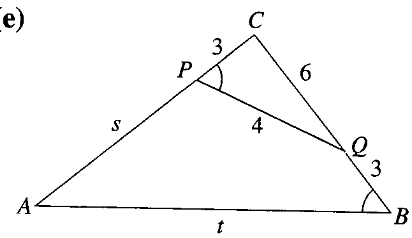
(c)



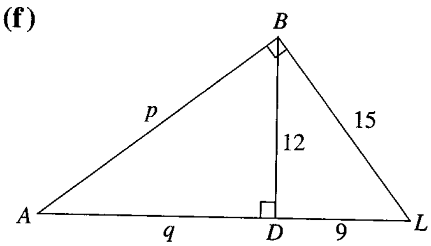
(d)



(e)

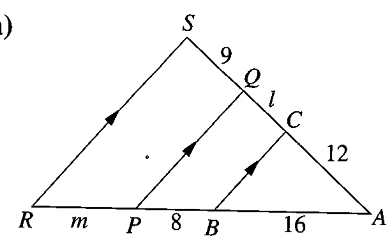


(f)

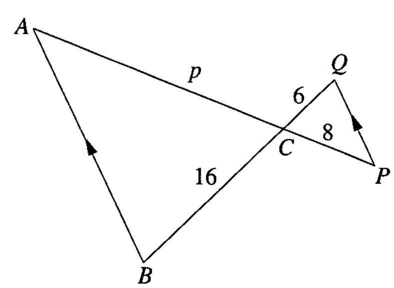


5. Find the values of l, m, p, q and r in the following figures:

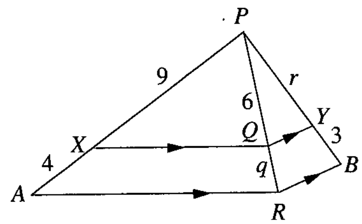
(a)



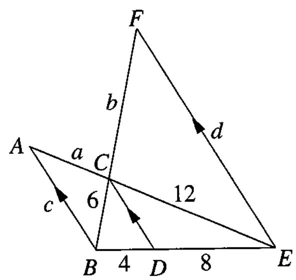
(b)



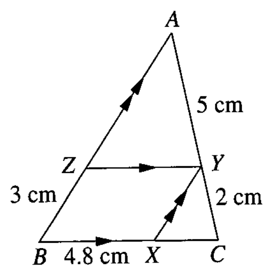
(c)



6. Find a and b as well as the ratio $c : d$ in the figure below.

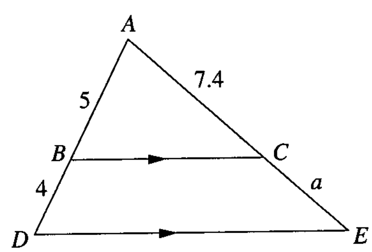


7. Find the lengths of AZ and CX in the following figure:

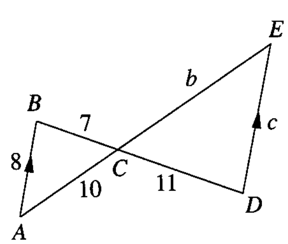


8. Find the values of a, b and c in the following figures:

(a)

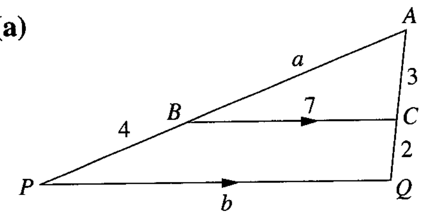


(b)

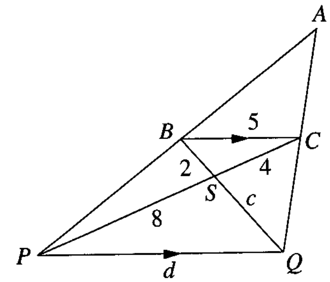


9. Find the lengths of the unknown sides marked with letters in the following figures:

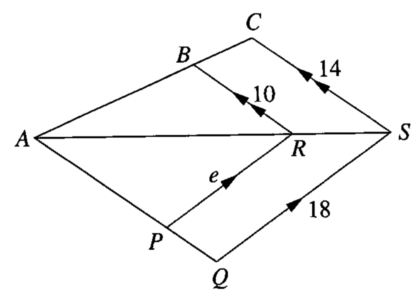
(a)



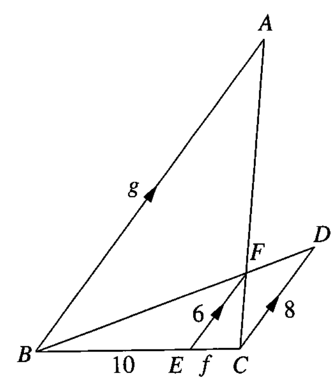
(b)



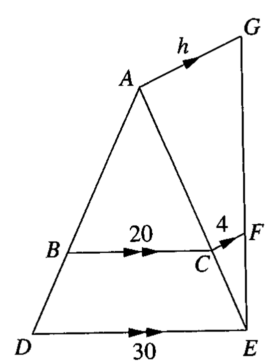
(c)



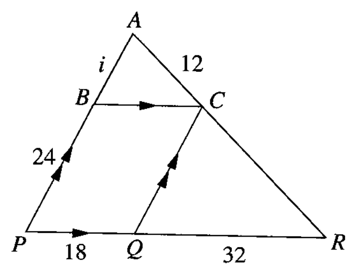
(d)



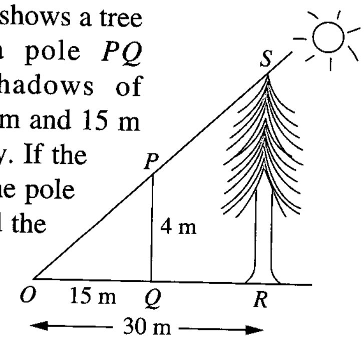
(e)



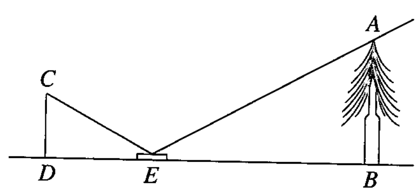
(f)



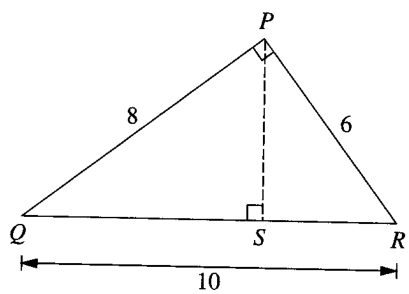
11. The figure shows a tree SR and a pole PQ casting shadows of lengths 30 m and 15 m respectively. If the length of the pole is 4 m, find the height of the tree.



10. To determine the height AB of a tree, a man places a mirror on the ground at E. From E, he walks backwards to a point D, where he is just able to see the top of the tree in the mirror. Given $BE = 18$ m, $ED = 2.1$ m and that his eyes are 1.4 m above the ground, find the height of the tree.



12. In the figure below, the angle $Q\hat{P}R$ is a right angle and PS is perpendicular to QR. $PQ = 8$ cm, $PR = 6$ cm and $QR = 10$ cm.
   (a) Name a triangle similar to triangle PQS.
   (b) Calculate the length of QS.



Summary

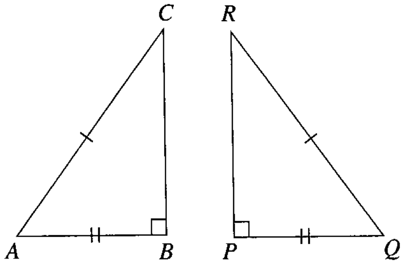
1. Two triangles are congruent if:
   (a) the three sides of one triangle are equal to the corresponding three sides of the other triangle **(SSS Property)**;
   (b) two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle **(SAS Property)**;
   (c) two angles and a side of one triangle are equal to two angles and the corresponding side of the other triangle **(AAS Property)**;
   (d) the hypotenuse and one side of one right-angled triangle are equal to the hypotenuse and one side of the other right-angled triangle **(RHS Property)**.

2. Two triangles are similar if:
   (a) two angles of one triangle are equal to two angles of the other triangle;
   (b) all the corresponding sides of the two triangles are proportional;
   (c) an angle of one triangle is equal to an angle of the other triangle, and the sides which include the equal angle of both triangles are proportional.

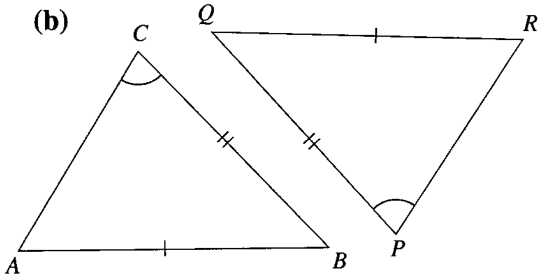
Review Questions 4

1. For each question below, determine whether the triangles are congruent. If so, state the case of congruency.

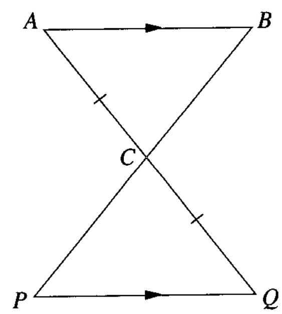
(a)



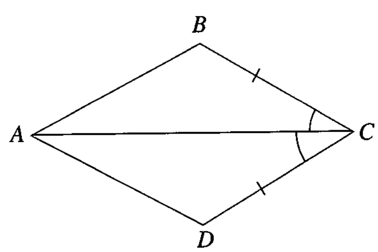
(b)



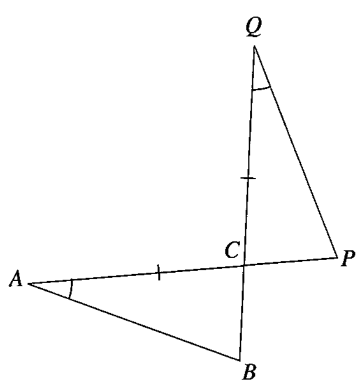
(c)



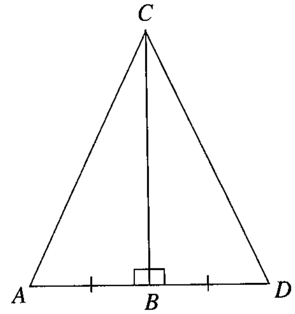
(d)



(e)

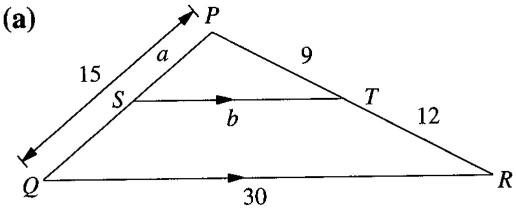


(f)

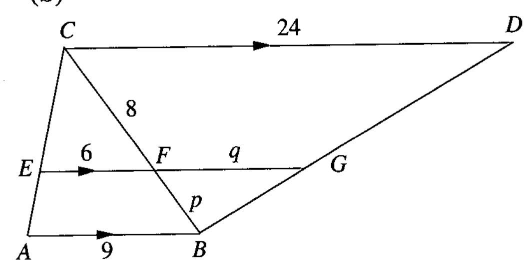


2. Find the lengths of the sides marked with letters in the following figures.
(All lengths are given in cm.)

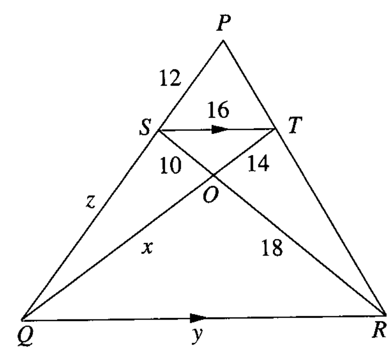
(a)



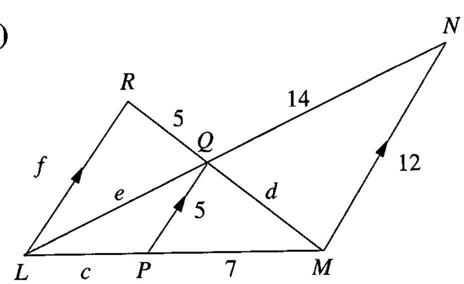
(b)



(c)

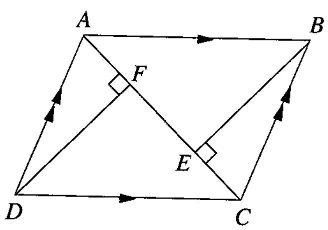


(d)

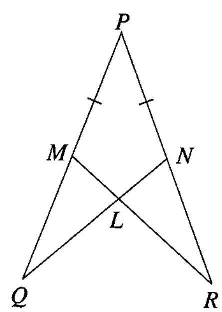


3. Name a pair of congruent triangles in each of the following and state the case of congruency.

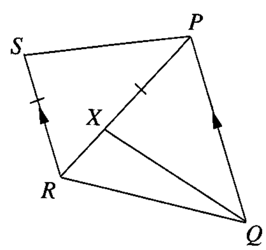
(a)



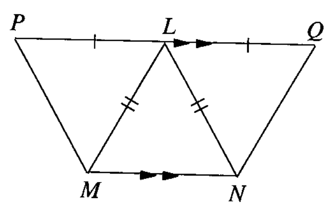
(b) (Given: $PM = PN$ and $PQ = PR$.)



(c) (Given: $PX = SR$ and $\triangle PQR$ is equilateral.)



(d)



4. In the following diagram, PLR and $QLMN$ are straight lines, PQ is parallel to NR and SP is parallel to RQ. $QL = 8$ cm, $LM = 4$ cm, $MN = 12$ cm, $QR = 18$ cm and $PL = 9$ cm.

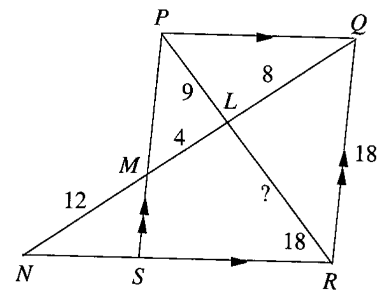
(a) (i) Name a triangle similar to triangle PLQ

(ii) Calculate the length of LR.

(b) (i) Name the triangle similar to triangle NQR.

(ii) Calculate the length of MS.

(c) Name all other pairs of similar triangles.

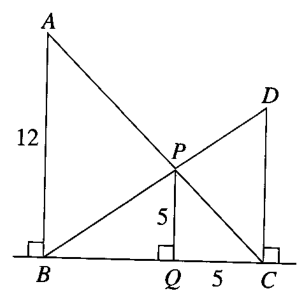


5. In the diagram below, APC and BQC are straight lines. AB, PQ and DC are perpendicular to BC. $AB = 12$ cm and $PQ = QC = 5$ cm.

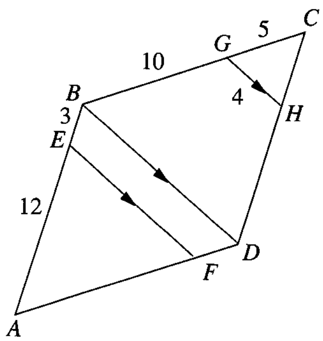
(a) Name two pairs of similar triangles.

(b) Calculate BQ and DC.

(c) Write down the ratio of AP to PC and also the ratio of BP to BD.

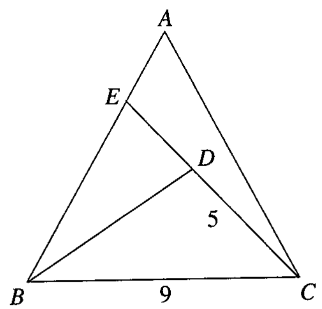


6. In the diagram, EF, BD and GH are parallel. $AE = 12$ cm, $EB = 3$ cm, $BG = 10$ cm, $GC = 5$ cm and $GH = 4$ cm. Find the lengths of BD and EF.

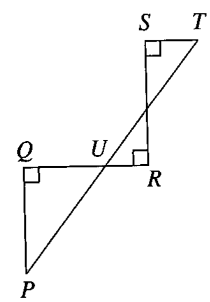


Problem Solving

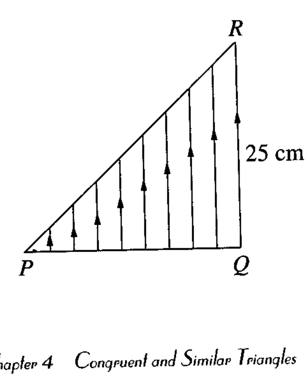
1. In the diagram below, $AB = AC$, $CB = CE$, $BD = BE$, $BC = 9$ cm and $CD = 5$ cm. Find AC.



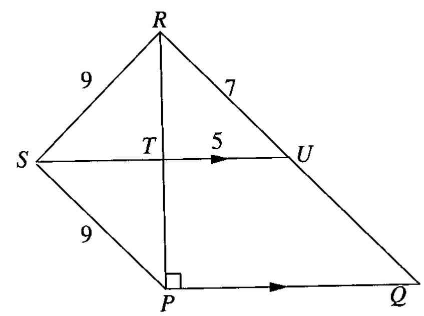
2. In the diagram, $P\hat{Q}R = Q\hat{R}S = R\hat{S}T = 90°$, $PQ = QR = RS = 5$ cm and $ST = 1$ cm. Find the length of QU.



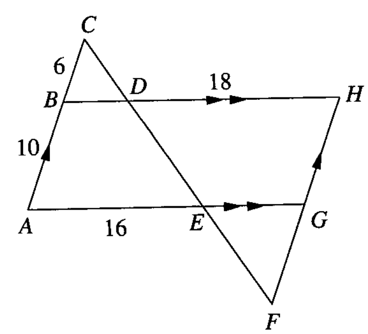
3. In the diagram shown below, $P\hat{Q}R = 90°$ and $QR = 25$ cm. PQ is divided into 9 equal parts. Eight line segments parallel to QR are drawn up to PR from the points of division. Find the sum of the lengths of the eight line segments.



4. In the following diagram, STU, RTP and RUQ are straight lines, SU is parallel to PQ. $R\hat{P}Q = 90°$, $SR = SP = 9$ cm, $TU = 5$ cm and $RU = 7$ cm.
   (a) Identify two triangles which are congruent.
   (b) Calculate the lengths of UQ and PQ.



5. In the diagram below, ABC, $CDEF$, FGH, BDH and AEG are straight lines. BH is parallel to AG and AC is parallel to FH, $AB = 10$ cm, $BC = 6$ cm, $AE = 16$ cm and $DH = 18$ cm.
   (a) **(i)** Identify two triangles similar to triangle BCD.
   (ii) Calculate the length of BD.
   (b) **(i)** Write down the length of HG.
   (ii) Calculate the length of EG.
   (iii) Identify two triangles similar to triangle EFG.
   (iv) Calculate the length of FH.
   (c) Prove that triangles ACE and HFD are similar.



CHAPTER 5

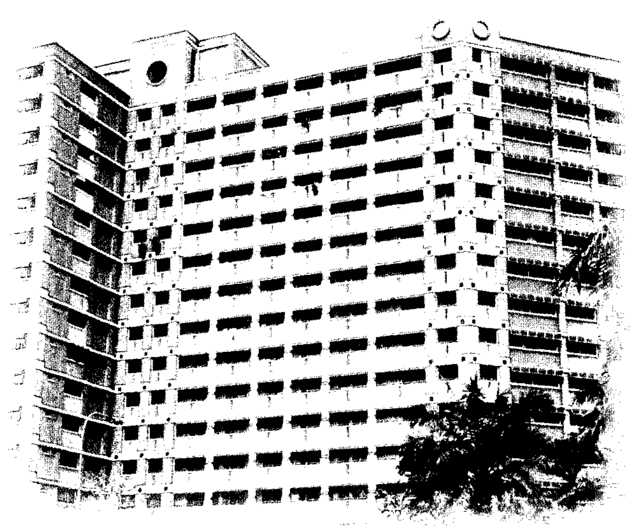
Coordinate Geometry

In this chapter, you will learn how to find

- △ the distance between two given points;
- △ the mid-point of two given points;
- △ the gradient of a straight line;
- △ the equation of a straight line.

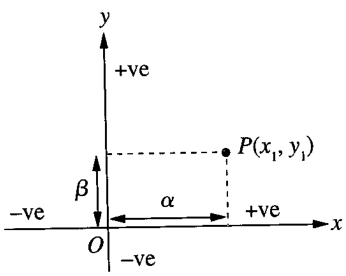
Preliminary Problem

Have you ever wondered how convenient it is to locate our seats in the cinema when we go to catch a movie? Do you know that the cinema operators use the idea of coordinate geometry in locating seats? This idea is similarly used in naming units in a block of flats.



Revision

In coordinate geometry, Ox and Oy are two lines intersecting at right angles at O, where Ox and Oy are the coordinate axes and O is the origin. Any point lying to the right of Oy has a positive x-coordinate and any point above the Ox axis has a positive y-coordinate, as indicated in Fig. 5.1 below. Any point P can be determined if its distances from the x and y axes are given. The distance of a point P from the y-axis is called the x-coordinate of the point and its distance from the x-axis, the y-coordinate.



A point P whose x-coordinate is x_1 and y-coordinate y_1 is represented by $\boldsymbol{P(x_1, y_1)}$.

Fig. 5.1

Back in Time

René Descartes, a French philosopher in the early 17th century, invented the coordinate system. His use of (x, y) as ordered pairs enhanced the inter-relationship between geometrical curves and algebraic equations. He was the first person to use the algebraic method to study geometry. He was also the first person to declare the words "I think, therefore I am." A true philosopher with a touch of genius, indeed.

Distance between Two Given Points

In this section, we shall find the distance between two given points.

For Your Information

*The x-coordinate is also known as the **abscissa** and the y-coordinate, the **ordinate**.*

Example 1

Find the distance between the points A(1, 1) and B(7, 9).

Solution

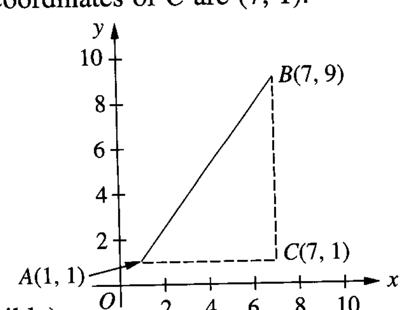
In the diagram, $\triangle ABC$ is drawn by drawing AC parallel to the x-axis and BC parallel to the y-axis. The coordinates of C are (7, 1).

Hence $BC = 9 - 1 = 8$ units
and $AC = 7 - 1 = 6$ units

Using Pythagoras' theorem,

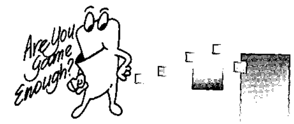
$AB^2 = BC^2 + AC^2$
$= 8^2 + 6^2$

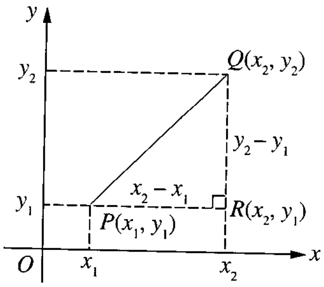
$AB = \sqrt{100}$
$= 10$ units ($AB < 0$ is inadmissible)



$\therefore$ the distance between the points A and B is 10 units.

In general, consider any two points P and Q with coordinates (x_1, y_1) and (x_2, y_2) respectively. By completing the right-angled triangle PQR, we have the coordinates of R as (x_2, y_1).





Hence $PR = x_2 - x_1$
and $QR = y_2 - y_1$

Using Pythagoras' theorem,

$PQ^2 = PR^2 + QR^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Would it make any difference if we wrote the general formula for the distance between any two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as

$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$?

NB: As lengths cannot be negative, we can ignore the negative square root.

The general formula for the distance between any two given points, $P(x_1, y_1)$ and $Q(x_2, y_2)$, is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 2

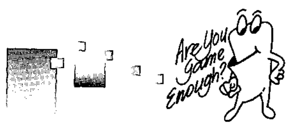
Find the distance between the points (–1, 5) and (6, –4).

Solution

Let (–1, 5) be denoted by (x_1, y_1) and (6, –4) be denoted by (x_2, y_2).

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[6 - (-1)]^2 + (-4 - 5)^2}$$
$$= \sqrt{49 + 81} = \sqrt{130}$$
$$= 11.4 \text{ (correct to 3 sig. fig.)}$$

$\therefore$ the required distance is 11.4 units.



Draw an equilateral triangle of sides 2 cm each.

(a) Mark 5 points, A, B, C, D and E on the triangle.

(b) Mark 4 points, W, X, Y and Z on the triangle.

Show that at least 2 of the points in (a) are less than 1 cm apart, and that at least 2 of the points in (b) are less than $\frac{2}{\sqrt{3}}$ cm apart.

(Hint: Divide the equilateral triangle into a number of identical parts.)

Example 3

A triangle has vertices A(0, –5), B(–2, 1) and C(10, 5). Show that it is a right-angled triangle.

Solution

$AB^2 = (-2 - 0)^2 + [1 - (-5)]^2 = 4 + 36 = 40$
$BC^2 = [10 - (-2)]^2 + (5 - 1)^2 = 144 + 16 = 160$
$AC^2 = (10 - 0)^2 + [5 - (-5)]^2 = 100 + 100 = 200$

Since $AB^2 + BC^2 = 40 + 160 = AC^2$ (Pythagoras' Theorem), so $A\hat{B}C = 90°$, this means that $\triangle ABC$ is a right-angled triangle.

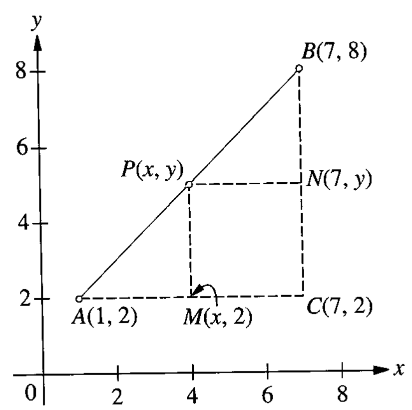
Mid-point of Two Given Points

In this section, we shall calculate the mid-point of two given points.

Example 4

Find the mid-point of the points A(1, 2) and B(7, 8).

Solution



Plot the points on a coordinate graph. Join AB and complete $\triangle ABC$ by drawing AC parallel to the x-axis, and BC parallel to the y-axis. Let $P(x, y)$ be the mid-point of AB. Draw PN parallel to the x-axis and PM parallel to the y-axis.

The coordinates of the points M, N and C are $(x, 2)$, $(7, y)$ and $(7, 2)$ respectively.

$P\hat{A}M = B\hat{P}N$, $A\hat{P}M = P\hat{B}N$ and $AP = PB$. Hence, $\triangle APM$ is congruent to $\triangle PBN$ (**AAS Property**). We have $AM = PN$ and $PM = BN$.

Now, $AM = (x - 1)$, $PN = (7 - x)$, $PM = (y - 2)$ and $BN = (8 - y)$.

$\therefore \quad x - 1 = 7 - x$ and $y - 2 = 8 - y$

$2x = 7 + 1$ and $2y = 8 + 2$

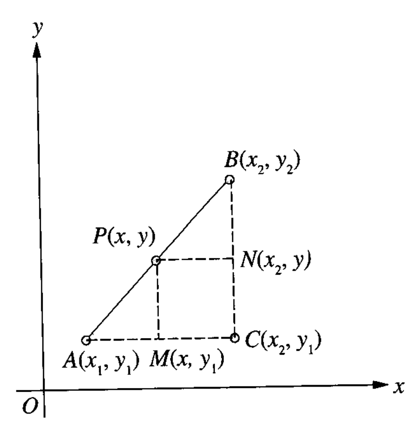
$x = 4$ and $y = 5$

$\therefore$ the mid-point of A and B is $(4, 5)$.

NB: Since M is the mid-point of A and C, and N is the mid-point of B and C, we can locate the point $P(x, y)$ easily. This is because M and P both have the *same* x-coordinate, and N and P both have the *same* y-coordinate.

Formula for Mid-point

Let A and B be the points (x_1, y_1) and (x_2, y_2) respectively. Let $P(x, y)$ be the mid-point of AB. Draw lines PN and PM parallel to the sides AC and BC respectively. The coordinates of the points C, M and N are (x_2, y_1), (x, y_1) and (x_2, y) respectively.



$B\hat{P}N = P\hat{A}M$, $P\hat{B}N = A\hat{P}M$ and $AP = BP$.

Hence, $\triangle PAM$ is congruent to $\triangle BPN$ (**AAS Property**).

We have $AM = PN$ and $PM = BN$. Now, $AM = (x - x_1)$, $PN = (x_2 - x)$, $PM = (y - y_1)$ and $BN = (y_2 - y)$.

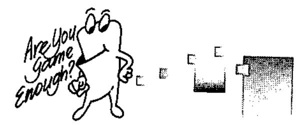
$\therefore \quad x - x_1 = x_2 - x \quad$ and $\quad y - y_1 = y_2 - y$

$2x = x_1 + x_2 \quad$ and $\quad 2y = y_1 + y_2$

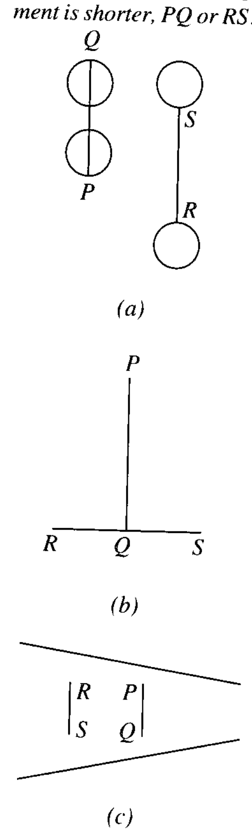
$x = \frac{x_1 + x_2}{2} \quad$ and $\quad y = \frac{y_1 + y_2}{2}$

$\therefore$ the mid-point of any two points, $A(x_1, y_1)$ and $B(x_2, y_2)$, is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



In each of the following diagrams, which line segment is shorter, PQ or RS?



Example 5

(a) Find the mid-point of (2, 5) and (–6, –1).
(b) If (3, 4) is the mid-point of A(–5, 3) and B(x, y), find the values of x and y.

Solution

(a) The mid-point $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2 + (-6)}{2}, \frac{5 + (-1)}{2}\right)$

$= (-2, 2)$

(b) $(3, 4) = \left(\frac{-5 + x}{2}, \frac{3 + y}{2}\right)$

i.e. $3 = \frac{-5 + x}{2}$ and $4 = \frac{3 + y}{2}$

$6 = -5 + x$ and $8 = 3 + y$

$\therefore \quad x = 11$ and $y = 5$

Exercise 5a

1. Plot the following pairs of points on a piece of graph paper and calculate the distance between them. Give your answers correct to 3 significant figures where necessary.

(a) (2, 3) and (9, 7) **(b)** (3, 6) and (–5, 9) **(c)** (–1, 4) and (8, –3)
(d) (–10, 2) and (–4, –7) **(e)** (–8, 2) and (6, 2) **(f)** (–8, 10), and (–8, –17)

2. Find the coordinates of the mid-point of the following pairs of points:

(a) (–4, –3) and (5, 7) **(b)** (–3, 8) and (5, –4) **(c)** (1, 7) and (–3, 2)
(d) (–1, 4) and (7, –3) **(e)** $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and $\left(4\frac{1}{2}, -4\frac{1}{2}\right)$ **(f)** $\left(2\frac{1}{2}, 5\frac{1}{4}\right)$ and (5, –1)
(g) (5, 8) and (0, 0) **(h)** $\left(ap, \frac{a}{p}\right)$ and $\left(aq, \frac{a}{q}\right)$

3. Find the perimeter and area of $\triangle ABC$, whose vertices are $A(-4, -2)$, $B(8, -2)$ and $C(2, 6)$. Hence, or otherwise, find the length of the perpendicular from A to BC.

4. Show that the points $A(3, 4)$, $B(3, 1)$ and $C(8, 4)$ are the vertices of a right-angled triangle. Find the length of the perpendicular from A to BC.

5. Given that $M(p, 7)$ is the mid-point of the line segment joining the points $A(-3, 1)$ and $B(11, q)$. Find the values of p and q.

6. Show that the points $A(-1, 2)$, $B(5, 2)$ and $C(2, 5)$ are the vertices of an isosceles triangle. Find the area of $\triangle ABC$.

7. Three of the vertices of a parallelogram $ABCD$ are $A(-3, 1)$, $B(4, 9)$ and $C(11, -3)$. Find
   (a) the mid-point of the diagonal AC; **(b)** the fourth vertex D;
   (c) the length of the diagonal AC; **(d)** the perimeter of $ABCD$.

8. Three of the vertices of a parallelogram are $A(2, 4)$, $B(6, 2)$ and $C(8, 6)$. Find the mid-point of AC and, hence, or otherwise, find the coordinates of the fourth vertex, D.

9. Three of the vertices of a parallelogram $ABCD$ are $A(3, 0)$, $B(7, 3)$ and $C(1, 7)$. Find the coordinates of the fourth vertex, D.

10. If the distance between the points $A(k, 0)$ and $B(0, k)$ is 10, find the possible values of k.

11. The coordinates of two points are $A(-2, 6)$ and $B(9, 3)$. Find the coordinates of the point C on the x-axis such that $AC = BC$.

12. The coordinates of the end-points of a line segment PQ are $P(3, 7)$ and $Q(11, -6)$. Find the coordinates of the point R on the y-axis such that $PR = QR$.

The Idea of a Gradient

When you walk or ride a bicycle up a hill, the hill will undoubtedly slow you down. The steeper the hill, the harder it is to climb. We call the measure of the steepness of a hill the **gradient**, or **slope**. This is the ratio of the vertical distance (the rise) to the horizontal distance (the run).

For Your Information

To help alleviate any confusion you might have about 'rise' or 'run', you can remember it in the following way:

When you run, you cover a horizontal distance. When you rise, you cover a vertical distance.

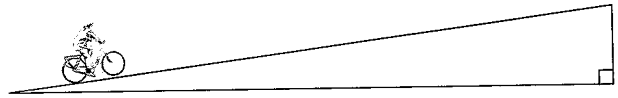


Fig. 5.2

Fig. 5.2 shows a hill that has a rise of 5 metres in 100 metres' run.

$\therefore$ the gradient of the hill $= \dfrac{\text{rise}}{\text{run}} = \dfrac{5\text{ m}}{100\text{ m}} = \dfrac{1}{20}$

Gradient of a Straight Line

In coordinate geometry, the gradient of a straight line is defined as:

$$\frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{the difference in } y\textbf{-coordinates between any 2 points on a straight line}}{\textbf{the difference in the corresponding } x\textbf{-coordinates}}$$

An alternative method of calculation that can be used to find out the gradient of AB is $\frac{2-8}{3-7} = \frac{-6}{-4} = \frac{3}{2}$.

In Fig. 5.3, the gradient of AB is

$$\frac{\text{rise}}{\text{run}} = \frac{\text{difference in the } y\text{-coordinates of } AB}{\text{difference in the corresponding } x\text{-coordinates}}$$

$$= \frac{8-2}{7-3} = \frac{3}{2}$$

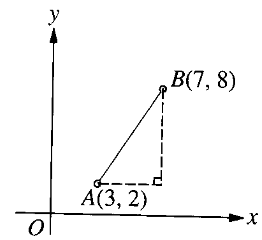
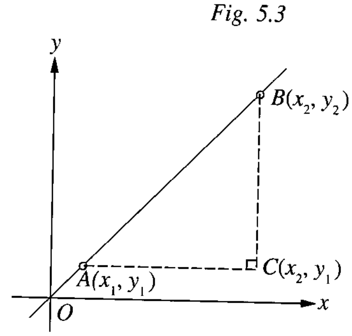


Fig. 5.3

In general, if $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a line, then the gradient of AB is

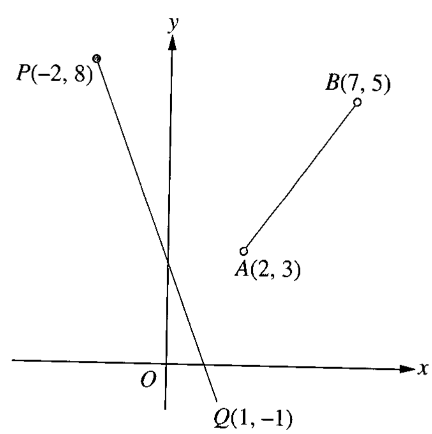
$$\frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y_1 - y_2}{x_1 - x_2}$$

NB: Gradient of $AB \neq \frac{y_2 - y_1}{x_1 - x_2}$ or $\frac{y_1 - y_2}{x_2 - x_1}$.



Example 6

Find the gradient of the line joining the points
(a) A(2, 3) and B(7, 5);
(b) P(−2, 8) and Q(1, −1).



Solution

(a) Gradient of $AB = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{7-2} = \frac{2}{5}$

or $m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{3-5}{2-7} = \frac{-2}{-5} = \frac{2}{5}$

(b) Gradient of $PQ = m = \frac{8-(-1)}{-2-1} = \frac{9}{-3} = -3$

or $m = \frac{-1-8}{1-(-2)} = \frac{-9}{3} = -3$

Angle of Slope of a Straight Line

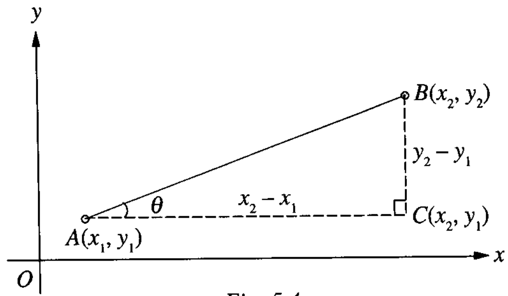


Fig. 5.4

In Fig. 5.4, $BC = y_2 - y_1$ and $AC = x_2 - x_1$. The angle of the slope of AB is defined as the angle θ made by the line AB with the positive horizontal axis, $0° < \theta < 180°$.

Now, $\tan \theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

= *gradient of the line*

Use Mathematica or Graphmatica to draw a line and observe how the angle of slope changes with different settings.

NB: The gradient of a straight line is the tangent of the angle which the line makes with the positive direction of the x-axis.

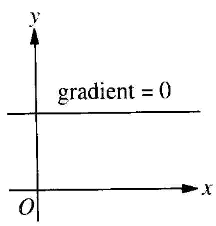
When $\theta = 0°$, $\tan \theta = 0$ and the gradient of the straight line is zero. This is the case when the line is parallel to the x-axis.

As θ tends to 90°, the gradient increases rapidly as the difference, $y_2 - y_1$, increases very much faster than $x_2 - x_1$ as can be seen in Fig. 5.4.

When $\theta = 90°$, $\tan \theta$ is not defined and the gradient of the straight line is thus not defined. This is the case when the line is parallel to the y-axis.

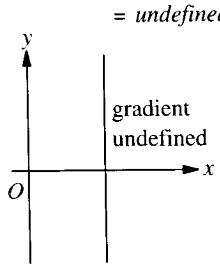
The gradient of a horizontal line = 0,

i.e. $\dfrac{y - y}{x_2 - x_1} = \dfrac{0}{x_2 - x_1} = 0$



The gradient of a vertical line is undefined,

i.e. $\dfrac{y_2 - y_1}{x - x} = \dfrac{y_2 - y_1}{0}$ = *undefined.*



Signs of the Gradient

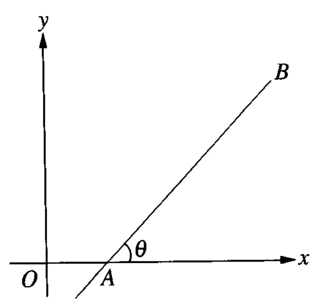


Fig. 5.5

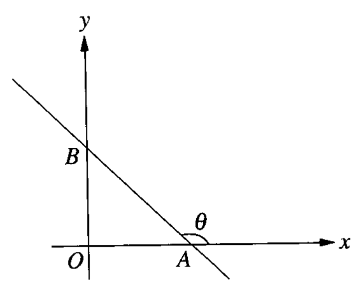
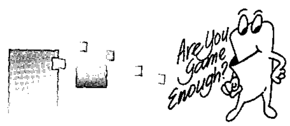


Fig. 5.6

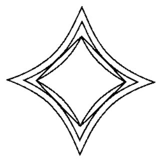
Fig. 5.5 shows a line AB making an angle of $\theta(0° < \theta < 90°)$ with the positive x-axis. In this case, as the value of x increases, the value of y increases and *the gradient is positive because tan θ is positive.*

Fig. 5.6 shows a line AB making an angle of $\theta(90° < \theta < 180°)$ with the positive x-axis. In this case, as the value of x increases, the value of y decreases and the *gradient is negative, because tan θ is negative.*

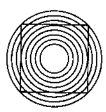
Gradients of Parallel Lines



Do you see a square in each of the following diagrams?



(a)



(b)

These two diagrams are enough to make you 'square-eyed'!

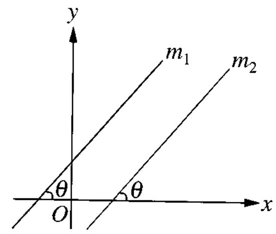


Fig. 5.7

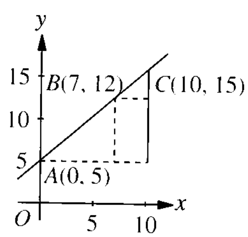
Two lines are parallel if the **angles** they make with the positive x-axis are **equal**. Fig. 5.7 shows that the angles made by the lines, m_1 and m_2, with the horizontal axis are equal. Thus m_1 is parallel to m_2 and their gradients are equal.

(1) If two non-vertical lines are parallel, their gradients are equal.

The converse is also true:

(2) If two lines have the same gradient, then they are parallel.

Collinear Points



The graph shows three points, A, B and C which lie on the same straight line.

Gradient of AB $= \frac{12-5}{7-0} = \frac{7}{7} = 1$

Gradient of AC $= \frac{15-5}{10-0} = \frac{10}{10} = 1$

Gradient of BC $= \frac{15-12}{10-7} = \frac{3}{3} = 1$

Hence, the lines containing common points have the same gradient which proves their collinearity.

When three or more points lie on the same straight line, they are said to be collinear.

The condition for collinearity is as follows:

> Three points A, B and C are said to be collinear if the *gradient* of AB = *gradient* of BC.

Will the points A, B and C be collinear if the gradient of AB is equal to the gradient of AC? Can you explain why or why not?

Example 7

Prove that A(0, 9), B(4, 7) and C(6, 6) are collinear.

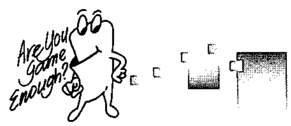
Solution

Gradient of $AB = \frac{9-7}{0-4} = -\frac{1}{2}$, Gradient of $BC = \frac{7-6}{4-6} = -\frac{1}{2}$

Since the gradient of AB is equal to the gradient of BC and B is a common point, A, B and C lie on the same line.

$\therefore$ A, B and C are collinear.

Exercise 5b



1. Find the gradient of the straight line determined by each of the following pairs of points:
   - **(a)** (0, 0) and (–2, 1)
   - **(b)** (2, –3) and (1, 7)
   - **(c)** (–2, 4) and (–5, 8)
   - **(d)** (–4, 7) and (1, 8)
   - **(e)** (–2, –5) and (2, 8)
   - **(f)** (3, 6) and (0, 9)
   - **(g)** (–3, 1) and (–2, 3)
   - **(h)** (1, 7) and (–5, 6)
   - **(i)** $(ak^2, 2ak)$ and $(ah^2, 2ah)$

2. If the gradient of the line joining the points (–3, –7) and (4, b) is equal to $\frac{3}{5}$, find b.

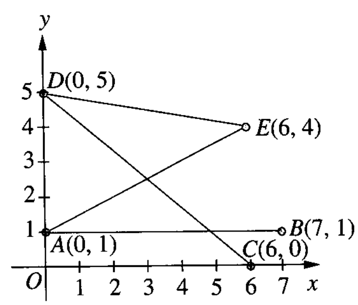
3. Find the value of k if the line joining the points $(3k, 8)$ and $(k, -3)$ has a gradient of 3.

4. If the points (2, 5), (1, 1) and (a, 3) are collinear, find the value of a.

5. The points (2, –3), (3, –2) and (8, k) lie on a straight line. Find k.

6. Show that the points (3, 6), (0, –2), (–7, –5) and (–4, 3) are the vertices of a parallelogram.

7. Find the gradient of each of the lines AB, AE, AD, DE and CD in the diagram.



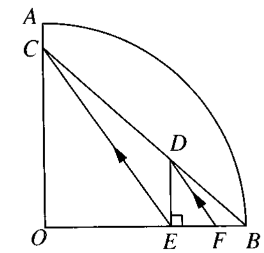
8. The points $A(1, -4)$, $B(-7, 2)$ and $C(k, -7)$ lie on a straight line. Find the value of k.

9. The points (3, 2), (5, k) and ($-k$, 4) are collinear. Find the possible values of k.

10. The coordinates of P, Q and R are (6, –11), (k, –9) and ($2k$, –3) respectively. If gradient of PQ = gradient of PR, find the value of k.

11. The line joining the points $A(2, t)$ and $B(7, 2t^2 + 7)$ has a gradient of 2. Find the possible values of t.

In a quadrant of unit radius, C and D are points such that $OC = \frac{7}{8}$ of the radius, $BD = \frac{1}{2}$ of the radius, DE is perpendicular to OB and DF is parallel to CE.



What is the length of BF in the above diagram? What is the length of (BF + 3OB), correct to six decimal places? What do you notice about this value?

If a line is parallel to the x-axis and its distance from the x-axis is a, where $a > 0$, then every point on the line has the same ordinate a, i.e., $MP = a = ON$ in Fig. 5.8. The equation of the line is $y = a$. How do you draw the line $y = -a$?

If a line is parallel to the y-axis and its distance from the y-axis is b where $b > 0$, then every point on the line has the same x-coordinate, b, i.e., $OM = b = NP$ in Fig. 5.8. The equation of the line is $x = b$. How do you draw the line $x = -b$?

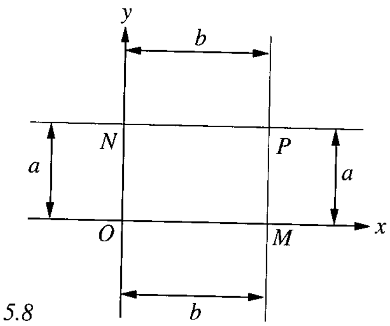
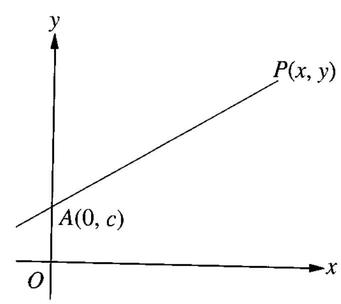


Fig. 5.8

Example 8

Find the equation of the straight line having gradient m and passing through the point A(0, c).



Solution

Let $P(x, y)$ lie on the straight line passing through the point $A(0, c)$.

$$\text{Gradient of } AP = m$$

$$\frac{y - c}{x - 0} = m$$

$$y - c = mx$$

$$\therefore \quad y = mx + c$$

$y = mx + c$ is known as the **gradient-intercept form** of the equation of a straight line. There is a linear relation between x and y, i.e., y varies directly as x, with an addition of a constant, c. In this equation, m gives the gradient of the straight line, c gives the intercept on the y-axis and $(0, c)$ is the point where the line cuts the y-axis.

For example, the line $y = 3x - 4$ has gradient 3 and cuts the y-axis at the point $(0, -4)$ and the line $y = \frac{x}{4} + \frac{1}{3}$ has gradient $\frac{1}{4}$ and cuts the y-axis at $\left(0, \frac{1}{3}\right)$.

The equation $2y + 3x - 4 = 0$ can be reduced to $y = -\frac{3}{2}x + 2$. Therefore, the line $2y + 3x - 4 = 0$ has gradient $-\frac{3}{2}$ and cuts the y-axis at $(0, 2)$

NB: The lines $y = 3x - 4$ and $y = 3x + 5$ are **parallel** because they have the **same** gradient and have no common point.

If an equation of a line is not in the gradient-intercept form, $y = mx + c$, then make y the subject in order to find the value of the gradient.

Example 9

(a) Given that $y = 3x + c$ passes through the point $A(3, 1)$, find the value of c.
(b) Find the equation of the line parallel to the line $5x - y = 2$ and passing through the point $(3, 3)$.

Solution

(a) Since $A(3, 1)$ lies on the line $y = 3x + c$, the coordinates $(3, 1)$ must satisfy the equation.

$$1 = 3(3) + c$$
$$c = -8$$

$\therefore$ The value of c is -8.

(b)
$$5x - y = 2$$
$$y = 5x - 2$$

Gradient of the line is 5.

The equation of the line with gradient 5 is of the form $y = 5x + c$.

Since $(3, 3)$ lies on the line,

$$3 = 5(3) + c$$
$$c = -12$$

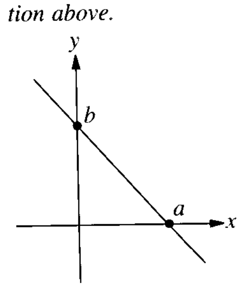
$\therefore$ the required equation is $y = 5x - 12$.

For your information

A linear equation is sometimes represented as shown below:

$$\frac{x}{a} + \frac{y}{b} = 1$$

This is known as the intercept form. The following diagram is a graphical presentation of the equation above.



From this, we can see that the line passes through 2 points, which are $(a, 0)$ and $(0, b)$.

Example 10

Find the equation of the straight line passing through the points
(a) $A(1, 2)$ and $B(3, 7)$; *(b) $P(2, 3)$ and $Q(7, 3)$;*
(c) $H(5, 1)$ and $K(5, 6)$.

Solution

(a) Gradient of $AB = \frac{7-2}{3-1} = \frac{5}{2}$

Equation of AB is of the form $y = \frac{5}{2}x + c$.

Since $(1, 2)$ lies on the line,

$$2 = \frac{5}{2}(1) + c$$
$$c = -\frac{1}{2}$$

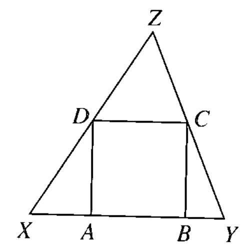
$\therefore$ equation of AB is $y = \frac{5}{2}x - \frac{1}{2}$.

(b) $P(2, 3)$ and $Q(7, 3)$ have the same y-coordinate of value 3. PQ is a horizontal line with equation $y = 3$.

(c) $H(5, 1)$ and $K(5, 6)$ have the same x-coordinate of value 5. HK is a vertical line with equation $x = 5$.

Are You Game Enough?

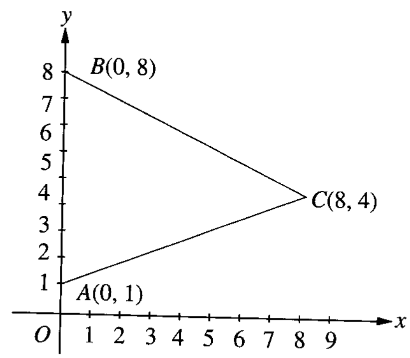
Given $\triangle XYZ$, draw a square ABCD inscribed in $\triangle XYZ$ as shown in the diagram below. (It is preferable that you do not use a trial and error method.)



Example 11

A triangle has vertices $A(0, 1)$, $B(0, 8)$ and $C(8, 4)$.

Find the equation of the straight line passing through the mid-point of BC, and which is parallel to the line $3y + 4x = 7$.



Solution

Mid-point of $BC = \left(\frac{8+0}{2}, \frac{4+8}{2}\right) = (4, 6)$

$$3y + 4x = 7$$

$$\Rightarrow \quad y = -\frac{4}{3}x + \frac{7}{3}$$

$\therefore$ gradient of the line is $-\frac{4}{3}$ and equation is of the form $y = -\frac{4}{3}x + c$.

Since (4, 6) lies on the line,

$$6 = -\frac{4}{3}(4) + c$$

$$c = 11\frac{1}{3}$$

$\therefore$ equation of the line is $y = -\frac{4}{3}x + 11\frac{1}{3}$, i.e., $3y + 4x = 34$.

Notes

Sometimes a graphical equation can look a bit "untidy" in the sense that it contains several fractions in it.

e.g. $y = \frac{1}{2}x + 12\frac{1}{4}$

In order to have an equation of integers, we can multiply both sides of the equation by the LCM of the denominators and the result would be as illustrated below:

$$4y = -2x + 49.$$

For Your Information

The equation of the x-axis is $y = 0$ and the equation of the y-axis is $x = 0$.

Exercise 5c

1. Find the equation of the straight line joining each of the following pairs of points:

(a) (0, 0) and (1, –1)
(b) (1, 3) and (2, 5)
(c) (2, 4) and (–2, 3)
(d) (–2, –4) and (1, –7)
(e) (3, 4) and (–1, 8)
(f) (–7, –5) and (–1, –1)

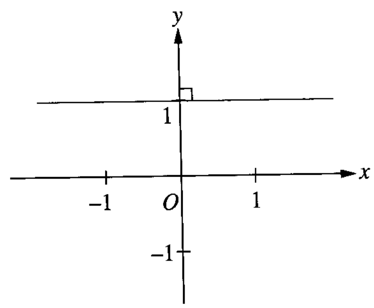
2. Find the equation of each of the straight lines below, given its gradient and the coordinates of a point on the straight line:

(a) $\frac{1}{3}$, (0, 0) **(b)** 3, (1, 1) **(c)** a, (0, a)

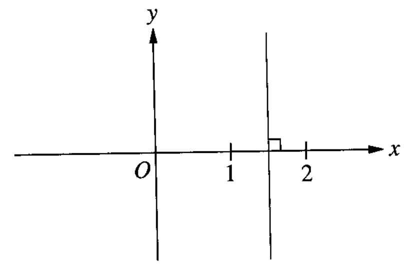
(d) −3, (2, −5) **(e)** $-\frac{1}{2}$, (5, 7) **(f)** 0, (5, 4)

3. In each of the following diagrams, find the gradient and the y-intercept of the line where possible, and write down the equation of each line:

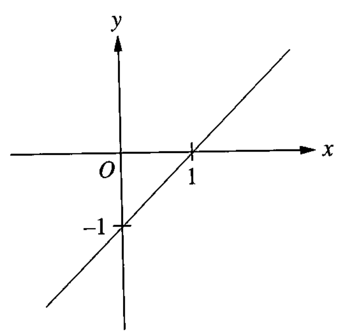
(a)



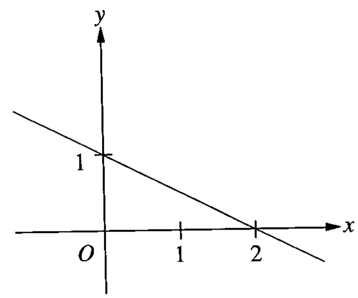
(b)



(c)



(d)



4. Write down the equation of the line which has gradient 2 and which passes through the origin.

5. Find the equation of the straight line which has gradient 3 and which passes through the point (3, 1).

6. Find the equation of the straight line that is parallel to $2y - x = 7$ and bisects the line segment joining the points (3, 1) and (1, −5).

7. Given that the point (1, 2) lies on the line $y = 5x + c$, find c.

8. Given the line $\frac{x}{3} + \frac{y}{2} = 1$, find

(a) its gradient;
(b) the coordinates of the point at which it cuts the x-axis.

9. Given that the line $y = 2x + c$ passes through the point (4, 4), find c.

10. A straight line passes through the points $A(0, 3)$ and $B(3, 12)$.

(a) Write down its **(i)** gradient; **(ii)** equation.
(b) The line $x = 3$ is the axis of symmetry of $\triangle ABC$. Find the coordinates of C.

11. The straight lines $kx = 4y + 5$ and $(2k + 2)x = 7 - 6y$ are parallel. Find k.

12. Find the values of k if the line $2x - 5 = ky$ is parallel to $(k + 1)x = 6y - 3$.

13. Find the equation of the straight line with gradient $-\frac{2}{3}$ and passing through $(-3, 5)$. If this line also passes through the point $(a, 3)$, find a.

14. Find the equation of the straight line passing through $(3, -2)$ and parallel to the line $2y = 5x + 7$.

15. The equation of a straight line l is $5x + 6y + 30 = 0$. K is the point $(3, -1)$. Find
    (a) the coordinates of the point where the line l crosses the x-axis;
    (b) the coordinates of the point M, at which the line l intersects the line $x = 2$;
    (c) the equation of the line passing through K and parallel to l;
    (d) the equation of the line passing through K and parallel to the line $5y - 10 = 0$.

16. P is the point $(2, 3)$ and Q is the point $(9, 5)$.
    (a) Find the equation of the line joining PQ.
    (b) Find the coordinates of the point where the line PQ intersects the x-axis.
    (c) The line $y = 5$ is the line of symmetry of $\triangle PQR$. Find the coordinates of R.
    (d) Calculate the area of $\triangle PQR$.
    (e) Calculate the length of PQ and hence calculate the perpendicular distance from R to the line PQ.

17. The line $mx = ny + 2$ is parallel to the x-axis. Find the value of m and state the condition for the line to be parallel to the y-axis.

18. The line $3x + 7y = 13$ is parallel to the line $kx + 8 = 3y$. Find the value of k.

Summary

1. The mid-point of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

2. The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ or $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

3. The gradient of the line joining (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{y_1 - y_2}{x_1 - x_2}$.

4. The gradient of a straight line is the tangent of the angle which the line makes with the positive direction of the x-axis.

5. If two lines are parallel, their gradients are equal. The converse is also true.

6. The equation of a straight line that is parallel to the y-axis and which passes through the point (a, b) is $x = a$.

7. The equation of a straight line that is parallel to the x-axis and which passes through the point (a, b) is $y = b$.

8. The equation of a straight line having a gradient m and passing through $(0, c)$ is $y = mx + c$. When $m = 0$, we have $y = c$, i.e. the line is parallel to the x-axis.

Review Questions 5

1. A straight line has a gradient of 2 and passes through the point (0, –3).

(a) Write down the equation of the straight line.
(b) Given that the line also passes through the point (4, k), find the value of k.

2. The equation of a straight line is $6x + 2y = 7$.

(a) Find the gradient of the line.
(b) Another line $y = mx + c$ is parallel to the given line and passes through the point (3, 5). Find the value of c.

3. The coordinates of the points O, A, B and C are (0, 0), (1, 5), (3, 4) and (2, –3) respectively. Find

(a) AB^2;
(b) the gradient of BC;
(c) the equation of the line passing through O and parallel to AC.

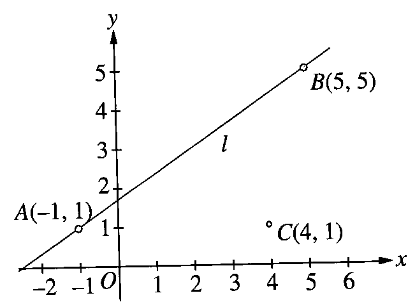
4. The equation of line l is $3x - 4y = 24$. The line intersects the x-axis at A and the y-axis at B. The mid-point of AB is M and O is the origin. Find

(a) the gradient of l; **(b)** the length of AB; **(c)** the coordinates of M;
(d) the equation of the line passing through B and parallel to OM.

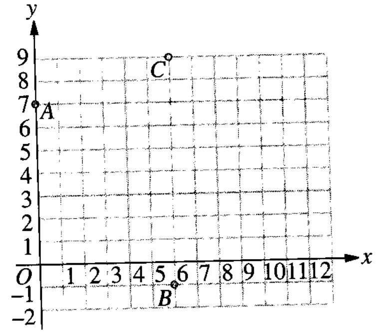
5. Find the equation of the line joining the points $A(0, 6)$ and $B(8, 0)$. Given that the line $y = x + 1$ cuts the line AB at the point M, find

(a) the coordinates of M;
(b) the equation of the line passing through M and parallel to the x-axis.
(c) the equation of the line passing through M and parallel to the y-axis.

6. The graph shows the line l passing through the points $A(-1, 1)$ and $B(5, 5)$. Given that C is the point (4, 1), find



(a) the gradient of l;
(b) the equation of l;
(c) the coordinates of the mid-point of AB;
(d) the area of $\triangle ABC$;
(e) the length of BC, giving your answer correct to 2 decimal places.



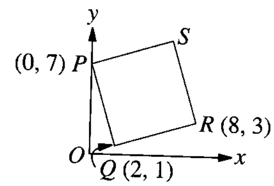
7. The points A, B and C have coordinates $A(0, 7)$, $B(6, -1)$ and $C(6, 9)$.

(a) Showing your working clearly, calculate the length of AB.
(b) Calculate the gradient of AC.
(c) Find the equation of the line AC.
(d) $ACPB$ is a quadrilateral with BC as its axis of symmetry. Find the coordinates of P.
(e) Calculate the area of the quadrilateral of $ACPB$. *(C)*

8. In the diagram, $PQRS$ is a square. P is the point (0, 7), Q is the point (2, 1) and R is the point (8, 3).

(a) Find (i) the coordinates of S;
(ii) the equation of PQ.

(b) Calculate the area of the square.



(C)

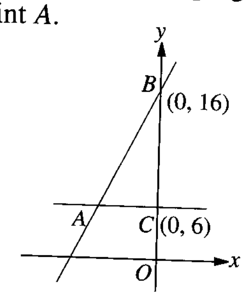
9. In the diagram, B is the point (0, 16) and C is the point (0, 6). The sloping line through B and the horizontal line that passes through C meet at the point A.

(a) Write down the equation of the line AC.

(b) Given that the gradient of the line AB is 2, find the equation of the line AB.

(c) Calculate the coordinates of the point A.

(d) Calculate the area of $\triangle ABC$.



(C)

Problems

1. Show that all the numbers in the following sequence are divisible by 6:

42, 402, 4 002, 40 002, 400 002, ...

2. John plays a guessing game based on his school's basketball tournament. He makes predictions on the Blue team, the Red team and the Green team. For every point he places to make a prediction, he will earn 2 points if the Blue team wins, 4 points if the Red team wins and 8 points if the Green team wins. He predicts that one of the above teams will win. How many points must he place on each of the three teams to ensure that he will earn exactly 100 points if his prediction is correct?

Problem Solving

Example 12

Mrs Lee gave birth to 4 children in 4 consecutive years. Today, she discovered that the product of the ages, in years, of her 4 children is 3 024. How old are her children now?

Solution

Method 1: Eliminate unlikely possibilities

Since the product of the ages of the 4 children is 3 024, the ages of any of the children cannot be multiples of 5, i.e., their ages cannot be 5, 10, 15 and so on. This is because the product of the ages does not end with a zero or five.

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, ...

Since the ages are consecutive, the set of possibilities are:

1, 2, 3, 4
6, 7, 8, 9,
11, 12, 13, 14 and so on.

Since the product of 11, 12, 13 and 14 is greater than 10 000, the ages of the children cannot be more than 11 years. Neither can the ages of the children be 1, 2, 3 and 4 years.

~~1~~, ~~2~~, ~~3~~, ~~4~~, ~~5~~, 6, 7, 8, 9, ~~10~~, ~~11~~, ~~12~~, ~~13~~, ~~14~~, ~~15~~, ...

Thus, the ages of the 4 children must be 6, 7, 8 and 9 years.

Method 2: **Use approximation**

Let y be the average age of the 4 children. Since the product of the ages is 3 024,

$$\therefore \quad y^4 = 3\ 024$$
$$y = \sqrt[4]{3\ 024} \quad 7.4$$

Since the 4 children were born in consecutive years, it is obvious from the number sequence below, that the ages of the 4 children are 6, 7, 8 and 9 years.

$$\begin{array}{c} 7.4 \\ \downarrow \end{array}$$
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, …

Method 3: **Use an equation**

Let the ages of the 4 children be a, $a + 1$, $a + 2$ and $a + 3$.

$$\therefore \quad a(a + 1)(a + 2)(a + 3) = 3\ 024$$
$$(a^2 + a)(a^2 + 5a + 6) = 3\ 024$$
$$a^4 + 5a^3 + 6a^2 + a^3 + 5a^2 + 6a = 3\ 024$$
$$a^4 + 6a^3 + 11a^2 + 6a = 3\ 024 \quad \text{———} \quad (1)$$

For equation (1) to be true, $a^4 < 3\ 024$

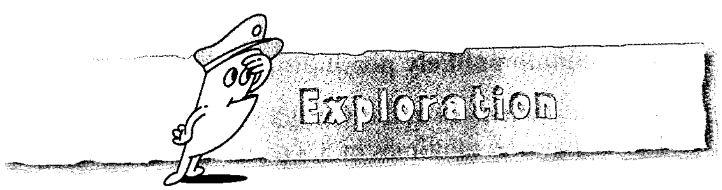
$$\therefore \quad a < 7.4$$

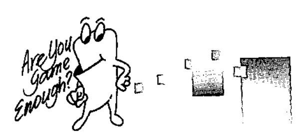
Since the ages of the 4 children must be a whole number, we will only consider integers for the above inequality.

Try $a = 7$, $7^4 + 6(7)^3 + 11(7)^2 + 6(7) = 5\ 040$

Try $a = 6$, $6^4 + 6(6)^3 + 11(6)^2 + 6(6) = 3\ 024$

From the above, $a = 6$. Thus, the ages of the 4 children are 6, 7, 8, 9.





Find out how you can position four golf balls such that each is equidistant from the others.

1. The line $y = 2 - x$ intersects the line segment joining the points $A(3, 2)$ and $B(0, 1)$ at the point P. Find the ratio of $AP : BP$.
2. The image of the point (2, 9), under a reflection in the line $y = 2x$, is (h, k). Calculate the values of h and k.
3. The line $x + 2y = 5$ intersects the curve $5x^2 + 4y^2 = 29 - 12x$ at points P and Q. Calculate the length of PQ, giving your answer correct to 2 decimal places.
4. The polar coordinate system is a useful means of locating the position of points on a plane. It is widely used in air traffic control. Have you seen pictures of a radar screen? Find out more about this system and how it works.
5. For all real values of m, the line $2y = mx + 4$ passes through a fixed point A. State the coordinates of A.

CHAPTER 6

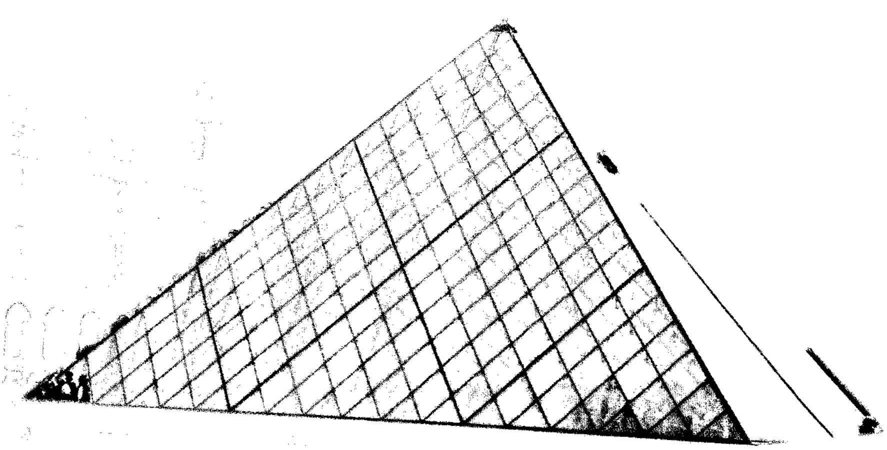
Area and Volume of Similar Figures and Solids

In this chapter, you will learn how to

- △ solve problems using the relationship between areas of similar figures;
- △ solve problems using the relationship between volumes of similar solids.

Preliminary Problem

The photograph shows two magnificent similar glass pyramids in Paris. Do you know that we can calculate the volume of the bigger pyramid by finding the volume of the smaller one?



Areas of Similar Figures

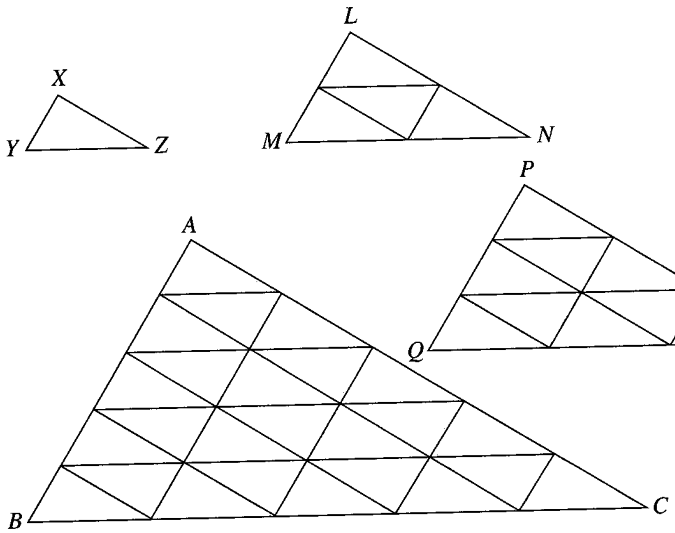
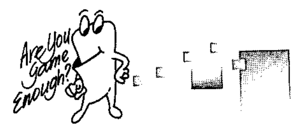


Fig. 6.1



Use 12 toothpicks to form 6 equilateral triangles of the same area. Move only 4 toothpicks from your figure so as to get 3 equilateral triangles, 2 of which are of the same area.

In Fig. 6.1, $\triangle ABC$ and $\triangle PQR$ are different in position and size but they have equal corresponding angles and their corresponding sides are in the ratio of 5 : 3. They are, thus, similar.

i.e., $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = \frac{5}{3}$

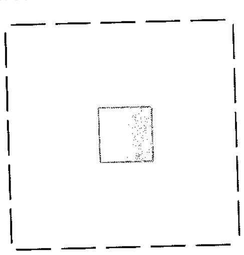
$\triangle ABC$ is made up of 25 little congruent triangles each identical in size to $\triangle XYZ$; while $\triangle PQR$ is made up of 9 such triangles. Hence, the ratio of the area of $\triangle ABC$ to the area of $\triangle PQR$ is 25 : 9.

$\triangle ABC$ and $\triangle LMN$ are also similar. Their sides are in the ratio of 5 : 2 and their areas are in the ratio of 25 : 4.

$\triangle PQR$ and $\triangle LMN$ are also similar. Their sides are in the ratio of 3 : 2 and their areas are in the ratio of 9 : 4.

From the above, it can be seen that the ratio of the areas of two triangles is the square of the ratio of their corresponding sides.

The figure below shows a square formed by 20 toothpicks. Inside the centre of the square is a square hole of area $\frac{1}{25}$ that of the whole square. By using 18 toothpicks, partition the space inside the square, excluding the hole, into 6 similar regions of equal area.



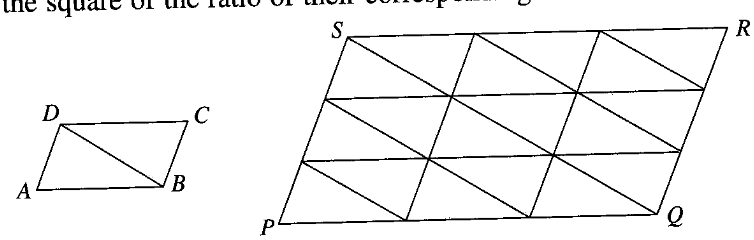


Fig. 6.2

In Fig. 6.2, the parallelogram $ABCD$ and the parallelogram $PQRS$ are similar. The ratio of their corresponding sides is 1 : 3. By counting the little congruent triangles, the ratio of their areas is found to be 2 : 18, i.e., 1 : 9. Thus, the ratio of their areas is the square of the ratio of their corresponding sides.

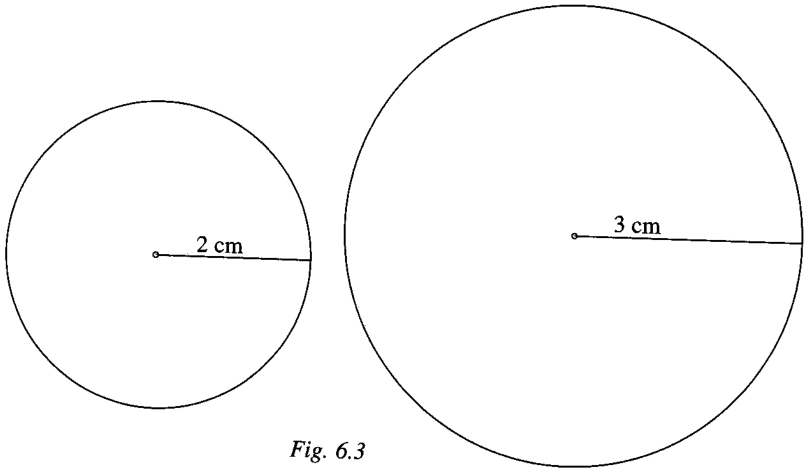
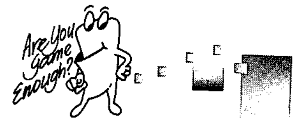


Fig. 6.3



The diagram below shows 4 small circles enclosed in a big circle. Find the ratio of the area of the shaded regions to that of the unshaded areas.

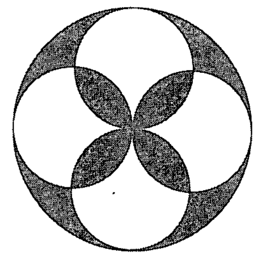


Fig. 6.3 shows two similar circles whose radii are in the ratio of 2 : 3. The ratio of their areas is $\pi \times 2^2 : \pi \times 3^2$, i.e., 4 : 9. In general, the ratio of the areas of two circles whose radii are in the ratio of $r : R$ is $r^2 : R^2$.

The above discussion leads to the following conclusion:

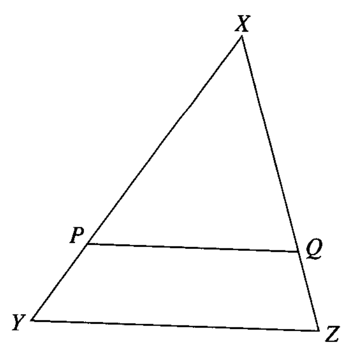
The ratio of the areas of any two similar figures is equal to the square of the ratio of any two corresponding lengths of the figures.

If A_1 and A_2 denote the areas of similar figures, and l_1 and l_2 denote their corresponding lengths, we have $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$.

If P_1 and P_0 denote the perimeters of two similar figure, what conclusion can you get about $\frac{P_1}{P_2}$?

Example 7

In the diagram, $\frac{XP}{PY} = \frac{XQ}{QZ} = \frac{3}{1}$. If the area of $\triangle XYZ$ is 32 cm², find the area of PYZQ.



Solution

$\triangle XPQ$ and $\triangle XYZ$ are similar.

$$\frac{XP}{XY} = \frac{XQ}{XZ} = \frac{3}{4}$$

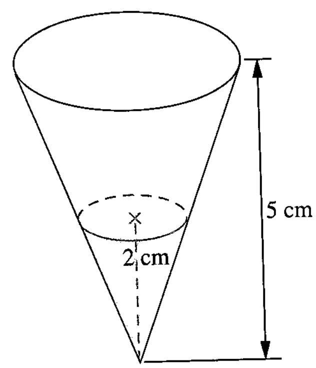
$$\therefore \quad \frac{\text{area of } \triangle XPQ}{\text{area of } \triangle XYZ} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$\text{Area of } \triangle XPQ = \frac{9}{16} \times \text{area of } \triangle XYZ$$
$$= \frac{9}{16} \times 32 \text{ cm}^2$$
$$= 18 \text{ cm}^2$$

$\therefore$ area of $PYZQ = 32 - 18 = 14$ cm^2

Example 2

The diagram shows an inverted open cone of height 5 cm. It contains water to a depth of 2 cm. If the internal surface area of the cone is 100 cm², find the surface area of the cone which is wet.

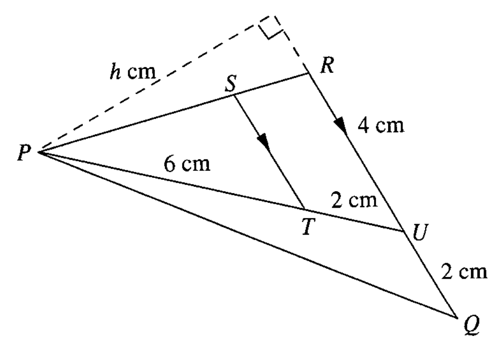


Solution

$$\text{The surface area of the wet part} = \left(\frac{2}{5}\right)^2 \times \text{internal surface area of cone}$$
$$= \frac{4}{25} \times 100 \text{ cm}^2$$
$$= 16 \text{ cm}^2$$

Example 3

In the diagram shown, ST is parallel to RU, RU = 4 cm, UQ = 2 cm, PT = 6 cm and TU = 2 cm. Given that the area of $\triangle PUR$ is 12 cm², calculate the area of (a) $\triangle PQU$; (b) $\triangle PTS$.



Solution

(a) Notice that $\triangle PUR$ and $\triangle PQU$ have a common height corresponding to the base RU and the base UQ respectively.

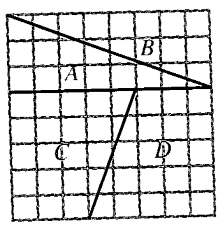
Diagram (a) shows a square of side 8 units. It is divided into two triangles (A, B) and two trapeziums (C, D).

Diagram (b) shows a rectangle which measures 13 units by 5 units divided into two triangle (A′, B′) and two trapeziums (C′, D′). A, B, C and D fit exactly into A′, B′, C′ and D′ respectively, which implies that (8 × 8) units² = (13 × 5) units².

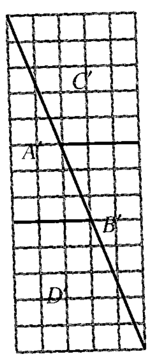
Is 8 × 8 = 13 × 5?

If not, how do you explain the result?

(a)



(b)



Let the common height be h cm.

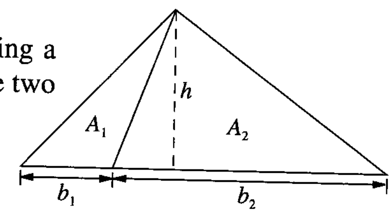
$$\frac{\text{Area of }\triangle PQU}{\text{Area of }\triangle PUR} = \frac{\frac{1}{2} \times UQ \times h}{\frac{1}{2} \times RU \times h} = \frac{UQ}{RU}$$

$$\text{Area of }\triangle PQU = \frac{UQ}{RU} \times \text{area of }\triangle PUR$$

$$= \frac{2}{4} \times 12 \text{ cm}^2 = 6 \text{ cm}^2$$

Note: $\frac{\text{Area of }\triangle PQU}{\text{Area of }\triangle PUR} \neq \left(\frac{UQ}{RU}\right)^2$. Why?

NB: In general, the ratio of the areas of two triangles having a **common height** is equal to the ratio of the bases of the two triangles.



e.g. $\frac{A_1}{A_2} = \frac{b_1}{b_2}$

(b) It can be shown that $\triangle PTS$ and $\triangle PUR$ are similar.

$$\frac{PT}{PU} = \frac{6}{8} = \frac{3}{4}$$

$$\therefore \quad \frac{\text{area of }\triangle PTS}{\text{area of }\triangle PUR} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$\text{Area of }\triangle PTS = \frac{9}{16} \times \text{area of }\triangle PUR$$

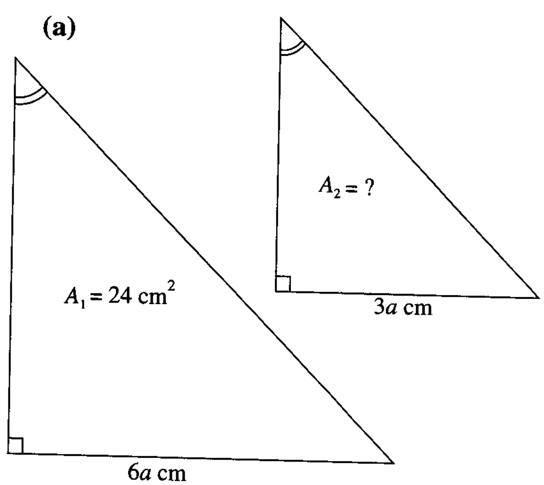
$$= \frac{9}{16} \times 12 \text{ cm}^2$$

$$= \frac{27}{4} = 6\frac{3}{4} \text{ cm}^2$$

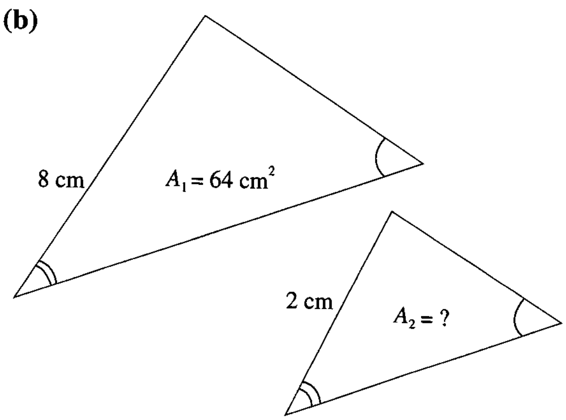
Exercise 6a

1. For questions **(a)** to **(f)**, find the unknown area A_2. In each case, the shapes are similar.

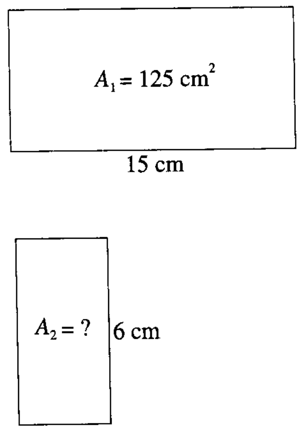
(a)



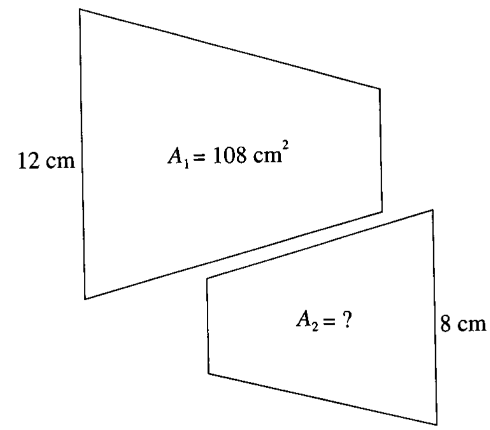
(b)



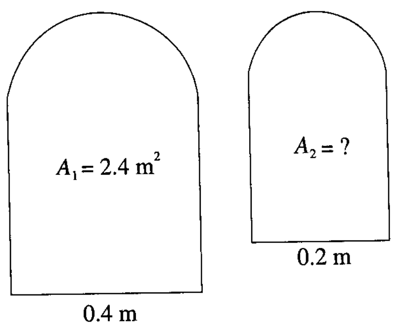
(c)



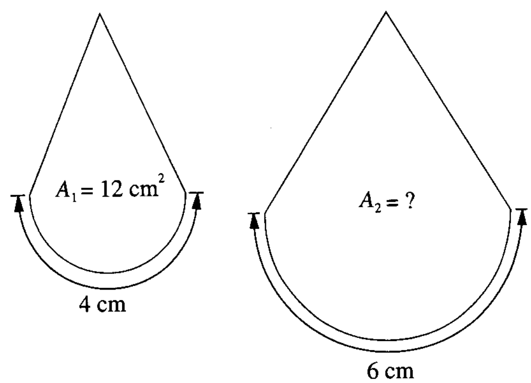
(d)



(e)

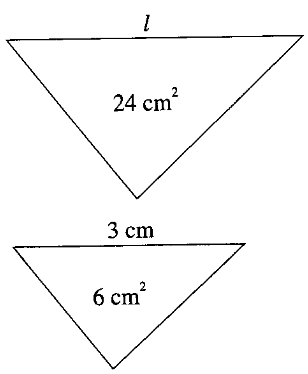


(f)

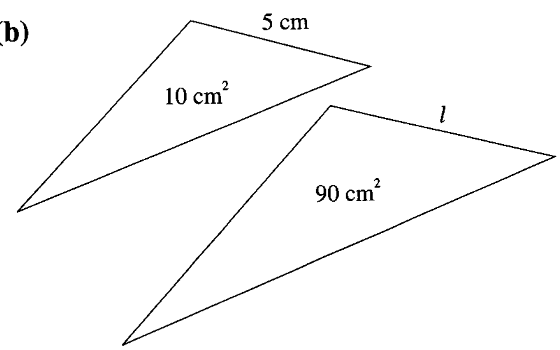


2. For questions **(a)** to **(d)**, find the lengths marked l for each pair of similar shapes.

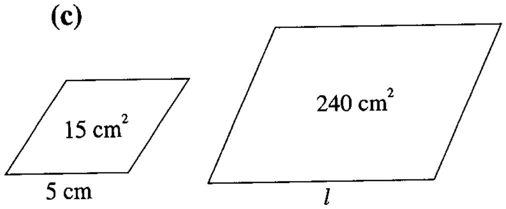
(a)



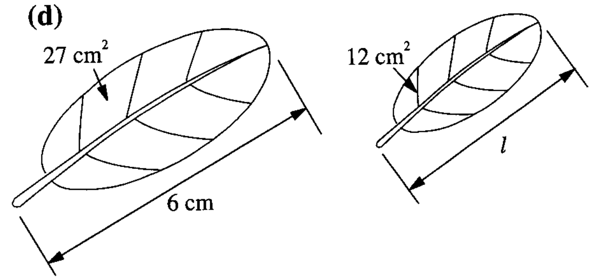
(b)



(c)



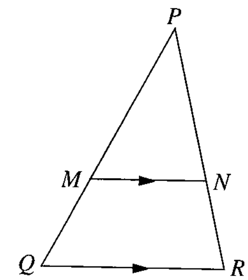
(d)



3. Find the ratio of the areas of two circles whose radii are 4 cm and 7 cm.

4. The lengths of the shortest sides of two similar hexagons are 10 cm and 8 cm. Given that the area of the larger hexagon is 200 cm^2, find the area of the smaller hexagon.

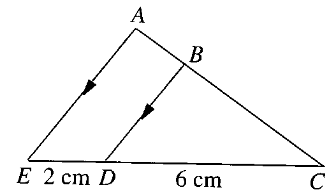
5. In the diagram, MN is parallel to QR. If the areas of triangle PMN and trapezium $MQRN$ are in the ratio of 9 : 16, find the value of the ratio $MN : QR$.



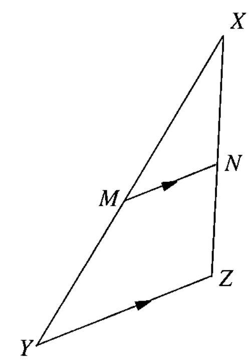
6. The areas of two trapeziums are 25 cm^2 and 36 cm^2. Find the ratio of their corresponding sides if they are similar.

7. In a plan of a house, the width, 150 cm, of a door is represented by a line 30 mm long. Find the area of the house if the corresponding area on the plan is 3 250 cm^2.

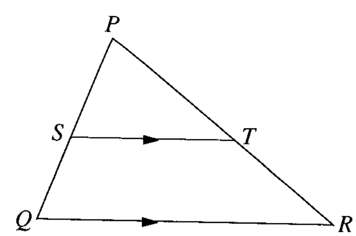
8. In the figure, BD is parallel to AE, ED = 2 cm and DC = 6 cm. Given that the area of $\triangle CBD$ = 9 cm^2, calculate the area of $ABDE$.



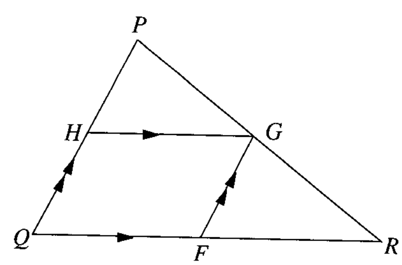
9. Given XM = 6 cm, area of $\triangle XMN$ = 14 cm^2 and area of $MYZN$ = 22 cm^2, find the length of MY.



10. In the diagram, PT = 6 cm, PR = 10 cm and area of $\triangle PST$ = 24 cm^2. Find
   (a) the area of $\triangle PQR$;
   (b) the area of $SQRT$.



11. In the figure, $HG \parallel QR$, $GF \parallel PQ$ and $QF : FR = p : q$. Find the ratio of the area of $\triangle PHG$ to the area of $\triangle PQR$.

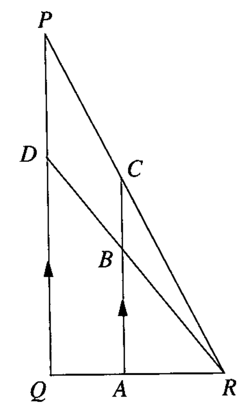


12. The radius of one sphere is $3\frac{1}{2}$ times that of another. Given that the surface area of the smaller sphere is 64 cm^2, find the surface area of the larger sphere.

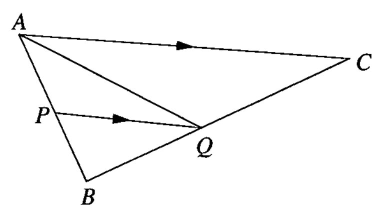
13. Two solid cones are geometrically similar and the height of one is $1\frac{1}{2}$ times that of the other.
   (a) Given that the height of the smaller cone is 12 cm, find the height of the bigger cone.
   (b) Write down the ratio of the total surface area of the smaller cone to that of the bigger cone, expressing your answer as a fraction.

14. Two prisms are geometrically similar and the height of one is $2\frac{1}{2}$ times that of the other.
   (a) Write down the ratio of the total surface area of the smaller prism to that of the larger prism.
   (b) Given that the total surface area of the larger prism is 625 cm^2, calculate the total surface area of the smaller prism.

15. In the diagram, $AR = 4$ cm, $QA = 3$ cm, $DQ = 7$ cm and $PD = 4$ cm. Find
   (a) the length of AB;
   (b) the length of BC;
   (c) the ratio of the areas of $\triangle ARB$ and $\triangle BRC$;
   (d) the ratio of the areas of $\triangle BRC$ and $\triangle PRD$.



★16. In the diagram, PQ is parallel to AC. Given that $BQ = 4$ cm, $BC = 10$ cm and area of $\triangle BPQ = 8$ cm^2, find
   (a) the area of $\triangle ABC$;
   (b) the area of $\triangle PQC$;
   (c) the area of $\triangle AQC$.



Volumes of Similar Figures

Two solids are said to be similar if they are alike in shape, i.e., the ratio of their corresponding lengths is a constant, called the **scale factor** of the solids.

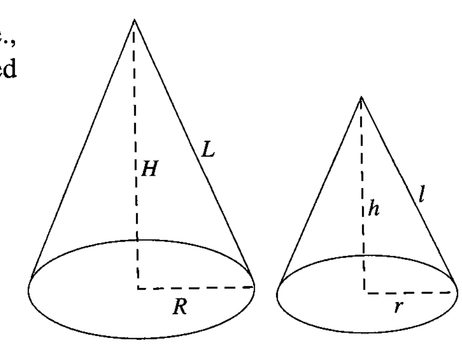


Fig. 6.4

In Fig. 6.4, if the two cones are similar, then

$$\frac{R}{r} = \frac{H}{h} = \frac{L}{l} = k, \text{ a constant.}$$

The volume, V, of a cone of radius R and height H is $\frac{1}{3}\pi R^2 H$.

The volume, v, of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

$$\frac{V}{v} = \frac{\frac{1}{3}\pi R^2 H}{\frac{1}{3}\pi r^2 h}$$

$$\frac{V}{v} = \left(\frac{R}{r}\right)^2 \times \frac{H}{h}$$

$$= k^2 \times k = k^3$$

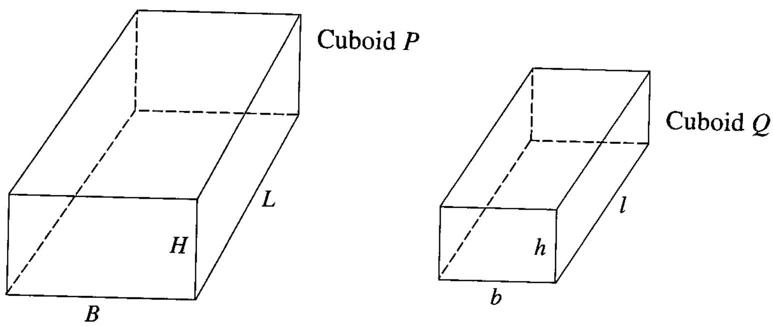


Fig. 6.5

In Fig. 6.5, the volume (V) of cuboid $P = BHL$ and the volume (v) of cuboid $Q = bhl$. If the two cuboids P and Q are similar and $\frac{B}{b} = \frac{H}{h} = \frac{L}{l} = k$, then

$$\frac{V}{v} = \frac{BHL}{bhl} = k \times k \times k = k^3$$

Also, if two spheres have radii R and r respectively, and $\frac{R}{r} = k$, then the volume of the first sphere is k^3 times the volume of the second sphere.

The above discussion suggests that, in general, for any similar solids, if the scale factor of the first solid to the second is k, then the volume of the first solid is k^3 times the volume of the second.

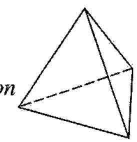
That is,

the ratio of the volumes of two similar solids is equal to the cube of the ratio of any two corresponding lengths of the two solids.

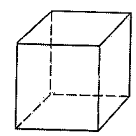
If any two corresponding lengths of two similar figures are l_1 and l_2, while the corresponding volumes are v_1 and v_2, then $\frac{v_1}{v_2} = \left(\frac{l_1}{l_2}\right)^3$.

For your information

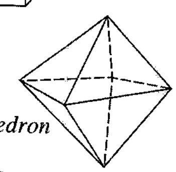
In geometry, a regular solid is a three-dimensional object with each plane having sides of equal length. The following are some of the examples of regular solids and their respective names:



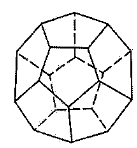
tetrahedron



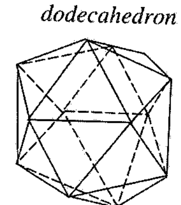
cube



octahedron



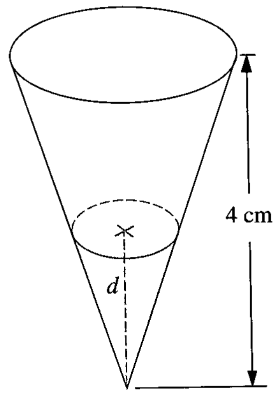
dodecahedron



icosahedron

Example 4

The diagram shows an inverted cone of height 4 cm. It contains a volume of water which is equal to one-eighth of its full capacity. Find the depth of the water.



Consider the dwarfs in the book "Gulliver's Travels" by Jonathan Swift. If they were $\frac{1}{10}$ the height of a normal human being, they would have $\frac{1}{100}$ the surface area and $\frac{1}{1\,000}$ the volume of a human being. Assuming that they functioned like normal human beings, do you think they could survive in our environment? Now let us turn to the giants in the same book. If the giants were 10 times the size of a human being, they would have 100 times the surface area and 1 000 times the body weight of a human being. Do you think their legs would be able to support their body weights?

Solution

The part containing water is similar to the whole cone.

$$\text{Ratio of volumes} = \frac{1}{8}$$

$$\text{Ratio of heights} = \frac{d}{4}$$

$$\left(\frac{d}{4}\right)^3 = \frac{1}{8}$$

$$\frac{d}{4} = \sqrt[3]{\frac{1}{8}}$$

$$\frac{d}{4} = \frac{1}{2}$$

$$d = 2$$

$\therefore$ the depth of the water = 2 cm.

NB: Can you find the ratio of the surface areas of the two cones involved in the above question? Explain your reasoning.

Example 5

Two similar prisms are of heights 2 cm and 10 cm. Given that the volume of the first prism is 80 cm³, find the volume of the second prism.

Solution

$$\text{Ratio of heights} = \frac{10}{2} = 5$$

$\therefore$ volume of the second prism $= 5^3 \times$ volume of the first prism

$= 125 \times 80 \text{ cm}^3$

$= 10\,000 \text{ cm}^3$

Example 6

Two solid spheres of diameters 4 cm and 5 cm are made of the same material. If the smaller sphere weighs 120 kg, find the weight of the larger sphere.

Solution

Ratio of weights = Ratios of volumes

$$\frac{\text{Weight of larger sphere}}{\text{Weight of smaller sphere}} = \frac{\text{Volume of larger sphere}}{\text{Volume of smaller sphere}}$$

$$= \left(\frac{\text{Diameter of larger sphere}}{\text{Diameter of smaller sphere}}\right)^3$$

$$\therefore \quad \text{weight of larger shape} = \left(\frac{5}{4}\right)^3 \times \text{weight of smaller sphere}$$

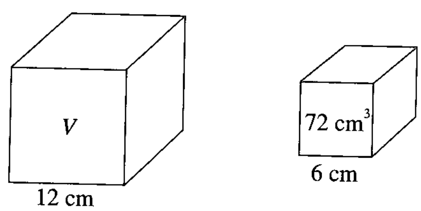
$$= \left(\frac{5}{4}\right)^3 \times 120 \text{ kg}$$

$$= 234\frac{3}{8} \text{ kg}$$

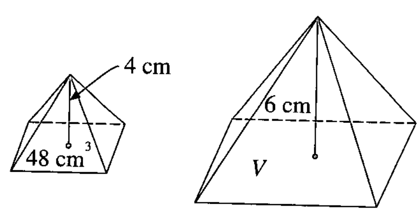
Exercise 6b

1. Find the unknown volume, V, of each of the following pairs of similar solids.

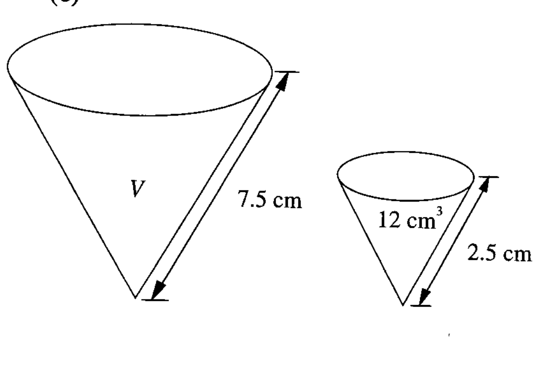
(a)



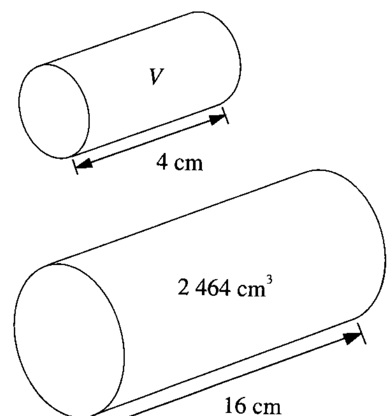
(b)



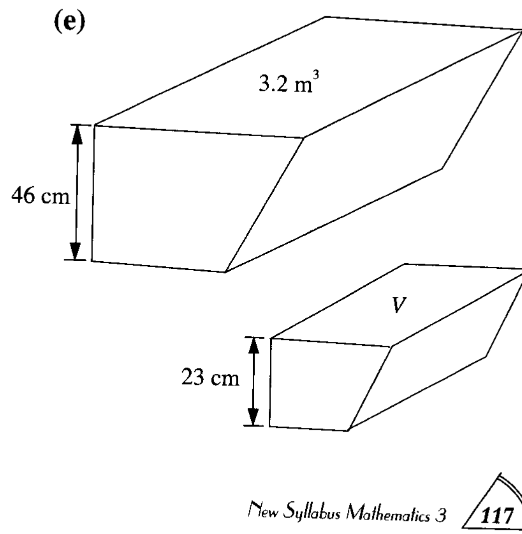
(c)



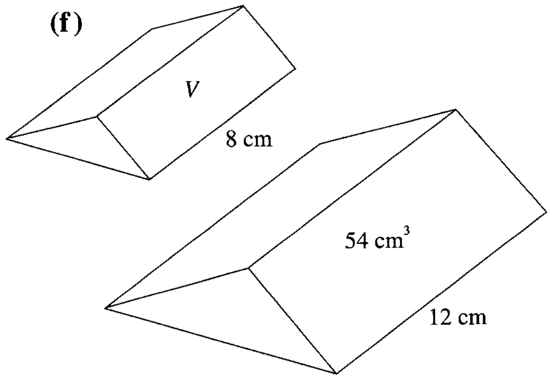
(d)



(e)

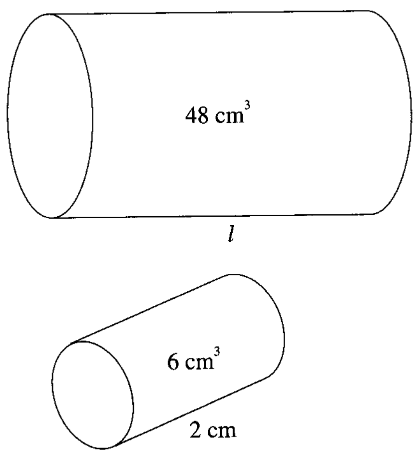


(f)

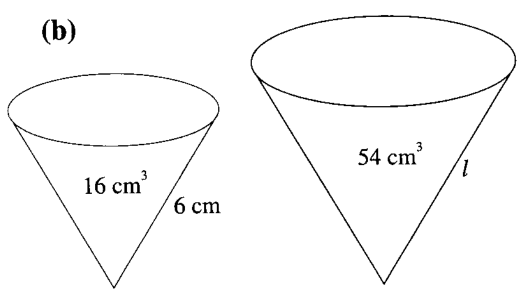


2. Find the unknown length, l, in each of the following pairs of similar objects.

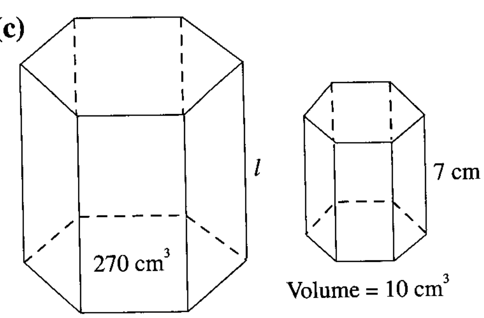
(a)



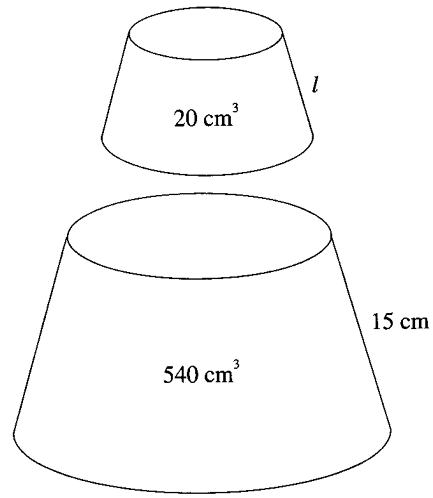
(b)



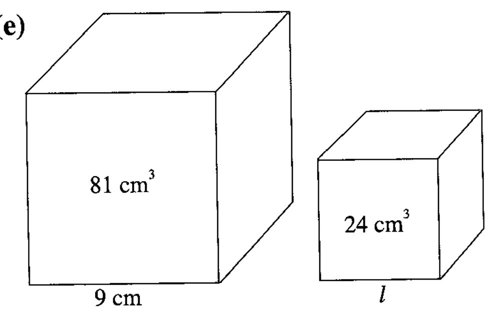
(c)



(d)



(e)



3. The areas of the bases of two similar cones are in the ratio of 9 : 16.

(a) Find the ratio of the heights of the cones.

(b) Given that the volume of the larger cone is 448 cm^3, find the volume of the smaller cone.

4. The areas of the bases of two similar pyramids are in the ratio 25 : 9.

(a) Find the ratio of the heights of the pyramids.

(b) Given that the volume of the smaller pyramid is 108 cm^3, find the volume of the larger pyramid.

5. The surface areas of two spheres are in the ratio of 4 : 25.

(a) Find the ratio of the radii of the spheres.

(b) Given that the volume of the larger sphere is 625 cm^2, find the volume of the smaller sphere.

6. Find the ratio of **(a)** the surface areas; **(b)** the volumes of

(i) two similar solid cylinders of circumferences 10 cm and 8 cm;
(ii) two similar solid cones of heights 9 cm and 12 cm;
(iii) two spheres of radii 4 cm and 6 cm.

7. The weight of a solid cube of side 6 cm is 240 g. Find the weight of a similar cube of side 10 cm.

8. The volumes of two spheres are 640 cm^3 and 270 cm^3. Find the ratio of

(a) their diameters;
(b) the areas of their surfaces.

9. Two similar glasses are of heights 6 cm and 9 cm. If the volume of the second glass is 54 cm^3, find the volume of the first glass.

10. A certain brand of soft drink comes in similar bottles of two sizes, the ordinary and family sizes. The ordinary-sized bottle has a height of 15 cm and a volume of 280 cm^3. If the family-sized bottle has a volume of 750 cm^3, what is its height?

11. The volume of one sphere is 4 times that of a second sphere. If the radius of the smaller sphere is 3 cm, what is the radius of the larger sphere?

12. The weight of a marble statue of height 6 cm is 500 g. What is the weight of a similar marble statue made of the same material if its height is 4 cm?

13. The masses of two similar marble toys of the same density are 8.58 kg and 4.29 kg. If the first toy is 12.94 cm high, what is the height of the second toy?

14. **(a)** A locomotive is 10 m long and weighs 72 tonnes. A similar model, made of the same material, is 40 cm long. find the mass of the model.
(b) Suppose the tank of the model locomotive contains 0.85 litres of water when full, find the capacity of the tank of the locomotive.

Summary

1. **Area of similar figures:**
If any two corresponding lengths of two similar figures are l_1 and l_2, while the corresponding areas are A_1 and A_2, then

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

2. **Volumes of similar figures:**
If any two corresponding lengths of two similar figures are l_1 and l_2, while the corresponding volumes are V_1 and V_2, then

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

Review Questions 6

1. Find the ratio of the areas of two similar triangles if their corresponding sides are
(a) 3 cm and 5 cm; **(b)** 4.5 cm and 9 cm; **(c)** 2 cm and 3 cm.

2. **(a)** If the length of each side of a pentagon is doubled, what will happen to its area?

(b) Given that the length of each side of a pentagon, whose area is 25 cm², is doubled, find the area of the enlarged pentagon.

3. Calculate the ratio of the areas of two similar triangles if

(a) two of their corresponding sides are 6 cm and 9 cm;

(b) their perimeters are 294 cm and 336 cm.

4. The volumes of two similar cylinders are in the ratio of 8 : 27.

(a) Find the ratio of the base radii of the cylinders.

(b) Given that the curved surface area of the smaller cylinder is 96 cm², find the curved surface area of the larger cylinder.

5. The volumes of two similar jugs are in the ratio of 27 : 64.

(a) Find the ratio of the heights of the jugs.

(b) Given that the area of the base of the larger jug is 80 cm², find the area of the base of the smaller jug.

6. A statue has a base of area 7.04 m². A similar model, made of material of the same density, has a base of area 800 cm². Find, in tonnes, the mass of the statue if the model weighs 42.5 kg.

7. A model of a marble statue of height 3.2 m is made of material of the same density. The height of the model is 40 cm. If its mass is 12 kg, find the mass of the statue.

8. A toy motor car, of the same material and density, is an exact model of a real one. The toy car has a 35 cm by 10 cm rectangular windscreen while the real car has a windscreen of area 0.315 m². If the toy car has a mass of 25 kg, find the mass of the real car.

9. A doll's house is a scale model of a real house. The volumes of air in the drawing rooms are 27 500 cm³ and 220 m³ respectively. If the area of the front door of the real house is 7 m², find the area of the front door of the doll's house.

☆10. Two similar spheres have surface areas A_1 and A_2, such that $A_1 = 16A_2$. If their volumes are such that $V_1 = kV_2$, write down the value of k.

☆11. In the diagram, $PQRS$ is a parallelogram. M is the mid-point of QR and PLM and SLQ are straight lines.

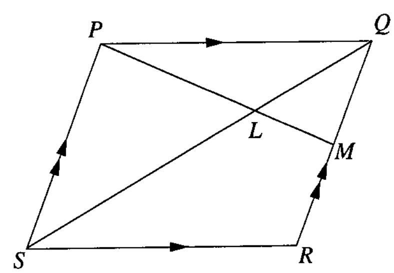
(a) Identify a triangle similar to triangle QLM.

(b) Write down the numerical value of

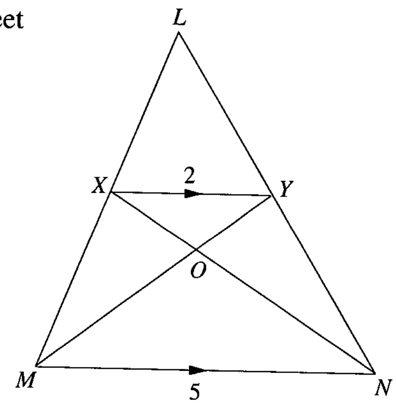
(i) $\dfrac{\text{area of } \triangle QLM}{\text{area of } \triangle SLP}$;

(ii) $\dfrac{LS}{QS}$;

(iii) $\dfrac{\text{area of } \triangle PLS}{\text{area of } \triangle PQS}$.



☆12. In the diagram, XY is parallel to MN while XN and YM meet at O. $XY = 2$ cm and $MN = 5$ cm.



(a) Find the numerical value of $\frac{\text{area of quadrilateral } XMNY}{\text{area of } \triangle LMN}$.

(b) (i) Identify a triangle similar to $\triangle XOY$.

(ii) Write down the value of $\frac{YO}{OM}$.

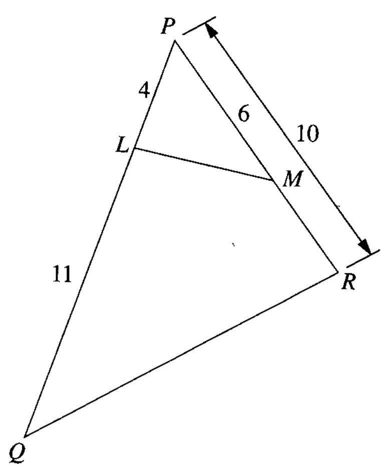
(iii) Show that area of $\triangle XOY$: area of $\triangle XOM$: area of $\triangle MON = 4 : 10 : 25$.

13. Two artificial ponds are similar in every aspect. The perimeter of the surface of the larger pond is three times that of the smaller pond.

(a) Write down the numerical value of the ratio of their surface areas.

(b) Given that the larger pond contains 10 800 litres of water, find the amount of water contained in the smaller pond.

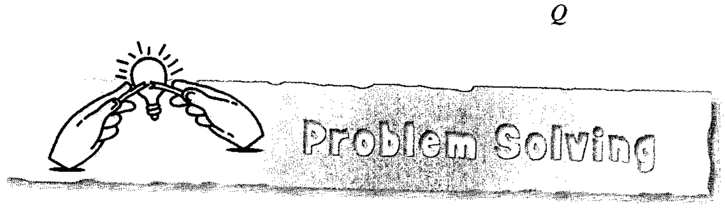
☆14. In the diagram, PLQ and PMR are straight lines. $PL = 4$ cm, $LQ = 11$ cm, $PM = 6$ cm and $PR = 10$ cm.



(a) Show that $\triangle PQR$ and $\triangle PML$ are similar.

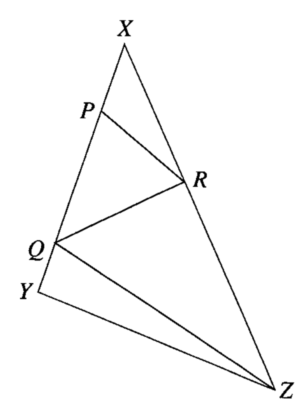
(b) Write down the numerical value of $\frac{\text{area of } \triangle PQR}{\text{area of } \triangle PML}$.

(c) Given that the area of triangle PML is 6 cm², find the area of the quadrilateral $LMRQ$.



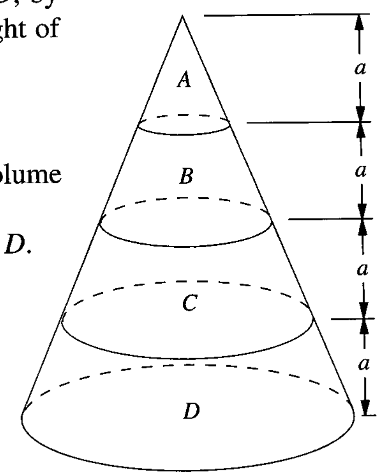
Problem Solving

1. In the diagram, $XP = \frac{1}{2}PQ$, $QY = \frac{1}{3}PQ$ and $XR = \frac{2}{5}XZ$.
Find the ratio of the area of $\triangle QRZ$ to the area of $\triangle QYZ$.



2. A right circular cone is divided into 4 portions, A, B, C and D, by planes parallel to the base, as shown in the diagram. The height of each portion is a units. Find

(a) the ratio of the volume of A to the volume of B;
(b) the ratio of the volume of B to the volume of C;
(c) the ratio of the sum of the volumes of A, B and C to the volume of D;
(d) the ratio of the area of the curved surface of C to that of D.

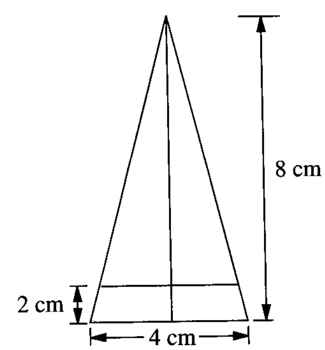


3. Two similar solids have volumes V and v such that $V = 27v$. If their surface areas, S and s respectively, are such that $S = ks$, write down the value of k.

4. The figure shows a vertical section through the axis of a solid paper-weight made in the shape of a right circular cone. Its height is 8 cm and the diameter of its base is 4 cm. The shaded portion is made of lead 2 cm thick while the unshaded portion is made of wood. Lead weighs 11.3 g/cm^3 and wood weighs 0.9 g/cm^3. Calculate

(a) the total volume of the paper-weight;
(b) the volume of the wooden portion;
(c) the total weight of the paper-weight.

[The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$]



Revision Exercise I No. 1

1. Evaluate the following without the use of a calculator:

(a) $8\frac{1}{2} - 3\frac{3}{4} \times \frac{5}{6}$ **(b)** $\left(\frac{25}{64}\right)^{-\frac{1}{2}}$

(c) $0.435\ 85 \div 0.23$ **(d)** 0.257×0.65

2. Multiply

(a) $(x + 1)$ by $(2x - 1)$;

(b) $(x^2 + 3x - 5)$ by $(2x - 3)$.

3. Solve

(a) $6x - 8 - 3x = 3x + 12 - 2x$;

(b) $4 - 2(x - 2) = 15$;

(c) $\frac{3x - 1}{4} - \frac{2x + 1}{5} = 4$.

4. Factorise:

(a) $3x^2 - 2x - 8$; **(b)** $12x^2 - x - 1$;

(c) $16x^2 - 40xy + 25y^2$.

5. (a) The simple interest on $1 462.50 for 4 years is $263.25. Find the rate of interest per annum.

(b) The perimeter of a rectangular cardboard is 500 cm and its area is 14 400 cm^2. Find the dimensions of the cardboard.

6. (a) Express the following as simple fractions:

(i) $\frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ **(ii)** $\frac{x}{3} + \frac{x}{4} + \frac{x}{5}$

(b) A map is drawn to the scale 1 : 50 000. A representation of a lake on the map has a perimeter of 32 cm.

(i) What is the actual perimeter of the lake? Express your answer in terms of kilometres.

(ii) If the lake is a square, calculate the surface area of the lake in square kilometres.

7. The line joining the points (1, 8) and (8, 1) cuts the x- and y-axes at points A and B respectively. Find the coordinates of A and B.

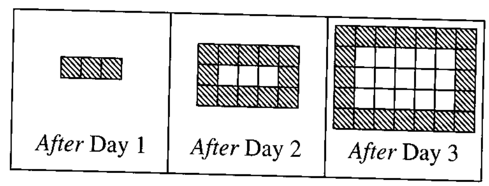
8. Simplify the following:

(a) $(xy^2)^3$ **(b)** $y^3 \times y^{-2} \div y^0$

(c) $\sqrt{a} \times a^{\frac{3}{2}}$ **(d)** $p^{\frac{1}{2}} \div p^{-1\frac{1}{2}}$

9. A large area is to be paved with blocks for every one metre square.

On Day 1, three blocks are placed in a line, as shown in the diagram below. Each following day, the paved region is enlarged by adding blocks to surround the previous day's region.



The perimeter and area of the total region covered by the blocks, after the completion of each day's work, are calculated as in the table.

Perimeter (m)	$2(1 + 3) = 8$	$2(3 + 5) = 16$	$2(5 + 7) = 24$
Area (m^2)	$1 \times 3 = 3$	$3 \times 5 = 15$	$5 \times 7 = 35$

By considering the paved regions as well as the patterns developing in the table, answer the questions below.

(a) Find

(i) the perimeter after Day 4 and after Day 5;

(ii) an expression, in terms of n, in its simplest form, for the perimeter after Day n.

(b) Find

(i) the area after Day 4 and after Day 5;

(ii) an expression, in the form $an^2 - b$, for the area after Day n;

(iii) the total number of blocks which will have been used after Day 15;

(iv) after which day will the area be 399 m^2.

(c) How many blocks will be added during Day 18? *(C)*

10. Solve the following inequalities and illustrate your answer with the number line in each case.

(a) $4x - 11 \leqslant 4 - x$
(b) $6 - 3x > 2x - 14$
(c) $x + 1 < 4x - 21 < 3x - 13$

Revision Exercise I No. 2

1. Evaluate the following, giving your answers correct to 3 significant figures.
   (a) 0.357×28
   (b) $3.576 \div 0.33$
   (c) $1\frac{1}{8} \times \frac{5}{7}$

2. Factorise
   (a) $4x^2 - 12x + 9$;
   (b) $(x - y)^2 - 16$;
   (c) $x(x - 2) - 3xy + 6y$.

3. (a) Solve the equation $\frac{x}{x+1} = \frac{3x}{x+2}$.
   (b) A man purchased 300 oranges at 6 for a dollar. He sold them at 5 for a dollar. What was his percentage gain?

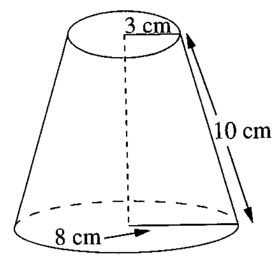
4. Evaluate
   (a) $\sqrt[3]{0.000\,027}$;
   (b) $999\,988^2 - 12^2$;
   (c) $(1.2 \times 10^2)^2 \div (8 \times 10^{-2})$;
   (d) $3\frac{3}{4} + 2\frac{2}{3} - 1\frac{5}{6}$.

5. (a) Make s the subject of the formula

$$v = \sqrt{u^2 + 2as}.$$

   (b) The perimeter of a square exceeds that of another by 100 cm and the area of the larger square exceeds three times the area of the smaller square by 325 cm². Find the length of a side of each of the squares.

6. The solid figure below shows the shape of a frustrum. Its circular top and base have radii 3 cm and 8 cm respectively, as indicated. The slant edge is 10 cm.



   (a) Find the height of the solid.
   (b) Calculate the volumes of 2 appropriate cylinders and use your results to estimate the volume of the solid.

7. (a) Find the equation of the line joining the point $A(5, 7)$ and $B(8, 12)$.
   (b) List the integral values of x which satisfy $2x - 5 < 8 \leqslant 3x - 1$.

8. Simplify the following algebraic fractions:
   (a) $\frac{(x-y)^2}{x^2 - y^2} + \frac{3}{x+y} - \frac{2}{x-y}$
   (b) $\frac{x^2 - 7x + 10}{x^2 - 5x} - \frac{4}{x-5} + \frac{2}{x}$
   (c) $\frac{x^2 - 2x - 15}{x^2 - 3x - 10} + \frac{4}{x-5} - \frac{3x}{x+2}$

9. A man sold two houses at $300 000 each. He made a profit of 20% on the first house. He lost 20% on the second house. Find the cost price of each of the two houses and, hence, find the percentage gain, or loss, on the combined sale of the two houses.

10. (a) Find the equation of the line passing through (2, –5) and parallel to the line $5x + 7y = 46$.
   (b) Solve the following equations
      (i) $10^{3x+2} = 0.01$
      (ii) $\sqrt{2x + 5} = 4$
      (iii) $x^{-2} = 9$

Revision Exercise I No. 3

1. Solve the following equations:

(a) $\frac{x+3}{2x-7} = \frac{2x-1}{x-3}$ **(b)** $\frac{5x+7}{3x+2} = x - 1$

2. Evaluate the following, giving your answers as a fraction in its lowest term.

(a) $1\frac{2}{5} + \frac{7}{10}$ **(b)** $\frac{3}{4} \div \frac{2}{5}$

(c) $1\frac{1}{3} - \frac{2}{5} \times \frac{6}{7}$

3. Solve the equation $\frac{1}{x-2} + \frac{1}{x+1} - \frac{1}{2} = 0$.

4. Factorise:

(a) $x^2 - 8x + 16$; **(b)** $x^6 + 2x^5 + x + 2$.

5. In how many years will a sum *double* itself at $7\frac{1}{2}\%$ simple interest?

6. A man travels 108 km at a speed of x km/h. He finds that he could have saved $4\frac{1}{2}$ hours if he had increased his speed by 2 km/h. Find x.

7. **(a)** The line joining the points (3, 4) and (8, 14) cuts the axes at A and B. Find the area of $\triangle AOB$, where O is the origin.

(b) Find the equation of the line passing through the point (1, 2), with gradient −3. Find the coordinates of the point where this line cuts the x-axis.

8. Simplify the following:

(a) $x^{\frac{2}{3}} \div 2x^{-\frac{1}{3}}$

(b) $\sqrt{a} \times \sqrt[3]{a} \div \sqrt[4]{a} \times a^0$

9. Given that $x = (1 + 2t)^2$ and $y = 2t + 5$,

(a) find the values of x and y when $t = 5$;

(b) express $\frac{x-4}{y-2}$ in terms of t, giving your answer in its simplest form.

10. A square of side x cm has the same area as a rectangle of length $(3x + 5)$ cm and width $(2x - 3)$ cm. Form an equation in x and show that it simplifies to $5x^2 + x - 15 = 0$.

Solve this equation, giving your answer correct to 2 decimal places. Which figure has a greater perimeter? State the difference in perimeter.

CHAPTER 7

Variations

In this chapter, you will learn how to

- △ write an equation connecting the two quantities involved in a variation;
- △ solve problems involving simple direct and inverse variations.

Preliminary Problem

The photograph shows many similar flower pots. Do you know that the volume of each of these flower pots is proportional to the cube of its height?

Direct Variation

In Book 1, you were introduced briefly to the idea of direct proportion. Here, we again consider the fine imposed on borrowers for returning library books late. Consider the case of one overdue book in the following table:

Number of days overdue (x)		*Fine in cents (y)*	
x increases ↓	1	20	y increases ↓
	2	40	
	3	60	
	4	80	
	5	100	
	6	120	
	⋮	⋮	

The two quantities x (the number of days a book is overdue) and y (the fine in cents) are related in such a way that when x is changed in any ratio, y is changed in the same ratio. That is, if x is doubled, then y is also doubled; if x is halved, then y is also halved; and so on. We say that y varies directly as x, or y is **directly proportional** to x.

If y varies directly as x, this relation is written as $\boldsymbol{y \propto x}$. The sign "$\propto$" is read as "**varies as**" and is called the *sign of variation*.

In the above example, we further notice that

$$\frac{y}{x} = \frac{20}{1} = \frac{40}{2} = \frac{60}{3} = \frac{80}{4} = \frac{100}{5} = \frac{120}{6} = \ldots = 20$$

i.e. $$\frac{\textbf{corresponding value of } y}{\textbf{value of } x} = \textbf{constant}$$

Hence, when $y \propto x$, the ratio $\frac{y}{x}$ is a constant, known as the **constant of variation**. If this constant is represented by k, then

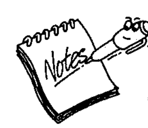
$$\frac{y}{x} = k \quad \text{or} \quad y = kx, \quad k \neq 0.$$

In our example, $k = 20$

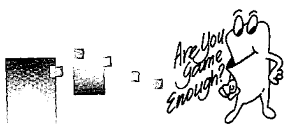
$\therefore \quad y = 20x$

This is an equation connecting x, the number of days a book is overdue and y, the fine in cents. Using this equation, we can determine the value of y corresponding to a given value of x and vice versa.

1. *A ship is anchored at harbour and over one of its sides hangs a ladder. There are altogether 50 steps from the surface of the water to the ship. The steps are at a distance of 50 cm apart. The tide rises at a rate of 50 cm/h. At the end of 5 hours, how many steps are above the water surface?*
2. *Find the least number of boys needed to form the following arrangement: Two boys in front of a boy, two boys behind a boy and a boy between two boys.*



In general, when we are given that a quantity y varies directly as another quantity x, we can write $y = kx$. The constant k can be determined if a pair of corresponding values of x and y is known.



1. A toy train measures $\frac{1}{4}$ m long. How long will it take for the train to pass a $\frac{1}{4}$ m long bridge if it moves at a constant speed of $\frac{1}{4}$ m/s?

2. Another toy train measures $\frac{1}{2}$ m long. If it move at a constant speed of $\frac{1}{2}$ m/s, how long will it take to pass a bridge that is $\frac{1}{2}$ m long?

3. A third toy train measures x m long. How long will it take for the train to pass a bridge that is x m long if it moves at a constant speed of x m/s?

Example 1

If y varies directly as x and if y = 8 when x = 5, find

(a) the equation connecting x and y;
(b) the value of y when x = 10.

Solution

(a) $y \propto x$, i.e. $y = kx$, where k is a constant.

Substitute $x = 5$ and $y = 8$ into the equation:

$$8 = 5k, \quad \text{i.e,} \quad k = \frac{8}{5}$$

$$\therefore \quad y = \frac{8}{5}x \quad \text{or} \quad 5y = 8x$$

(b) Substitute $x = 10$ into the equation: $y = \frac{8}{5} \times 10 = 16$

Example 2

The expenses (E) at a tea party vary directly with the number of guests (N) present. For 40 guests, the expenses are \$320. Find the expenses of 120 guests.

Solution

$E \propto N$, i.e. $E = kN$, where k is a constant.

Substitute $E = 320$ and $N = 40$ into the equation:

$$320 = 40k, \quad \text{i.e. } k = \frac{320}{40} = 8$$

$$\therefore \quad E = 8N.$$

Substitute $N = 120$ into the equation: $E = 8(120) = 960$

$\therefore$ the expenses for 120 guests are \$960.

Exercise 7a

1. If $y \propto x$ and if $y = 6$ when $x = 9$, find y when $x = 15$.

2. If x varies as y and if $x = 20$ when $y = 9$, find x when $y = 18$.

3. If y varies as x and if $y = 16$ when $x = 8$, find y when $x = 30$.

4. If P varies directly as Q and if $P = 4$ when $Q = 2$, find the relation between P and Q.

5. If $M \propto L$ and if $M = 7$ when $L = 3$, find the law connecting M and L. Find also the value of L when $M = 12$.

6. Given $y \propto x$ and $y = 3$ when $x = 4$, find the value of y when $x = 2$ and the value of x when $y = 6$.

7. If A varies directly as B and if $A = 3$ when $B = 27$, find the value of A when $B = 3$ and the value of B when $A = 10$.

8. If y varies directly as x and if $y = 18$ when $x = 6$, find y when $x = 5$, 7 and 13.

9. Given y varies directly as x, copy and complete the tables below:

(a)

x	3	6	8		
y		24		40	60

(b)

x	8			17	22	
y	20	27.3	35			74

(c)

x	1	3	6		
y	2.3			20.7	29.9

10. The horizontal force, F newtons, needed to push a block of weight, W newtons, along a rough horizontal surface is such that F varies directly as W. When $W = 325$, $F = 75$. Find the equation connecting F and W.
Calculate
(a) F when $W = 240$; **(b)** W when $F = 78$.

11. The interest, \$$I$, on a loan of \$$P$ for a year at a rate of 6% varies directly as the loan. Find the formula relating I and P.
Calculate
(a) I when $P = 800$; **(b)** P when $I = 72$.

12. The cost, \$$C$, of transporting goods varies directly as the distance, d km. Given that $C = 50$ when $d = 120$, find
(a) C when $d = 156$; **(b)** d when $C = 85$.

Other Forms of Direct Variation

The area A of a circle of radius r is given by the equation

$$A = \pi r^2, \text{ where } \pi \text{ is a constant.}$$

The volume V of a sphere of radius r is given by the equation

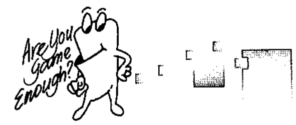
$$V = \frac{4}{3}\pi r^3, \text{ where the expression } \frac{4}{3}\pi \text{ is a constant.}$$

NB: A is not directly proportional to r, but to r^2. We say that 'A varies directly as the square of r', i.e, $A \propto r^2$.

Similarly, we say that 'V varies directly as the cube of r', or $V \propto r^3$.

Example 3

Given that y varies directly as the square of x and that $y = 16$ when $x = 2$, calculate the value of y when $x = 5$.



What about SHOPPING $\propto \frac{1}{MONEY}$? What kind of a relationship do these variable share?

Solution

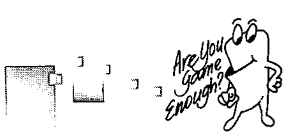
$y \propto x^2$, i.e. $y = kx^2$, where k is a constant.

Substitute $y = 16$ and $x = 2$ into the equation:

$$16 = k(2)^2 \quad \text{or} \quad k = \frac{16}{4} = 4$$

$$\therefore \quad y = 4x^2.$$

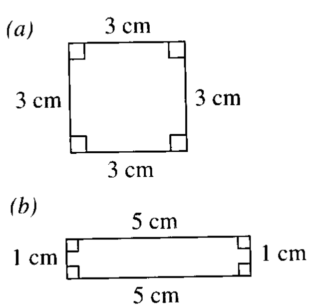
When $x = 5$, $\quad y = 4(5)^2 = 100$



You are given a string of length 12 cm. You are to form a quadrilateral whose sides are integers and whose areas are

(a) 9 cm²; (b) 5 cm²;
(c) 4 cm²; (d) 3 cm².

The first two have been done for you as shown below:



Example 4

The total monthly cost, \$C, of running a kindergarten with an enrolment of n children consists of a fixed amount of \$5 000 and a variable amount which depends on the enrolment. For every child enrolled, the variable amount is \$41.

(a) Calculate the total monthly cost of running the kindergarten if the enrolment is 200.
(b) If the total monthly cost of running the kindergarten is \$20 580, calculate the number of children attending the kindergarten.
(c) Write down a formula connecting C and n.

Solution

(a) Total monthly cost of running the kindergarten if the enrolment is 200 = \$5 000 + \$41 × 200
= \$5 000 + \$8 200 = \$13 200

(b) The variable amount = \$20 580 − \$5 000 = \$15 580

The number of children in the kindergarten = $\frac{15\,580}{41} = 380$

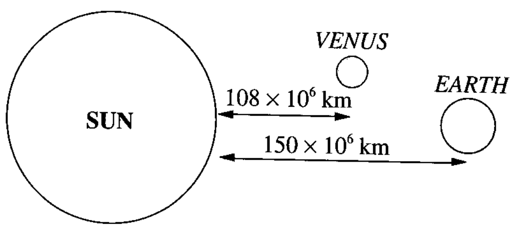
(c) The variable amount for n children = \$41n

$$\therefore \quad C = 5\,000 + 41n$$

Example 5

Kepler's law states that the squares of the time taken for a planet to revolve around the sun once varies directly as the cube of the distance of the planet from the sun. It is known that the distances of Earth and Venus from the sun are 150 and 108 million kilometres respectively. Find the time taken for Venus to revolve once around the sun, if the time taken for Earth to do the same is 365 days. Give your answer to the nearest whole number.

Solution



Let P be the time taken in days, for a planet to revolve once around the sun and D be the distance, in million kilometres, of a planet from the sun.

$P^2 \propto D^3$, i.e. $P^2 = kD^3$, where k is a constant.

Earth: $P = 365$, $D = 150$.

$(365)^2 = k \times (150)^3$ or $k \approx 0.039\,5$

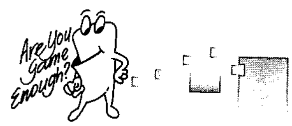
$\therefore$ $P^2 = 0.039\,5D^3$

Venus: $P = \text{?}$, $D = 108$

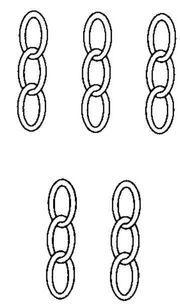
$P^2 = 0.039\,5\,(108)^3$

$P = \sqrt{0.039\,5\,(108)^3} \approx 223$ (since $P > 0$)

$\therefore$ Venus takes approximately 223 days to revolve once around the sun.



1. *For every 3 kg of aluminium trash that Mr Green recycles, he can make 20 new biscuit tins. How many biscuit tins can he obtain from 11 kg of aluminium trash and how many kilograms of aluminium material will be left over?*
2. *If it costs $1 to break a chain and $2 to weld it back, what is the least cost to join the following 5 chains to form a long chain?*



Exercise 7b

1. Given that $y \propto x^2$ and that $y = 6$ when $x = 2$, find the value of y when $x = 3$ and the value of x when $y = 4$.

2. Given that P varies directly as Q^2 and that $P = 13\frac{1}{2}$ when $Q = 3$, calculate the value of P when $Q = 2\frac{1}{2}$.

3. Given that $w \propto d^2$ and that $w = 36$ when $d = 3$, find the relation connecting w and d. Find also the value of w when $d = 4$.

4. Given that the volume of a sphere is directly proportional to the cube of its radius, copy and complete the table below:

Volume				113
Radius	1	1.5	2	3

5. Given that the mass, m g, of a sphere varies directly as the cube of its radius, r mm, and that $m = 67.5$ when $r = 3$, calculate m when $r = 4$.

6. Given that y varies directly as x^2 and that $y = 12.5$ when $x = 2.5$, find an equation expressing y in terms of x. Find the value of y when $x = 3$.

7. If v varies directly as $\sqrt{h}$ and if $v = 71$ when $h = 25$, find v when $h = 9$.

8. If $y \propto \sqrt{x}$ and if $y = 0.4$ when $x = 4$, find y in terms of x. Also find y when $x = 100$.

9. Given that $V \propto r^3$, copy and complete the table below:

r		2		$1\frac{1}{3}$
V	3	24	375	

10. The length, l cm, of a simple pendulum varies directly as the square of its period T (time to swing to and fro in seconds). A pendulum with period 3 s is 220.5 cm long.
    (a) Find the length of a pendulum whose period is 5 s.
    (b) What is the period of a pendulum whose length is 0.98 m?
    (c) What equation connects l and T?

11. A company pays a salesman $$D$ per month to sell tyres. The amount is made up of a basic salary of $600 plus $8 for each of the n tyres he sells each month.
    (a) Calculate the salesman's income for the month when he sold 95 tyres.
    (b) If in a particular month he received $1 680, find the number of tyres he sold.
    (c) Write down a formula connecting D and n.

12. If y varies directly as $(x - 1)^2$ and if $y = 18$ when $x = 4$, find
    (a) the value of y when $x = 5$; **(b)** the value of x when $y = 8$.

13. Within a certain period of its life, the length, L, of an earthworm varies directly as the square root of N, the number of hours after its birth. If a worm is $2\frac{1}{2}$ cm long after 1 hour, how long will it be after 4 hours? How long will it take to grow to a length of 15 cm?

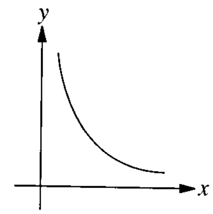
14. An ice manufacturing machine requires 10 minutes to warm up before the production of ice begins. The weight, in tonnes, of ice produced varies directly as the number of hours of production. Given that 20 tonnes of ice are produced when the machine is run for half an hour, find the weight of ice manufactured when the machine is run for $1\frac{3}{4}$ hours.

Inverse Variation

The times taken by a car to travel a distance of 120 km at various speeds are as follows:

	Speed, x (in km/h)		*Time taken, y (in hours)*	
x increases ↓	20	⟷	6	y decreases ↓
	30	⟷	4	
	40	⟷	3	
	60	⟷	2	
	120	⟷	1	
	⋮	⋮	⋮	

In general, when y is inversely proportional to x, the graph of y against x is a hyperbola as shown below:



The two quantities, x (speed) and y (time taken), are related in such a way that when one quantity increases, the other decreases. We notice that when x is doubled, y is halved and when x is halved, y is doubled, and so on. We further notice that

$$\frac{y}{\left(\frac{1}{x}\right)} = xy = 20 \times 6 = 30 \times 4 = 40 \times 3 = 60 \times 2 = 120 \times 1 = \ldots = 120.$$

i.e. $\dfrac{\textbf{corresponding value of } y}{\textbf{value of } \frac{1}{x}}$ **= value of x × corresponding value of y = constant**

We say that y **varies inversely** as x and this relation is written as $y \propto \frac{1}{x}$.

NB: If y varies inversely as x, then y varies directly as $\frac{1}{x}$,

i.e. $\dfrac{y}{\frac{1}{x}} = k$, or $xy = k$, or $y = \dfrac{k}{x}$, k is a non-zero constant.

Have you heard of Boyle's Law? This law states that for a fixed mass of gas at constant temperature, its volume is inversely proportional to its pressure, or $V \propto \frac{1}{P}$. *If its pressure is doubled, then its volume is halved.*

Example 6

Given that y varies inversely as x and that y = 6 when x = 5, calculate the value of y when x = 12.

Solution

$y \propto \frac{1}{x}$, i.e. $xy = k$ or $y = \frac{k}{x}$, where k is a constant.

Substitute $x = 5$ and $y = 6$ into the equation:

$$5 \times 6 = k \Rightarrow k = 30$$

$$\therefore \quad y = \frac{30}{x}$$

When $x = 12$, $\quad y = \dfrac{30}{12} = 2\dfrac{1}{2}$

Example 7

Six men can complete a certain job in 8 hours. Suppose all the men work at the same speed, how long will 18 men take to complete the same job?

Solution

Suppose x men take y hours to finish the job.

It is then obvious that more men will take less time to complete a given job. Thus, y varies inversely as x.

We have $y \propto \frac{1}{x}$

i.e. $xy = k$ or $y = \frac{k}{x}$, where k is a constant.

Substitute $x = 6$ and $y = 8$ into the equation:

$$8 \times 6 = k \Rightarrow k = 48$$

$$\therefore \quad y = \frac{48}{x}$$

When $x = 18$, $y = \frac{48}{18} = 2\frac{2}{3}$

$\therefore$ 18 men will take $2\frac{2}{3}$ hours or 2 hours 40 minutes to finish the job.

Example 8

If y varies inversely as x^2 and if $y = 8$ when $x = \frac{3}{2}$, find the equation connecting x and y. Find also the value of y when $x = 3$.

Solution

$y \propto \frac{1}{x^2}$, i.e. $x^2y = k$ or $y = \frac{k}{x^2}$, where k is a constant.

Substitute $x = \frac{3}{2}$ and $y = 8$ into the equation:

$$k = 8 \times \left(\frac{3}{2}\right)^2 = 8 \times \frac{9}{4} = 18$$

$$\therefore \quad y = \frac{18}{x^2}$$

When $x = 3$, $y = \frac{18}{9} = 2$

Example 9

In an experiment, a drug is added to identical flasks, each containing the same amount of a certain bacteria. The drug is then allowed to react with the bacteria for various times in t hours. It is found that the amount of bacteria left, s units, varies inversely as $(t - 2)$ hours. In one flask, there were 6 units of bacteria left after 5 hours. Calculate the amount of units of bacteria left in another flask after 7 hours.

Solution

$s \propto \frac{1}{t-2}$, i.e. $s(t - 2) = k$ or $s = \frac{k}{t-2}$, where k is a constant.

Substitute $s = 6$ and $t = 5$ into the equation:

$$6(5 - 2) = k \Rightarrow k = 18$$

$$\therefore \quad s = \frac{18}{t - 2}$$

When $t = 7$, $s = \frac{18}{7 - 2} = \frac{18}{5} = 3\frac{3}{5}$

Hence, the amount of bacteria left in the flask after 7 hours is $3\frac{3}{5}$ units.

Exercise 7c

1. If y varies inversely as x and if $y = 120$ when $x = 2$, form an equation connecting x and y. Calculate y when $x = 5$.

2. If y varies inversely as x and if $y = 2$ when $x = 9$, find the equation connecting x and y. Calculate the value of y when $x = 6$.

3. If $p \propto \frac{1}{v}$ and if $v = 30$ when $p = 10$, find p when $v = 35$.

4. If $y \propto \frac{1}{x}$ and if $y = 12$ when $x = 6$, find y in terms of x. Calculate y when $x = 8$ and x when $y = 0.5$.

5. If y varies inversely as $(x - 3)$ and if $y = 5$ when $x = 7$, find y when $x = 5$.

6. If $y \propto \frac{1}{x^2}$, and if $y = 9$ when $x = 11$, find y in terms of x. Calculate
   (a) y when $x = 6$; **(b)** x when $y = 4$.

7. Given that y varies inversely as x^2 and that $y = 4\frac{1}{2}$ when $x = 2$, find the equation connecting x and y. Find the value of y when $x = -\frac{3}{2}$.

8. If y varies inversely as the square root of x and if $y = 6$ when $x = 4$, find y in terms of x. Find also x when $y = 9$ and y when $x = 8$.

9. Given that $P \propto \frac{1}{Q}$, copy and complete the table below:

Q	2	3		$\frac{1}{3}$
P	6		12	

10. Given that $y \propto \frac{1}{x^2}$, copy and complete the table below:

x	2	4			8
y		3	$\frac{1}{3}$	27	

11. Given that $S \propto \frac{1}{\sqrt{t}}$, copy and complete the table below:

t	1	4			16
S	8		$1\frac{1}{3}$	16	

12. Given that $y = \frac{k}{x^n}$, find k and n, then copy and complete the table:

x	1	2	4	
y	80	10		$\frac{1}{100}$

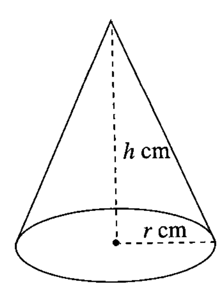
13. The frequency f of a radio wave varies inversely as its wavelength W. When $W = 3\ 000$ m and $f = 100$ kHz, find the frequency when the wavelength is 500 m. Find the wavelength when the frequency is 800 kHz (kHz = kilohertz).

14. The pressure P of a given mass of gas at constant temperature is inversely proportional to its volume V. When $P = 250$ N/m^2 and $V = 4$ m^3, find

(a) P when $V = 5$ m^3;　　**(b)** V when $P = 750$ N/m^2.

15. The number of hours, n, required to complete a certain job is inversely proportional to the number of workers available, x. When 8 men are working, the job takes 3 hours. If we need to complete the job in $\frac{3}{4}$ hour, how many workers are required? Find the time taken to complete the job when 6 men are available.

16. For a given volume, the height h cm of a cone varies inversely as the square of the base radius r cm. Find the height of a cone of base radius 3 cm if it has the same volume as a cone of base radius 6 cm and of height 5 cm.



Summary

1. If y varies directly as x, we write $y \propto x$ and $y = kx$, where k is a non-zero constant.
2. If y varies inversely as x, we write $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$, where k is a non-zero constant.
3. The constant k in the above cases can be determined if one set of values, each of x and y, is given.

Review Questions 7

1. Given that $R \propto S$ and that $R = 140$ when $S = -5$, find the constant of the variation in this case, and calculate the value of S when $R = 170$.
2. If $x \propto \sqrt[3]{v}$ and if $x = 4$ when $v = 64$, find the value of x when $v = 125$ and the value of v when $x = 2$.
3. If y is directly proportional to x and if $y = 6$ when $x = 2$,
   (a) express y in terms of x; **(b)** find the value of x when $y = 12$.
4. Given that x is directly proportional to y^2, and that $x = 9\frac{3}{8}$ when $y = 2\frac{1}{2}$, calculate the value of x when $y = 3$.
5. Given that $A \propto B$ and that $A = 1\frac{2}{3}$ when $B = \frac{5}{6}$, find A when $B = \frac{1}{3}$ and B when $A = 7\frac{1}{2}$.
6. If $V \propto x^3$ and if $V = 108$ when $x = 3$, find V in terms of x. Find also V when $x = 6$ and x when $V = 4\ 000$.
7. The monthly telephone charges for a household, \$$C$, is given by the formula

$$C = a + bn,$$

   where n is the number of units of time during which the telephone is used, and a and b are constants. When 300 units of time are used, the charges are \$29 and when 700 units of time are used, the charges are \$57.
   (a) Write down two equations in a and b.
   (b) Find the values of a and b by solving these equations.
   (c) Find the monthly charges if the telephone is used for 320 units of time.
8. Given that y varies directly as the square of x and that $y = 112$ when $x = 4$, calculate
   (a) the value of y when $x = 2$; **(b)** the value of x when $y = 700$.
9. If d is directly proportional to t^2 and if $d = 27$ when $t = 3$, find an equation giving d in terms of t.
10. Given that the mass m of a cube varies directly as the cube of its edge x and that $m = 24$ when $x = 2$, find m when $x = 3$.
11. Given that N is inversely proportional to r^2 and that $N = 3$ when $r = 5$, find the value of N when $r = 10$.
12. The value of a mirror exceeding 10 m^2 in area varies directly as the square of its area. Given that a 60-m^2 mirror costs \$400, find the price of a 45-m^2 mirror.

☆13. Given the table of values for s and t, write down a formula expressing s in terms of t for these values:

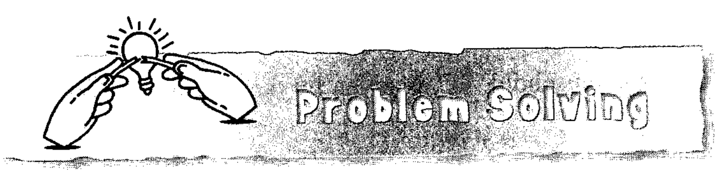
t	0	1	2	3
s	0	5	20	45

☆14. It is given that the force between two particles is inversely proportional to the square of the distance between them. If the force is F when the distance between them is r, and cF when the distance is $5r$, write down the value of c.

☆15. If y varies directly as x^2 and if the difference in the values of y when $x = 1$ and $x = 3$ is 32, find the value of y when $x = -2$.

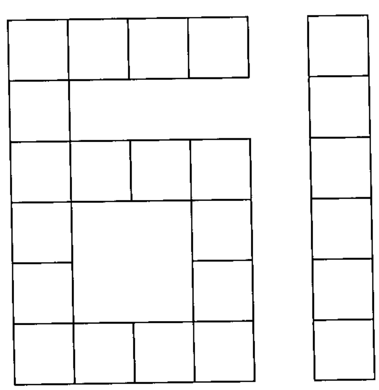
☆16. 5 men are hired to complete a certain job. If an additional man is hired, the job can be completed 8 days earlier. Given that the number of days required to complete the job is inversely proportional to the number of men hired, find how many additional men must be hired in order for the job to be completed 28 days earlier.

☆17. A man donates a certain amount of money to a charity organisation each month. His monthly donation is directly proportional to the square of his monthly savings. Given that he saves \$900 and \$1 200 in January and February 1998 respectively, and that his donation increased by \$35 in February, find the amounts he donated to the charity organisation in January and February 1998.



Example 10

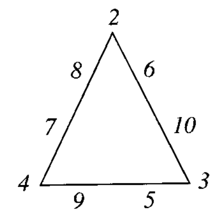
Fill in the numbers from 1 to 23 into the 23 squares so that the sum of each row and each column with a minimum of 5 squares is equal to 61.



Solution

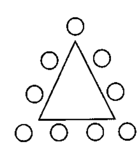
There are 3 rows and 3 columns with a minimum of 5 squares in all. The sum of the rows and columns will be $61 \times 6 = 366$.

1. *The sum of the numbers on each side of the triangle given below is 21.*

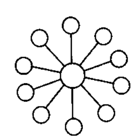


Rearrange the numbers such that the sum on each side is now 24.

2. *Write each of the digits 1 to 9 in each circle below so that the sum of the numbers on each side of the triangle is 20.*



3. *Write each of the digits 1 to 11 in each circle below so that the sum of the numbers along any straight line is 18.*



The sum of the numbers (1 + 2 + 3 + … + 21 + 22 + 23) is equal to [(1 + 23) + (2 + 22) + (3 + 21) + … + (11 + 13) + 12] = (11 × 24) + 12 = 276.

The difference between 366 and 276 is 90. As there are 9 intersecting points where the numbers need to be counted twice, we must find 9 numbers whose sum is 90. There are a few combinations of numbers whose sum is 90. One such set of numbers is 2, 3, 4, 5, 10, 14, 15, 17 and 20.

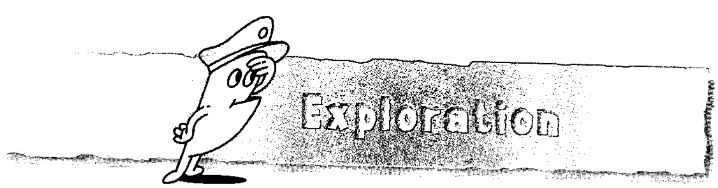
15			4		2
10			20		17
3			14		5

Once these five numbers are filled in, we proceed to do the rest. Starting with the columns on the left, proceed with the rows and leave the right column till the last.

The final answer is shown below:

15	19	21	4		2
7					22
10	1	13	20		17
18			11		6
8			12		9
3	16	23	14		5

We have used *logical reasoning* and *trial and error* to achieve our result. Can you give an alternative solution to the above?



1. If $A \propto C$ and $B \propto C$, prove that $A + B$, $A - B$ and $\sqrt{AB}$ each varies directly as C.

2. A varies directly as x and B is inversely proportional to x^2. Given that $y = A + B$ and that $y = 19$ when $x = 2$ or 3, express y in terms of x.

3. The cost, $\$C$, of printing n copies of a book is partly constant and varies partly as n. The selling price of each copy of the book is fixed as $\$b$. If 630 copies were printed and sold, there would be a loss of 10%. If 980 copies were printed and sold, there would be a profit of 12%. How many copies would have to be printed and sold in order to break even (i.e. neither gain nor lose)?

4. A developer estimates that he needs 96 men to build a house in 14 days. If he is asked to complete the building in 12 days, how many more men must he hire, assuming that the men work at the same rate?

5. If y varies as x^n, the relationship between x and y can be expressed by the equation $y = kx^n$, where k and n are both constants. Write down the value of n if

   (a) y varies as the cube of x;
   (b) y varies inversely as the square root of x;
   (c) y varies as the fourth root of x.

   If y varies as the square of x and if $y = \frac{3}{4}$ when $x = \frac{1}{4}$, find the value of y when $x = 2.5$.

CHAPTER 8

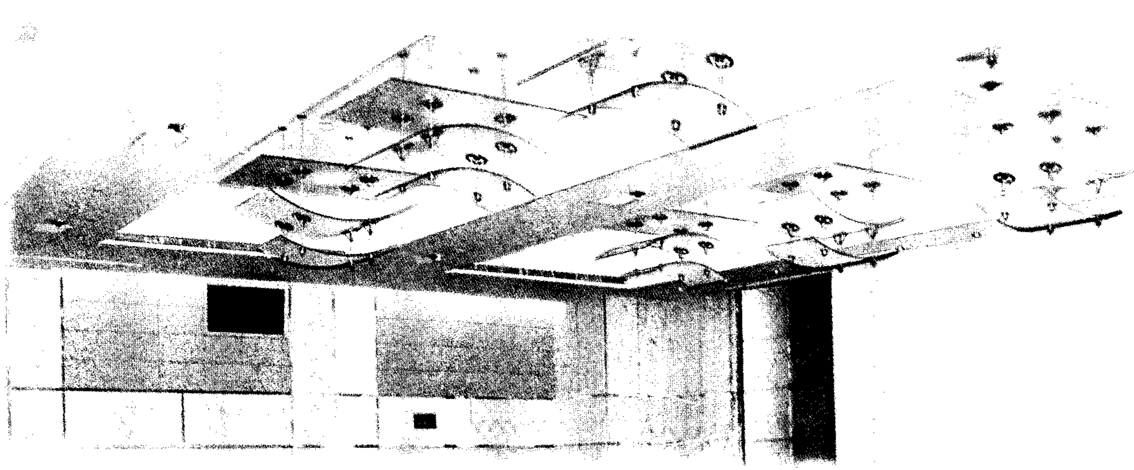
Graphical Solution of Equations

In this chapter, you will learn how to

- △ solve simultaneous linear equations graphically;
- △ solve quadratic, cubic, hyperbolic and exponential equations graphically.

Preliminary Problem

The photograph shows the decorative shapes of glass panels hanging from the ceiling of a function hall in a local hotel. Each part of the glass panel resembles a quadratic graph.



Graphs of Cubic Functions

In Book 2, you learnt how to draw graphs of linear and quadratic functions, or parabolas. Now, you will learn how to draw graphs of cubic, hyperbolic, reciprocal and exponential functions.

The general form of a **cubic function** is $y = ax^3 + bx^2 + cx + d$ where a, b, c and d are **real numbers** and a, which is the coefficient of the highest power of x, is not zero. In addition, a, b and c, which are constants, are called the coefficients of the respective powers of the variable x.

Example 1

Draw the graph of $y = x^3 - 15x + 5$ for values of x from -4 to 4 and use your graph to find

(a) the value of y when $x = 2.5$; *(b) the value of x when $y = 5$.*

Solution

A table displaying the values of x and y is set up as shown below:

x	−4	−3	−2	−1	0	1	2	3	4
x^3	−64	−27	−8	−1	0	1	8	27	64
$-15x$	60	45	30	15	0	−15	−30	−45	−60
5	5	5	5	5	5	5	5	5	5
$y = x^3 - 15x + 5$	1	23	27	19	5	−9	−17	−13	9

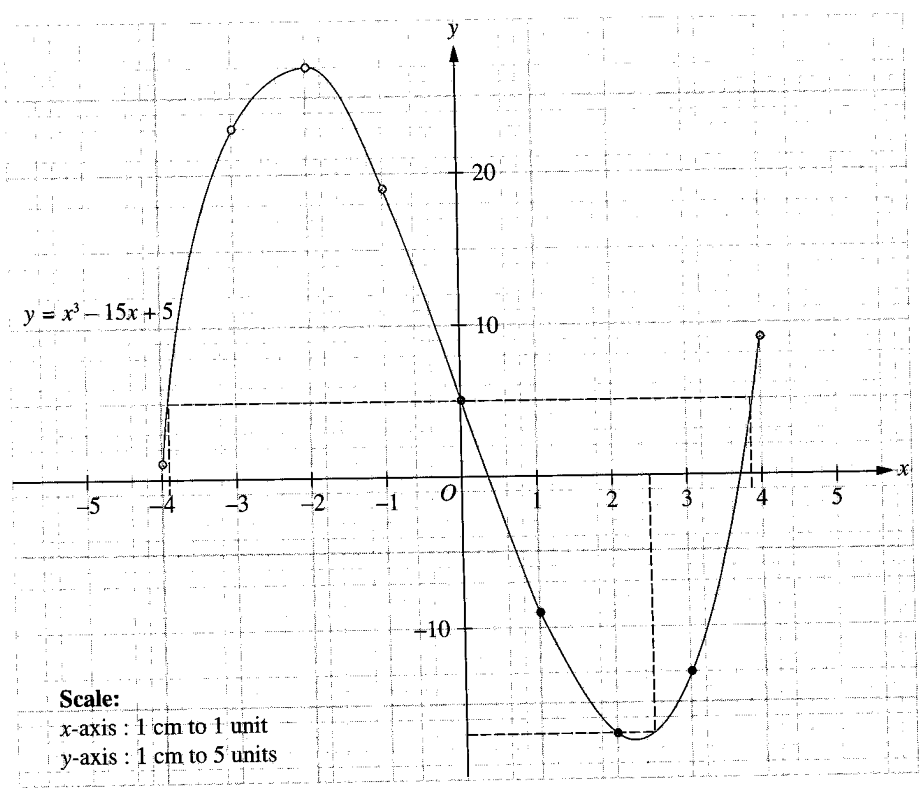


Fig. 8.1

Before plotting the graph, choose a suitable scale so that the graph covers at least half of the graph paper. For this graph, we use a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis. The coordinates of the points are plotted and joined by a smooth curve as shown in Fig. 8.1.

From the graph,

(a) when $x = 2.5$, $y \approx -17$

(b) when $y = 5$, $x \approx -3.9$, 0 or 3.9.

Example 2

Draw the graph of $y = x^3 + 3$ for values of x from -3 to 3 and use your graph to find

(a) the value of y when $x = 1.5$;

(b) the value of x when $y = 20$.

Solution

The table displaying values of x and y is shown below:

x	-3	-2	-1	0	1	2	3
$y = x^3 + 3$	-24	-5	2	3	4	11	30

The graph of $y = x^3 + 3$ is plotted as shown in Fig. 8.2.

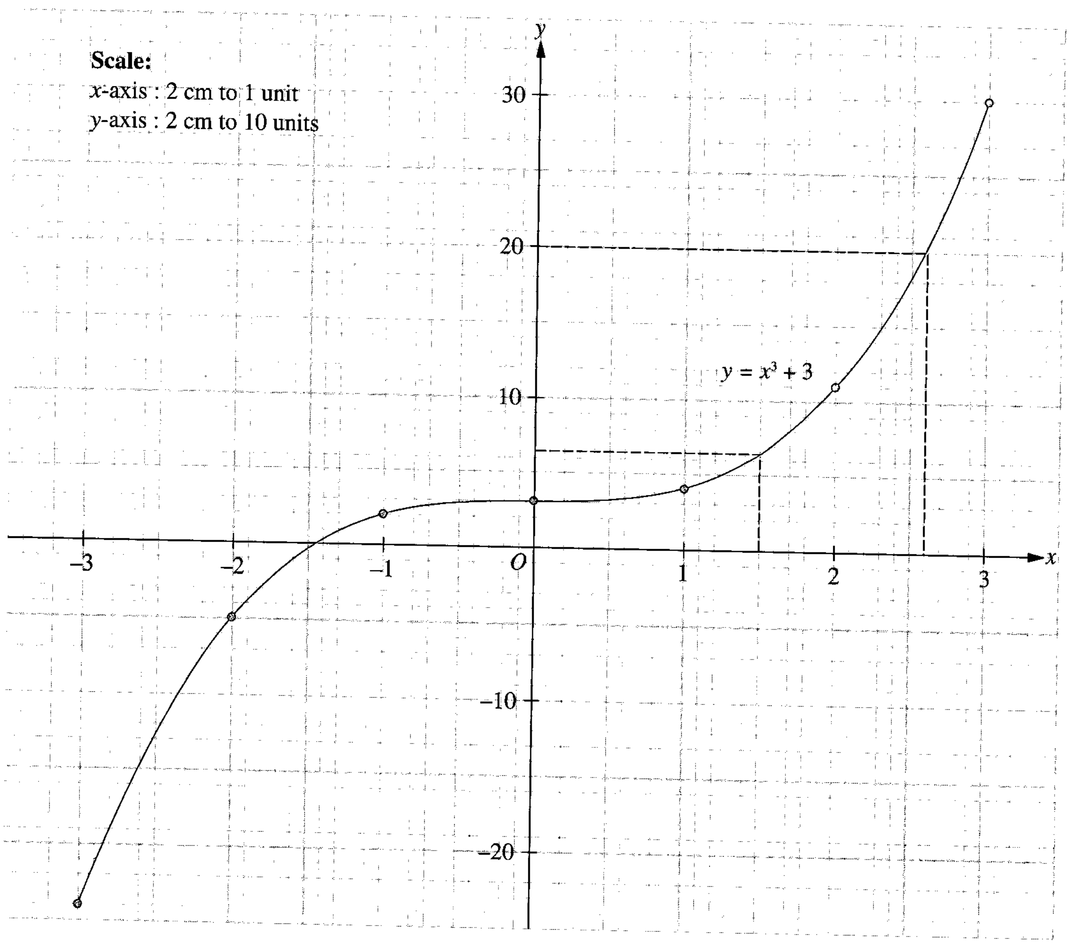


Fig. 8.2

From the graph,

(a) when $x = 1.5$, $y \approx 6.5$ (b) when $y = 20$, $x \approx 2.6$.

A reciprocal function is a special kind of hyperbolic function, which has a more general form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Graphs of Reciprocal Functions

The general form of a **reciprocal function** or **rectangular hyperbolic function** is $y = \frac{k}{x}$, where k is a **real number**. The function $y = \frac{k}{x}$ is defined for all values of x, except when $x = 0$.

Example 3

Draw the graph of $y = \frac{6}{x}$ for $-5 \leqslant x \leqslant 5$.

Solution

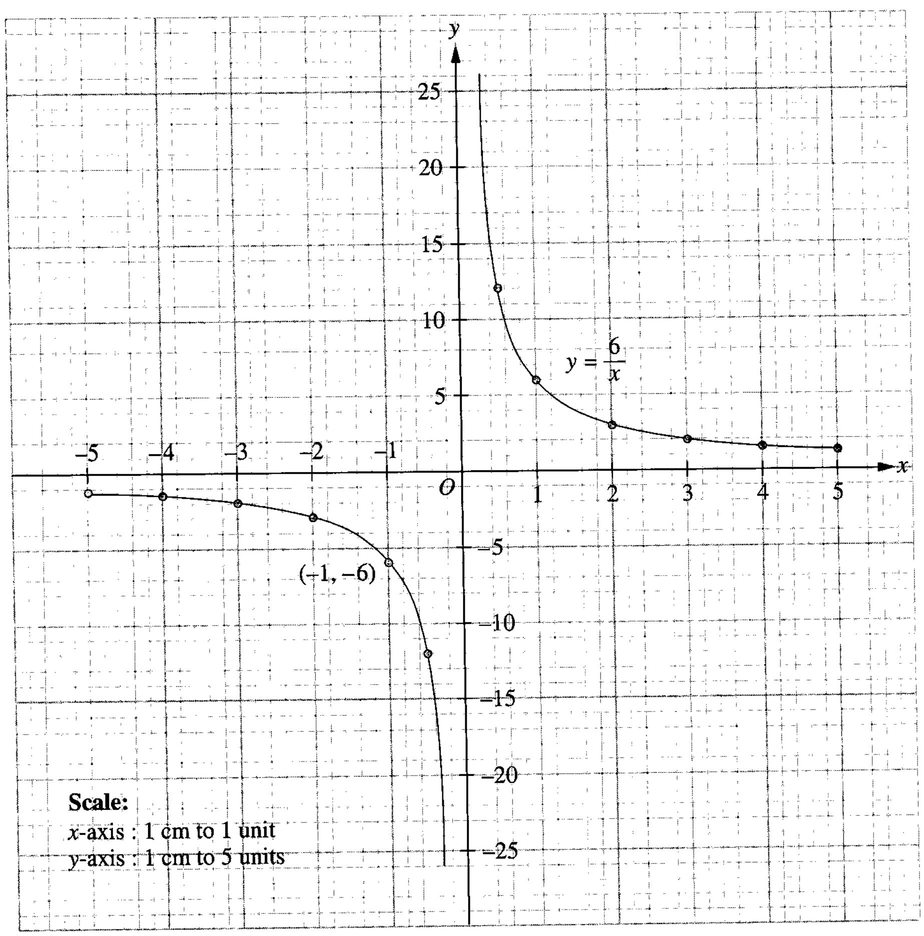


Fig. 8.3

The table displaying values of x and y for positive x is

x	0.5	1	2	3	4	5
y	12	6	3	2	1.5	1.2

and that for negative x is

x	−0.5	−1	−2	−3	−4	−5
y	−12	−6	−3	−2	−1.5	−1.2

The coordinates of the points are plotted as shown in Fig. 8.3.

NB: Drawing graphs may seem complicated at first but after the plotting of tables and sketching, it becomes a much simpler process.

Graphs of the Function $y = \frac{a}{x^2}$

The following example involves a graph of the function $y = \frac{a}{x^2}$, where a is a constant and $x \neq 0$.

Example 4

Draw the graph of $y = \frac{1}{x^2}$ for values of x between −4 and 4, except for $x = 0$.

Solution

The table displaying the values of $y = \frac{1}{x^2}$ is given below:

x	−4	−3	−2	−1	−0.5	0.5	1	2	3	4
x^2	16	9	4	1	0.25	0.25	1	4	9	16
$y = \frac{1}{x^2}$	0.06	0.11	0.25	1	4	4	1	0.25	0.11	0.06

Notes

(1) Example 3 shows that although the graph occurs in two separate parts, it must be regarded as a single graph and not as two separate graphs.

*(2) As the positive value of x increases, the value of y decreases. The curve gets very close to the x-axis but never touches it. As the positive value of x decreases, the value of y increases rapidly and it gets very close to the y-axis. The x and y axes are known as the **asymptotes**.*
*When x is negative, the value of y becomes larger as x becomes smaller and when x gets very close to 0, the value of y decreases rapidly. In other words, as x approaches zero, y approaches **infinity** (symbol: ∞).*

(3) When $x = 0$, the function $y = \frac{6}{x}$ is not defined. This means that there is a break when $x = 0$.

(4) The two parts of the graph are mirror images of each other. The equation of the mirror line of symmetry is $y = x$ or $y = -x$.

*(5) We call a graph of this kind a **rectangular hyperbola**, which has many applications in the field of Physics.*

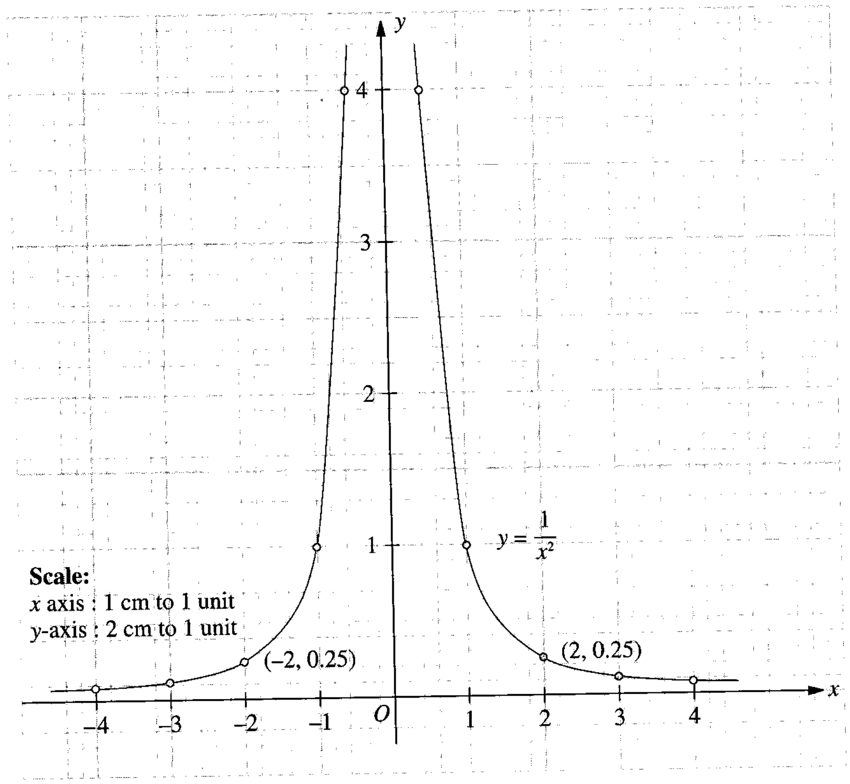


Fig. 8.4

From Fig. 8.4,

(1) the function is made up of two parts and should be treated as a single graph.
(2) the values of y are always positive. Therefore the curve lies entirely above the x-axis.
(3) the curve is symmetrical about the y-axis and thus the y-axis is the line of symmetry.
(4) the graph is not defined when $x = 0$.

Graphs of Exponential Functions

The general form of an **exponential function** is $y = a^x$, where a is a positive number. The function $y = a^x$ is defined for all real values of x.

Notes

From Example 5, we observe that:

(1) there is no negative value of y for all real values of x. Therefore the graph lies entirely above the x-axis.
(2) as the value for negative x increases, the value of y increases very slowly. As x increases its value in the positive range, the value of y increases very rapidly.

Example 5

Given that $y = 2^x$, copy and complete the following table of values. Give all values of x and y correct to 1 decimal place where necessary.

x	−1	−0.5	0	0.5	1	1.5	2	2.5
y	0.5		1	1.4	2			

(a) Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 2^x$ *for* $-1 \leqslant x \leqslant 2.5$.
(b) Use your graph to estimate the values of
(i) y *when* $x = -0.8$, *0.8 and 1.8;*
(ii) x *when* $y = 0.8$, *2.6 and 4.6.*

Solution

x	–1	–0.5	0	0.5	1	1.5	2	2.5
y	0.5	0.7	1	1.4	2	2.8	4	5.7

(a) The graph is shown below:

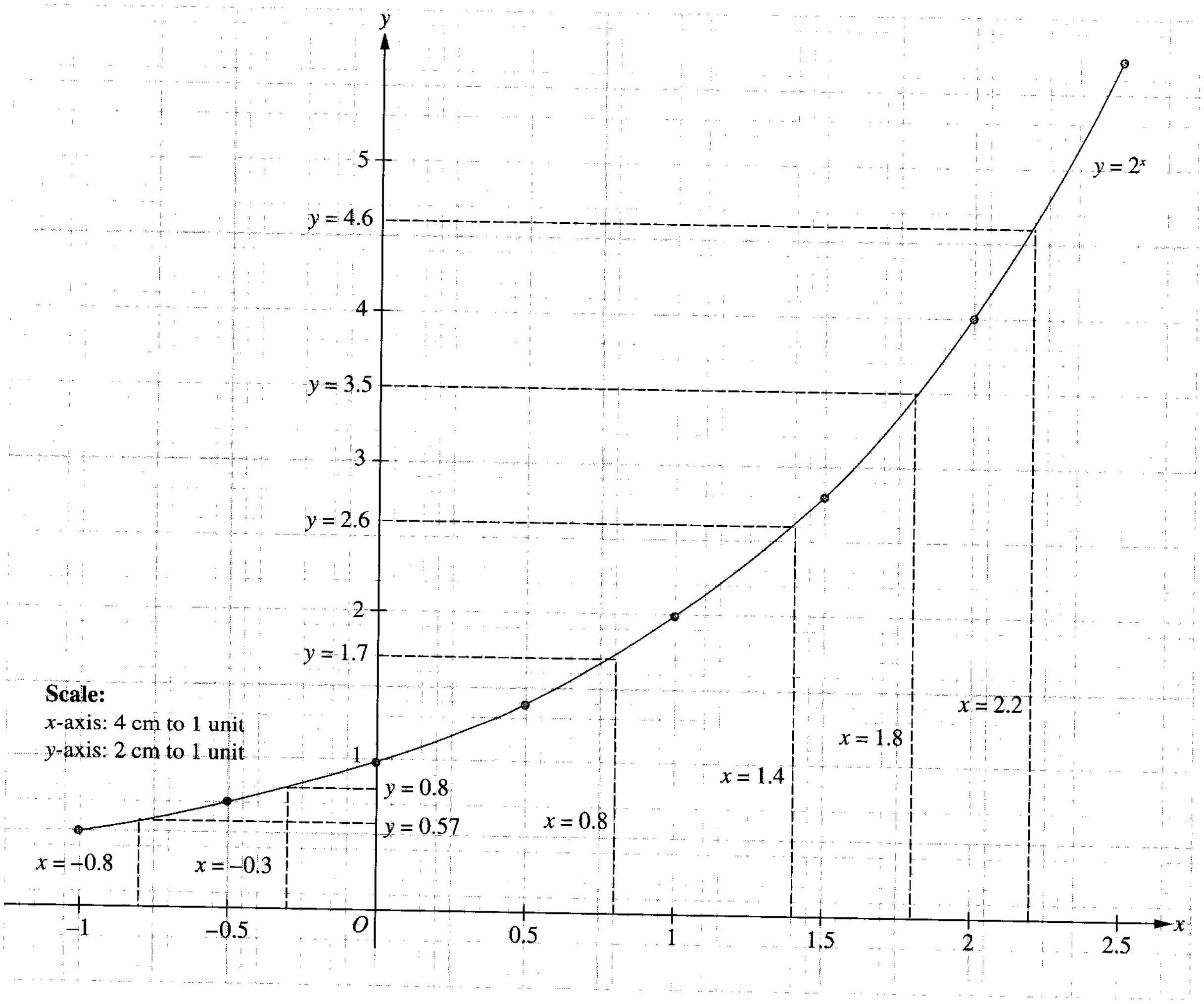


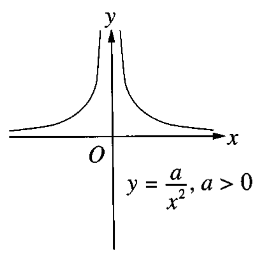
Fig. 8.5

(b) (i) When $x = -0.8$, $y \approx 0.57$; when $x = 0.8$, $y \approx 1.7$; when $x = 1.8$, $y \approx 3.5$.
(ii) When $y = 0.8$, $x \approx -0.3$; when $y = 2.6$; $x \approx 1.4$, when $y = 4.6$, $x \approx 2.2$

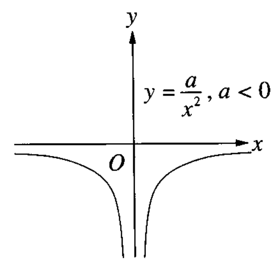
Sketches of Some Important Graphs

The following are graphs of the functions with the form $y = ax^n$, where $n = -2, -1, 1, 2$ and 3.

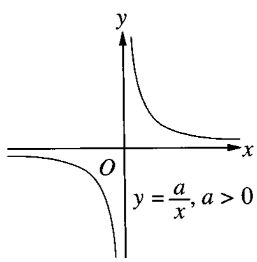
(a) $a > 0, n = -2$



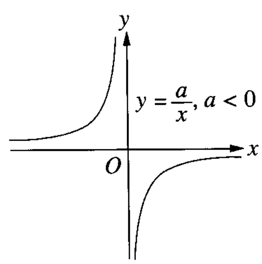
(b) $a < 0, n = -2$



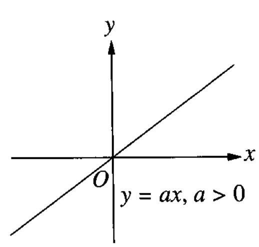
(c) $a > 0, n = -1$



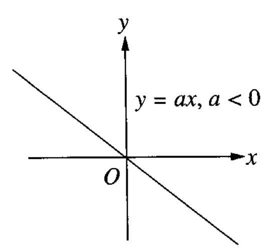
(d) $a < 0, n = -1$



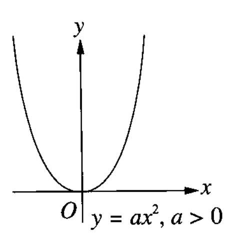
(e) $a > 0, n = 1$



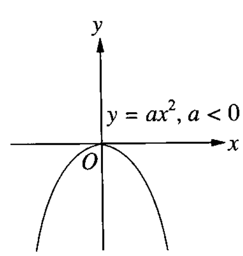
(f) $a < 0, n = 1$



(g) $a > 0, n = 2$

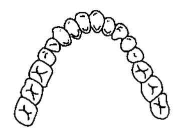


(h) $a < 0, n = 2$

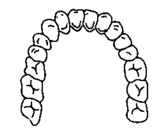


CHECK THIS OUT

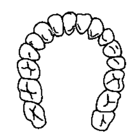
The arch of our upper jaws can be described by the shapes of the graphs of some functions.



(a) hyperbolic



(b) parabolic



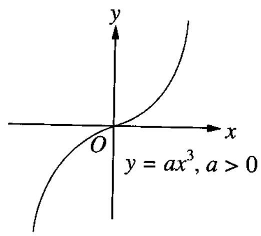
(c) elliptic

Which one of the above best describes the shape of your upper jaw? Find out yourself.

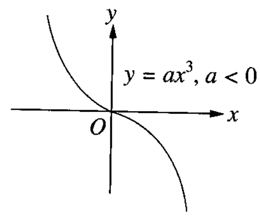
For Your Information

Before plotting a graph, it is useful to draw a table (if not given) of the values of x and y for a given range of x. Check if the range includes the end points. The table of values will help you to decide the correct scale to use for each axis.

(i) $a > 0, n = 3$

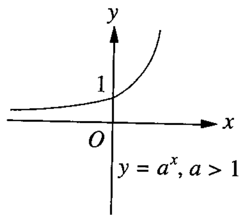


(j) $a < 0, n = 3$

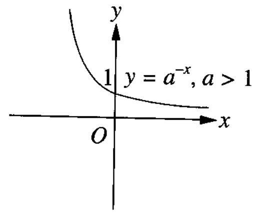


The following are graphs of the functions with the form $y = a^x$ and $y = a^{-x}$, where $a > 1$.

(a)



(b)



Do you know the shape of a^x and a^{-x}, when $0 < a < 1$?

Exercise 8a

1. Copy and complete the following table which gives values of $y = x^3$ for values of x between -3 and 3 inclusive.

x	-3	-2	-1	0	1	2	3
y		-8			1		27

Taking 2 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis, draw the graph of $y = x^3$. Use your graph to find

(a) the value of y when $x = 1.5$; **(b)** the value of x when $y = -12$.

2. Copy and complete the following table which gives values of $y = x^3 - 5$ for values of x between -3 and 3 inclusive.

x	-3	-2	-1	0	1	2	3
y	-32		-6				22

Using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis, draw the graph of $y = x^3 - 5$. Use your graph to find

(a) the value of y when $x = -2.2$; **(b)** the value of x when $y = 14$.

3. Copy and complete the following table which gives the values of $y = 3x - x^3$ for $-3 \leqslant x < 3$.

x	-3	-2	-1	0	1	2	3
y	18				2		-18

Using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis, draw the graph of $y = x^3 - 3x$. Use your graph to find

(a) the value of y when $x = 1.7$; **(b)** the values of x when $y = -6.6$.

4. Copy and complete the following table which gives values of $y = \frac{4}{x}$ for $\frac{1}{4} < x < 5$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4	5
y	16			2		1	

Using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = \frac{4}{x}$. Use your graph to find

(a) the value of y when $x = 3.6$; **(b)** the value of x when $y = 1.5$.

5. Copy and complete the following table which gives values of $y = \frac{10}{x^2}$ for $1 \leqslant x \leqslant 5$.

x	1	2	3	4	5
y	10			0.6	

Using 2 cm to represent 1 unit on both axes, draw the graph of $y = \frac{10}{x^2}$. Use your graph to find

(a) the value of y when $x = 2.8$; **(b)** the value of x when $y = 4.4$.

6. Given that $y = -\frac{2}{x} - 1$, copy and complete the following table for $\frac{1}{2} \leqslant x \leqslant 4$.

x	$\frac{1}{2}$	1	1.5	2	3	4
y		-3		-2		

Taking 2 cm to represent 1 unit in the x- and y-axis, plot the graph of $y = -\frac{2}{x} - 1$ for $\frac{1}{2} \leqslant x \leqslant 4$. Use your graph to find

(a) the value of y when $x = 2.5$; **(b)** the value of x when $y = -1.6$.

7. Copy and complete the following table which gives values of $y = 2 - \frac{3}{x^2}$ for $1 \leqslant x \leqslant 5$.

x	1	2	3	4	5
y		1.25	1.67		

Using 2 cm to represent 1 unit on the x-axis and 4 cm to represent 1 unit on the y-axis, draw the graph of $y = 2 - \frac{3}{x^2}$. Use your graph to find

(a) the value of y when $x = 1.5$; **(b)** the value of x when $y = 1.5$.

8. **(a)** Copy and complete the following table which gives values of $y = x^3 - 2x - 1$ for $-3 \leqslant x \leqslant 3$.

x	-3	-2	-1	0	1	2	3
y	-22		0			3	

Taking 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = x^3 - 2x - 1$.

(b) Use your graph to find

(i) the values of x when $y = 0$, -10 and 15;

(ii) the values of y when $x = -2.5$, 0.5 and 2.2.

9. **(a)** Copy and complete the following table which gives values of $y = x^3 - 6x^2 + 13x$ for values of x between 0 and 5 inclusive.

x	0	1	2	3	4	$4\frac{1}{2}$	5
y		8		12		$28\frac{1}{8}$	40

Using 2 cm to represent 1 unit of x and 2 cm to represent 5 units of y, draw the graph of $y = x^3 - 6x^2 + 13x$.

(b) Use your graph to find

(i) the values of y when $x = 1.5$, 3.5 and 4.45;

(ii) the values of x when $y = 7$, 15 and 22.

10. Using 2 cm to represent 1 unit on the x-axis and the y-axis, draw the graph of $y = x - \frac{3}{x}$ for $\frac{1}{2} \leqslant x \leqslant 6$. Use your graph to find

(a) the values of x when $y = -2.5$, 0 and 4.6; **(b)** the values of y when $x = 1.6$, 3.4 and 5.3.

11. **(a)** Copy and complete the following table which gives values of $y = 16 + \frac{16}{x}$ for values of x between 1 and 6.

x	1	2	3	4	5	6
y	32		21.3			18.7

Using 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = 16 + \frac{16}{x}$ for values of x between 1 and 6 inclusive.

(b) Use your graph to find

(i) the values of x when $y = 19$, 21 and 30;

(ii) the values of y when $x = 1.5$, 3.2 and 5.4.

12. Given that $y = 2 + 2^x$, copy and complete the following table of values:

x	-1	-0.5	0	1	1.5	2	2.5	3
y		2.7	3	4	4.8	6		10

(a) Using a scale of 4 cm for 1 unit on the x-axis and 2 cm for 1 unit on the y-axis, draw the graph of $y = 2 + 2^x$ for $-1 \leqslant x \leqslant 3$.

(b) Use your graph to find the values of

(i) y when $x = -0.7$ and 2.7; **(ii)** x when $y = 5.3$ and 7.5.

13. **(a)** Given that $y = 2x + 2^x$, copy and complete the following table of values:

x	–1	–0.5	0	0.5	1	1.5	2	2.5	3
y	–1.5	–0.3	1		4		8		14

Using a scale of 4 cm for 1 unit on the x-axis and 1 cm for 1 unit on the y-axis, plot the graph of $y = 2x + 2^x$ for $-1 \leqslant x \leqslant 3$.

(b) Use your graph to find the values of

(i) y when $x = 0.7$ and 2.3; **(ii)** x when $y = 1.5$ and 6.3.

14. Plot and draw the graph of $y = 2^{-x} - 2$ for $-3 \leqslant x \leqslant 3$. Use a scale of 2 cm to represent 1 unit on both the x-axis and the y-axis. From your graph, find the values of

(a) y when $x = -0.5, 0.5$; **(b)** x when $y = -1.5, 2.8$.

Graphical Solution of Simultaneous Equations

In Book 2, you learned how to solve simultaneous equations by the graphical method. We shall now revise this before proceeding to solve quadratic and cubic equations by the graphical method.

Example 6

Using a scale of 1 cm to represent 1 unit on both axes, draw the graphs of $y = x + 1$ and $x + y = 3$ for values of x between 0 and 3. Write down the coordinates of the point of intersection of the two lines.

Solution

To draw the graph of $y = x + 1$, three pairs of values of (x, y) are calculated. Two of the values are used to draw the line and the third to check that the calculations are correct.

x	0	1	2
y	1	2	3

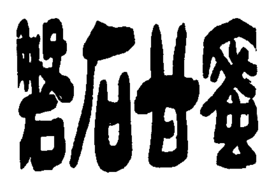
For the graph of $x + y = 3$, the table shows three pairs of values of (x, y).

x	0	1	2
y	3	2	1

Back in Time

In ancient times, many practical problems on simultaneous equations were studied in China. Small identical rods made of bamboo or bones were used to form simultaneous equations.

Ancient way of writing simultaneous euquations:



Modern way:

$3x + 2y + z = 39$ — (1)
$2x + 3y + z = 34$ — (2)
$x + 2y + 3z = 26$ — (3)

The graph of these two straight lines are as shown in Fig. 8.6.

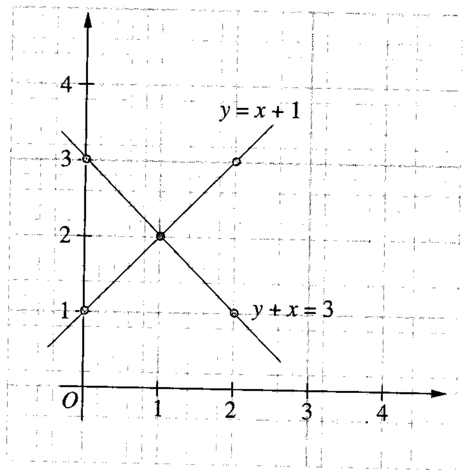
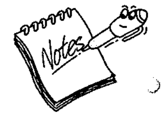


Fig. 8.6



Notice that $x = 1$ and $y = 2$ satisfy the equation $y = x + 1$ and $x + y = 3$ simultaneously. Hence, the solution of the pair of simultaneous equations $y = x + 1$ and $x + y = 3$ can be obtained by plotting the graphs of $y = x + 1$ and $x + y = 3$, and then finding their point of intersection.

The coordinates of the point of intersection of the two lines are (1, 2), i.e., $x = 1$, $y = 2$.

Exercise 8b

Obtain, graphically, the solutions of the following simultaneous equations:

1. $x + y = 1$, $x - y = 9$

2. $2x + y = 3$, $4x - y = 0$

3. $x + 3y = 1$, $2x + 5y = 1$

4. $5x - y = -2$, $4x - 3y = 5$

5. $3x + 2y = 4$, $5x + y = 2$

6. $5x - 3y = 2$, $7x + 2y = 9$

7. Draw the graphs of $y = 2x + 1$ and $2y = 4x + 5$. Are you able to find the point of intersection of the graph $y = 2x + 1$ and $2y = 4x + 5$? Can you explain the reason for your answer?

8. Draw the graphs of $3y = 2x + 4$ and $6y - 4x = 8$. How many lines do you get? Do the coordinates (1, 2) satisfy the simultaneous equations? Can you find another point that satisfies the simultaneous equations? How many points of intersection do the two graphs have?

9. The equations of the sides of a triangle ABC are given as follows:

$$AB : y = x, \quad BC : x + y = 4, \quad AC : 2y + x = 3.$$

Find the coordinates of A, B and C by the graphical method.

10. The equations of the sides of a quadrilateral $ABCD$ are given as follows:

$$AB : 4y + x = 20, \quad BC : x = 0, \quad CD : y + 2x = 2, \quad AD : 3y = 4x - 4.$$

Find the coordinates of A, B, C and D by the graphical method.

11. The equations of the sides of a pentagon $ABCDE$ are given as follows:

$$AB : y = 1, \quad BC : x = 4, \quad CD : 2y + 3x = 16,$$
$$DE : 2y = 3x + 4, \quad AE : x = 0.$$

Find the coordinates of A, B, C, D and E by the graphical method.

Graphical Solution of Quadratic Equations

A quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, can be solved by an algebraic method which we learned in Chapter 1. We can also use a *graphical method* to solve quadratic equations and simultaneous equations involving a linear equation and a quadratic equation.

Example 7

Draw the graphs of $y = x^2 - x - 6$ for $-4 \leqslant x \leqslant 4$ and $y = x + 2$ and use them to solve the simultaneous equations $y = x^2 - x - 6$ and $y = x + 2$.

Solution

To plot the curve, a table of corresponding values of x and y is required.

x	−4	−3	−2	−1	0	1	2	3	4
$y = x^2 - x - 6$	14	6	0	−4	−6	−6	−4	0	6

A table of values satisfying $y = x + 2$ is also drawn up.

x	0	1	2
y	2	3	4

For greater accuracy, scales chosen are usually as large as possible. The points of the two tables are then plotted and joined as follows:

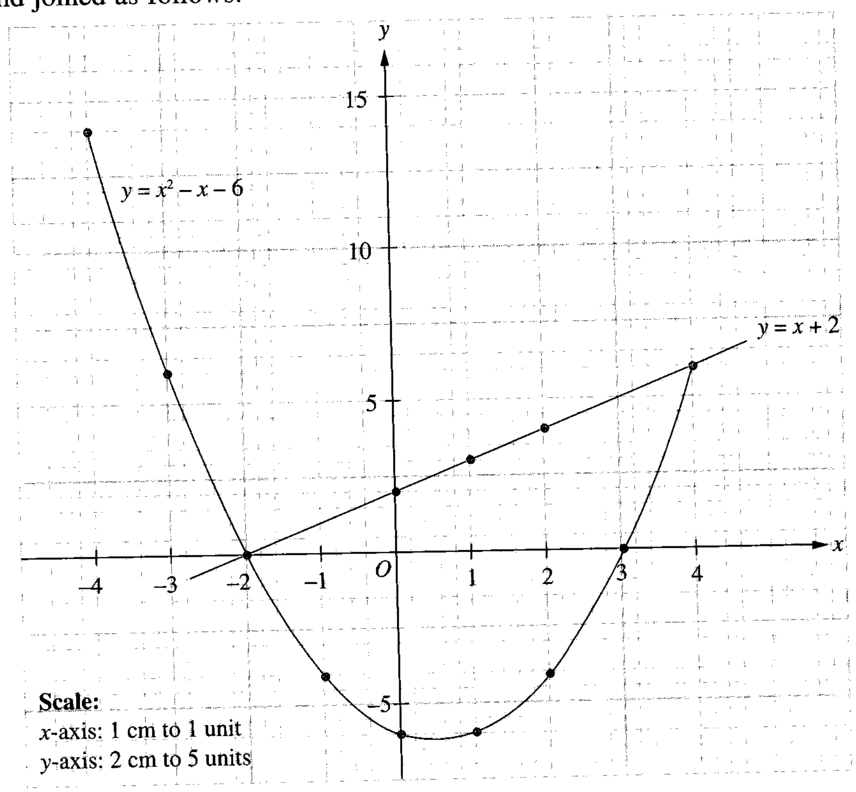


Fig. 8.7

The values of x and y satisfying the pair of simultaneous equations are $x = -2$, $y = 0$ and $x = 4$, $y = 6$.

Example 8

Draw the graph of $y = x^2 - 2x + 1$ for $-4 \leqslant x \leqslant 5$ and use it to solve the equation $x^2 - 2x + 1 = 10$. What is the least value of y? State the value of x when this occurs.

Solution

A table of values is first constructed. The graph is then plotted accordingly, as shown in Fig. 8.8 below.

x	−4	−3	−2	−1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16

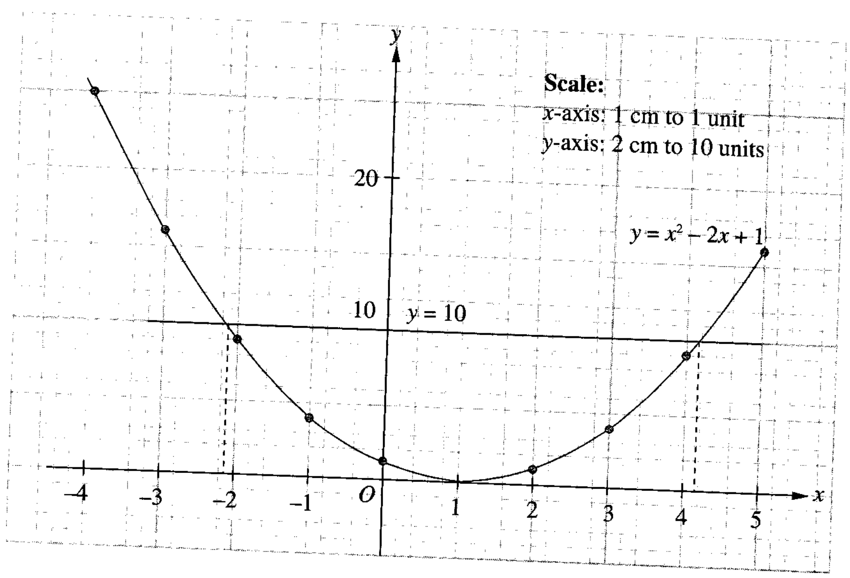


Fig. 8.8

The *solutions* to $x^2 - 2x + 1 = 10$ can be found by the *points of intersection* of the graph $y = x^2 - 2x + 1$ and $y = 10$. Thus $x \approx -2.2$ and $x \approx 4.2$ are the solutions to the equation $x^2 - 2x + 1 = 10$.

From the graph the least value of y is 0 and it occurs when $x = 1$.

Notes

Notice that the values $x = -2$ and $x = 4$ which we obtained graphically also satisfy the equation $x^2 - 2x - 8 = 0$. Hence we can solve the equation $x^2 - x - 6 = x + 2$ or $x^2 - 2x - 8 = 0$ by finding the x-coordinates of the points of intersection of the graphs $y = x^2 - x - 6$ and $y = x + 2$.

Are You Game Enough?

Plot and draw the graph of $y = x^2$. Locate the points on the curve that have −3 and 4 as their x-coordinates. Join the two points by a straight line. What do you notice about the y-intercept of the line? Use other pairs of points on the curve. What do you notice?

Example 9

Draw the graph of $y = x^2 + x + 4$ for $-4 \leqslant x \leqslant 4$ and use it to solve the equations $x^2 + x + 4 = 0$. State the minimum value of y and the value of x when this occurs.

Solution

A table of values is constructed and the graph plotted as shown in Fig. 8.9.

x	−4	−3	−2	−1	0	1	2	3	4
y	16	10	6	4	4	6	10	16	24

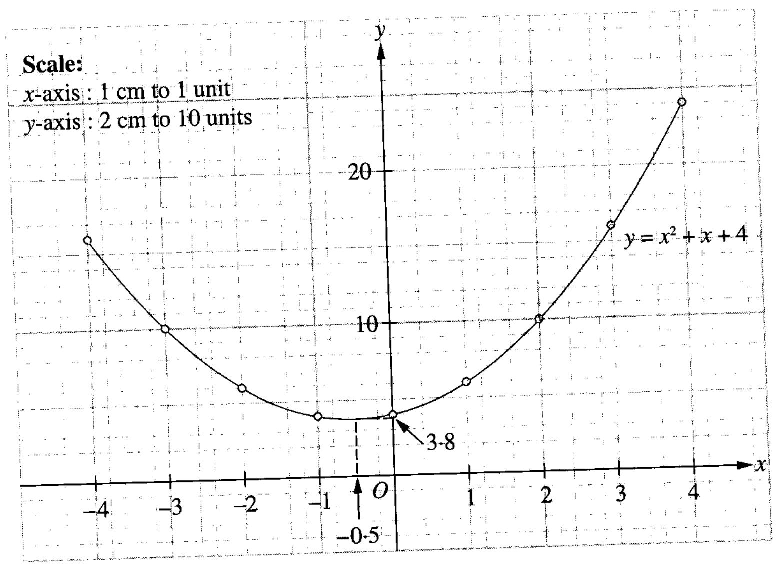
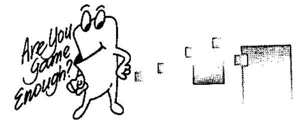


Fig. 8.9

The solution to $x^2 + x + 4 = 0$ can be obtained from the point of intersection of the curve $y = x^2 + x + 4$ and the line $y = 0$. Since the curve $y = x^2 + x + 4$ does not cut the line $y = 0$, i.e., the x-axis, there is no real solution to $x^2 + x + 4 = 0$.

From the graph, the minimum value of y is about 3.8 and it occurs when $x = -0.5$.

NB: The accurate minimum value of y is 3.75. Can you explain why?



Let $a = b$. Then $a^2 = ab$.

$$a^2 - b^2 = ab - b^2$$
$$(a + b)(a - b) = b(a - b)$$
$$a + b = b$$

Since $a = b$,
then $2b = b$,
$2 = 1$.

But is 2 equal to 1? Where is the mistake?

Carl Friedrich Gauss, a German mathematician (1777–1855), is often considered to be the greatest mathematician of all time. He proved that every algebraic equation in one unknown has a root. This is known as the ***fundamental theorem of algebra.***

Example 10

Draw the graph of $y = 4x^2 - 4x - 15$ for $-3 \leqslant x \leqslant 4$ and use it to solve the following equations:

(a) $4x^2 - 4x - 15 = 0$
(b) $4x^2 - 4x - 15 = 20$
(c) $4x^2 - 4x - 25 = 0$
(d) $4x^2 - 6x - 20 = 0$
(e) $4x^2 - 3x + 3 = 0$

For Your Information

When there is no real solution to an equation, the roots of the equation are known to be complex, or imaginary, or simply ... unreal.

Solution

A table of values is constructed as follows:

x	-3	-2	-1	0	1	2	3	4
y	33	9	-7	-15	-15	-7	9	33

The values are plotted as shown in Fig. 8.10.

Use a graph plotter to get more accurate values for the points of intersection.

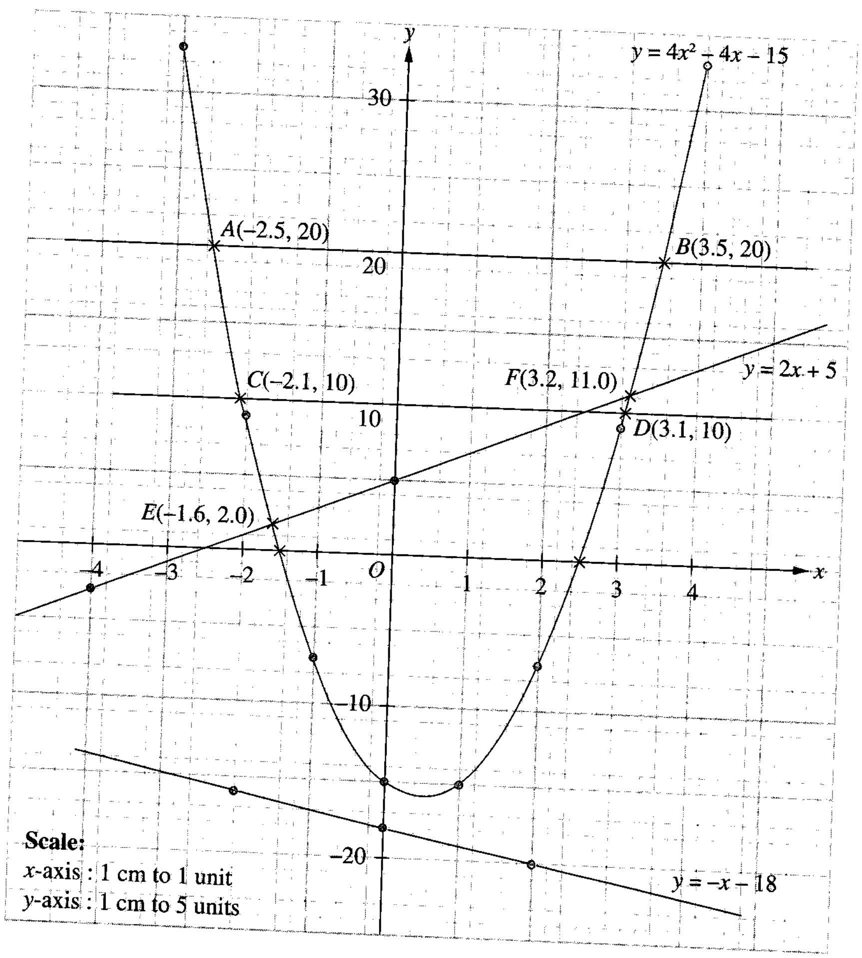


Fig. 8.10

Are You Game Enough?

Given that $f(x) = (x+1)(x-2)(x-3)$, solve $f(x) = 0$, $f(x) > 0$ and $f(x) < 0$.

Can you guess the shape of the curve of $y = f(x)$ on the coordinate plane?

From the above exercise, we see that the graphical solution of two simple functions derived from a given complicated equation is a very useful method.

(a) The solution to $4x^2 - 4x - 15 = 0$ is derived from the points where the curve cuts the line $y = 0$, i.e., the x-axis. From the graph, the solutions of $4x^2 - 4x - 15 = 0$ are $x = -1.5$ and 2.5.

(b) The solution to $4x^2 - 4x - 15 = 20$ is obtained from the points of intersection, A and B, of the curve and the line $y = 20$. From the graph, $x = -2.5$ and 3.5.

(c) To solve the equation $4x^2 - 4x - 25 = 0$ graphically, we make the LHS of the equation equal to the function y, i.e., $4x^2 - 4x - 15$.

$4x^2 - 4x - 25 = 0$ can be written as $4x^2 - 4x - 25 + 10 = 0 + 10$, i.e., $4x^2 - 4x - 15 = 10$.

The solution to the above equation is found from the points of intersection of the curve and the line $y = 10$.

From the graph, it is seen that the line $y = 10$ cuts the curve at approximately $C(-2.1, 10)$ and $D(3.1, 10)$. Hence the solutions to $4x^2 - 4x - 25 = 0$ are $x \approx -2.1$ and 3.1.

(d) $4x^2 - 6x - 20 = 0$ can similarly be expressed as

$$4x^2 - 6x + (2x) - 20 + (5) = 0 + (2x) + (5),$$

i.e. $$4x^2 - 4x - 15 = 2x + 5.$$

To find the solution, the points of intersection of $y = 4x^2 - 4x - 15$ and $y = 2x + 5$ must be located.

A table of values for $y = 2x + 5$ is constructed.

x	-4	0	4
y	-3	5	13

The straight line is drawn on the same graph and is found to intersect the curve approximately at $E(-1.6, 2.0)$ and $F(3.2, 11.0)$. Thus, the solutions to $4x^2 - 6x - 20 = 0$ are $x \approx -1.6$ and $x \approx 3.2$.

Back in Time

Evariste Galois (1811–1832), a French mathematician, developed a new branch of mathematics known as the group theory. This theory generalised an important result in algebra, i.e., there is no general solution to equations of the 5th degree.

(e) $4x^2 - 3x + 3 = 0$ is written as

$$4x^2 - 3x - (x) + 3 - (18) = 0 - (x) - (18).$$

i.e. $$4x^2 - 4x - 15 = -x - 18.$$

The straight line function $y = -x - 18$ is plotted.

x	-2	0	2
$y = -x - 18$	-16	-18	-20

From the graph, it is seen that the line $y = -x - 18$ does not cut the curve $y = 4x^2 - 4x - 15$. Thus, there is no real solution to $4x^2 - 3x + 3 = 0$, i.e., the roots of the equation are complex, or imaginary.

Exercise 8c

1. Draw the graphs of the following functions for the range of values of x as indicated. Then use each graph to solve the given equation for which $y = 0$.

   (a) $y = x^2 - 3x + 2$ $(-1 \leqslant x \leqslant 4)$ (b) $2y = -x^2 + 6x$ $(-1 \leqslant x \leqslant 8)$
   (c) $y = -x^2 - 6$ $(-1 \leqslant x \leqslant 3)$ (d) $y = (2x - 1)(4 - x)$ $(0 \leqslant x \leqslant 6)$
   (e) $\frac{1}{2}y + 2x^2 - 3x + 1 = 0$ $(-1 \leqslant x \leqslant 2)$ (f) $2 - 9x + 6x^2 - 3y = 0$ $(-1 \leqslant x \leqslant 3)$

2. Draw the graph of $y = 2x^2 - 7x + 1$ for $-1 \leqslant x \leqslant 4$ and use it to solve the following equations:

   (a) $2x^2 - 7x - 2 = 0$ (b) $2x^2 - 7x + 3 = 0$ (c) $2x^2 = 0$ (d) $2x^2 - 5x + 2 = 0$

3. Draw the graph of $y = x^2 + 2x - 8$ for $-5 \leqslant x \leqslant 3$ and use it to solve the following equations.

   (a) $x^2 + 2x - 8 = 0$ (b) $x^2 + 2x = 5$ (c) $x^2 = 8$ (d) $x^2 + 5x - 9 = 0$

4. Draw the graph of $y = 6 + x - x^2$ for $-3 \leqslant x \leqslant 4$ and use it to solve the equations.

   (a) $6 + x - x^2 = 0$ (b) $6 + x - x^2 = 1$ (c) $6 + x - x^2 = -2$ (d) $2 - 3x - x^2 = 0$

5. Solve the following equations graphically.

   (a) $x^2 - 5x - 4 = 0$ (b) $x^2 + x - 3 = 0$ (c) $x^2 + 7x = 0$

6. Draw the graph of $y = x^2$ for $-4 \leqslant x \leqslant 4$. By further plotting suitable straight lines on the graph, solve the following equations if real roots do exist.

   (a) $x^2 - 4x + 3 = 0$ (b) $x^2 + 2x - 3 = 0$ (c) $x^2 - 3x - 1 = 0$ (d) $x^2 - 2x + 5 = 0$
   (e) $x^2 + 3x - 1 = 0$

7. Plot the graph of $y = x^2 - 4x$ for $-2 \leqslant x \leqslant 6$ and use it to find the real roots, where they exist, of the following equations:

   (a) $x^2 - 4x = 3$ (b) $x^2 - 4x + 2 = 0$ (c) $x^2 - 4x - 7 = 0$ (d) $3x^2 - 12x + 10 = 0$
   (e) $2x^2 - 9x + 4 = 0$

8. Draw the graph of $y = (3 + 2x)(2 - x)$ for $-3 \leqslant x \leqslant 3$ and use it to find

   (a) the greatest value of y;
   (b) the solution to the equation $(3 + 2x)(2 - x) = 2$.

9. The perimeter of a rectangular lawn is 60 m and its diagonal is 25 m. If the length of the lawn is x m, show that $x^2 + (30 - x)^2 - 25^2 = 0$. Hence solve this equation graphically.

10. Taking 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = 2x^2 - 5x - 2$ for values of x from -2 to 5. Use your graph to find

    (a) the least value of y;
    (b) the solution of the equation $2x^2 - 5x - 2 = 0$;
    (c) the solution of the equation $2x^2 - 7x - 6 = 0$.

More Graphical Solutions

In the following examples, the graphical solutions of more complicated functions will be illustrated.

Example 11

Draw the graph of $y = x^3 - 2x^2 - 5x + 6$ for $-3 \leqslant x \leqslant 4$ and use it to solve the following equations:

(a) $x^3 - 2x^2 - 5x + 6 = 0$ *(b)* $x^3 - 2x^2 - 5x + 6 = 7$

(c) $x^3 - 2x^2 - 5x + 6 = x + 2$ *(d)* $x^3 - 2x^2 - 8x + 15 = 0$

Solution

A table of values of $y = x^3 - 2x^2 - 5x + 6$ is constructed.

x	–3	–2	–1	0	1	2	3	4
y	–24	0	8	6	0	–4	0	18

The graph of the curve $y = x^3 - 2x^2 - 5x + 6$ is shown below. It illustrates a typical cubic curve.

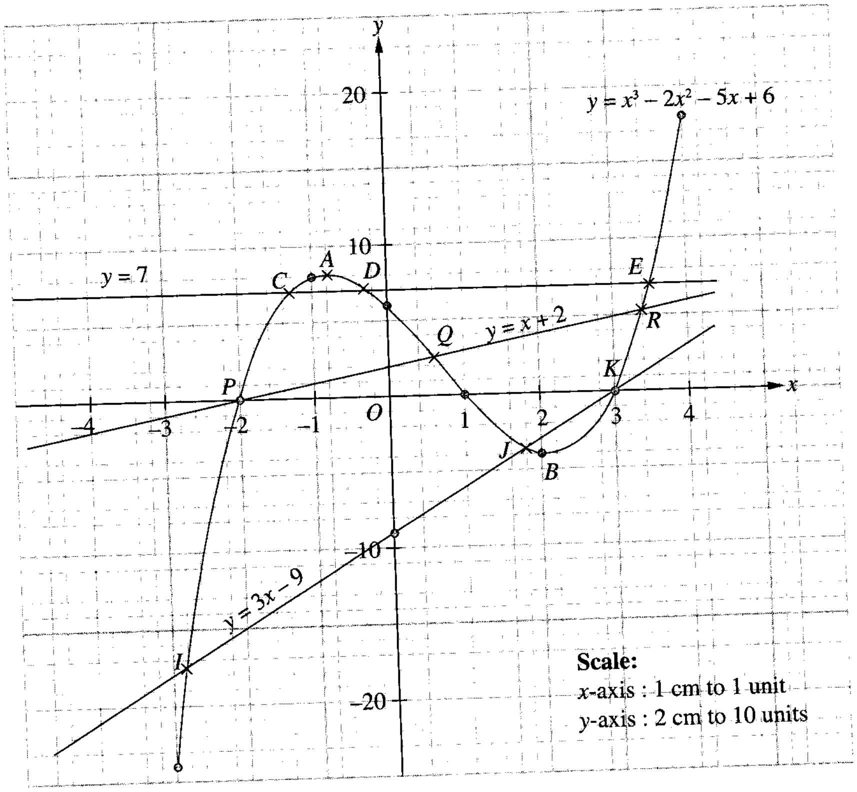


Fig. 8.11

Just for fun

Use a piece of string to form a 20-cm loop.

(a) Investigate how

(i) rectangles of various lengths from 0 cm to 10 cm can be constructed from the string.

(ii) the width, perimeter and area of the rectangle vary as the length increases.

(b) Sketch the relationship between the length of the rectangle and its width, perimeter and area.

There are two turning points, namely at A and B. As the value of y at A is greater than any of the neighbouring points, point A is termed a **maximum** point of the curve. As the value of y at B is the lowest among the neighbouring points, point B is termed a **minimum** point of the curve.

It should be noted that the value of y at A is not necessarily the largest for the whole curve. It is the largest only *with respect to* all other neighbouring points about A. (Complicated curves may have more than one such point.)

(a) The curve cuts the x-axis at the points where $x = -2$, $x = 1$ and $x = 3$. Thus the solutions to $x^3 - 2x^2 - 5x + 6 = 0$ are $x = -2$, 1 and 3.

(b) $x^3 - 2x^2 - 5x + 6 = 7 \quad \Rightarrow \quad y = 7$
When the straight line $y = 7$ is plotted, it intersects the curve $y = x^3 - 2x^2 - 5x + 6$ at points C, D and E where $x = -1.3$, $x = -0.3$ and $x = 3.5$ respectively. Thus, the solutions to $x^3 - 2x^2 - 5x + 6 = 7$ are $x = -1.3$, -0.3 and 3.5.

(c) $x^3 - 2x^2 - 5x + 6 = x + 2$
The straight line $y = x + 2$ is plotted. It intersects the curve $y = x^3 - 2x^2 - 5x + 6$ at points P, Q and R where $x = -2$, $x = 0.6$ and $x = 3.4$ respectively. Thus, the solutions to $x^3 - 2x^2 - 5x + 6 = x + 2$ are $x = -2$, 0.6 and 3.4.

(d) $x^3 - 2x^2 - 8x + 15 = 0$ is written as

$$x^3 - 2x^2 - 8x + (3x) + 15 - (9) = 0 + (3x) - (9),$$

i.e.

$$x^3 - 2x^2 - 5x + 6 = 3x - 9$$

The straight line $y = 3x - 9$ is plotted. It intersects the curve at points I, J and K where $x = -2.8$, $x = 1.8$ and $x = 3$, respectively. Thus, the solutions of $x^3 - 2x^2 - 8x + 15 = 0$ are $x = -2.8$, 1.8 and 3.

For Your Information

Sometimes, points such as A and B are termed local maxima and local minima respectively. This indicates that they are maximum and minimum points only in their respective vicinity.

Example 12

Find, graphically, the roots of the equation $x^3 - 6x + 3 = 0$.

Solution

Several methods may be used to solve this equation graphically. However, only three methods will be discussed here.

Method 1

Construct a table of values of x and y for $y = x^3 - 6x + 3$ and plot them on graph paper. The solution to $x^3 - 6x + 3 = 0$ is read off the graph where the curve cuts the x-axis. This is left as an exercise for you.

Are You Game Enough?

There are 10 bank notes altogether. They consist of \$10, \$20 and \$50 notes. If the total value of the notes is \$180, find the number of each type of note. Can the problem be solved by a system of simultaneous equations?

Method 2

$$x^3 - 6x + 3 = 0$$
$$x^3 = 6x - 3$$

This gives two functions: $y_1 = x^3$ and $y_2 = 6x - 3$. When y_1 and y_2 are plotted on the same graph, the points of intersection of these two functions give the solutions to the equation $x^3 = 6x - 3$, i.e., $x^3 - 6x + 3 = 0$.

The tables of values are:

x	−3	−2	−1	0	1	2	3
$y_1 = x^3$	−27	−8	−1	0	1	8	27

x	−3	0	3
$y_2 = 6x - 3$	−21	−3	15

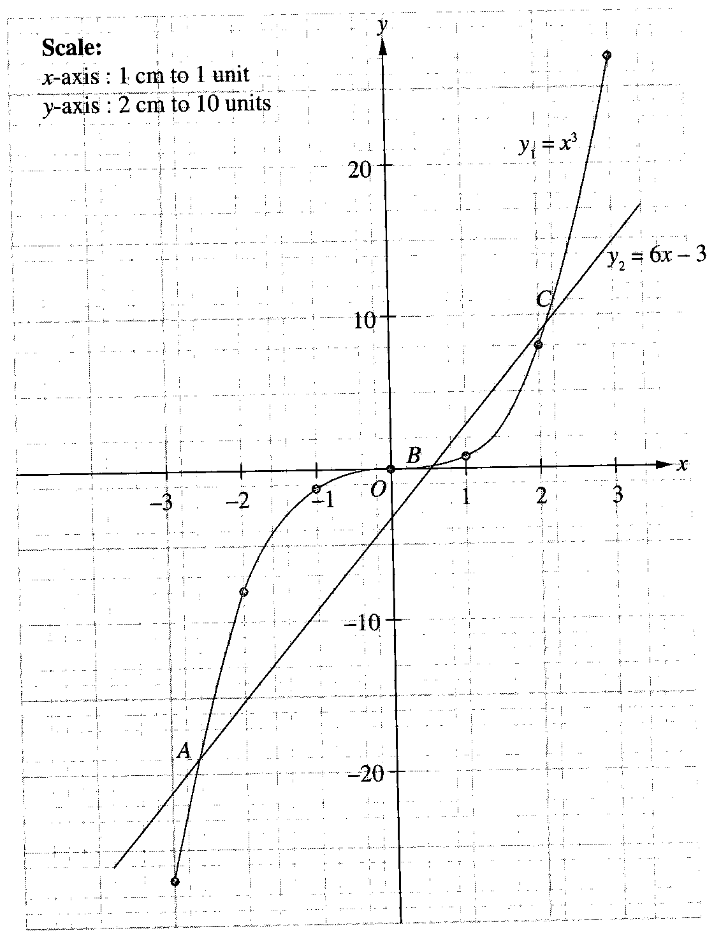


Fig. 8.12(a)

The two graphs intersect at points A, B and C, where $x \approx -2.6$, 0.5 and 2.1 respectively (see Fig. 8.12(a)). Thus, the solutions of $x^3 - 6x + 3 = 0$ are $x \approx -2.6$, 0.5 and 2.1.

Method 3

$$x^3 - 6x + 3 = 0$$

Since $x \neq 0$,

dividing by x throughout, we have $x^2 - 6 + \frac{3}{x} = 0$

$$x^2 - 6 = -\frac{3}{x}$$

Again, there are two functions $y_1 = x^2 - 6$ and $y_2 = -\frac{3}{x}$.

y_1 is a quadratic function while y_2 is a reciprocal function. The points of intersection of these two curves will yield the roots of the equation $x^2 - 6 = -\frac{3}{x}$, i.e., $x^3 - 6x + 3 = 0$. The table of values is constructed.

x	−3	−2	−1	0	1	2	3
$y_1 = x^2 - 6$	3	−2	−5	−6	−5	−2	3
$y_2 = -\frac{3}{x}$	1	1.5	3	∞	−3	−1.5	−1

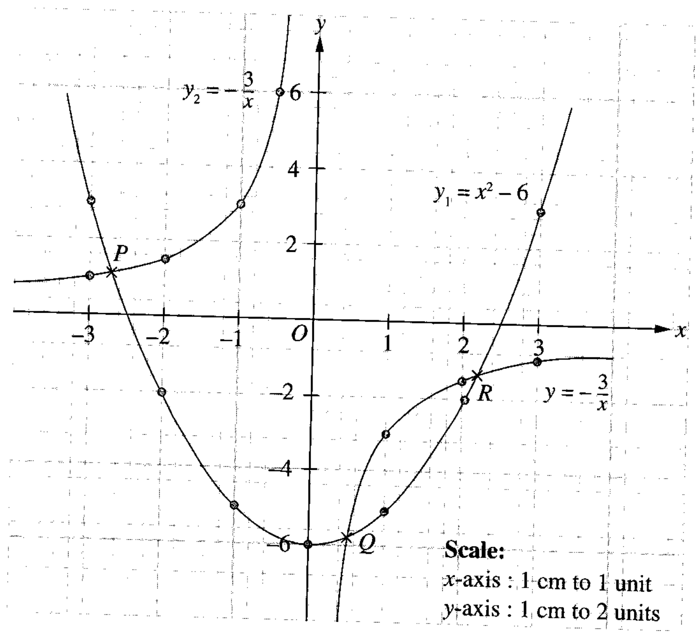


Fig. 8.12(b)

In Fig. 8.12(b), the two curves intersect at points P, Q and R where $x = -2.7$, 0.5, and 2.2 respectively. Thus, it can be concluded that the solutions to $x^3 - 6x + 3 = 0$ are $x = -2.7$, 0.5 and 2.2.

For Your Information

For questions which involve the plotting of graphical equations, it is always a good idea to indicate the ***scale*** *used for the x-axis and the y-axis respectively. On the other hand, if the scale has already been indicated in the question, then there is no need to repeat it in the solution.*

Notes

Examples 11 and 12 have demonstrated that the graphical solution of two simple functions derived from a complicated equation is a useful method.

Example 13

Given that $y = x + 2^x$,

(a) Copy and complete the following table. Give your answers correct to 1 decimal place.

x	-1	-0.5	0	0.5	1	1.5	2	2.5
y		0.2	1		3		6	

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x + 2^x$ *for* $-1 \leqslant x \leqslant 2\frac{1}{2}$.

(c) Use your graph to solve the equations

(i) $x + 2^x = 7$; *(ii)* $x + 2^x = 3 - x$; *(iii)* $2^x = 4 - \frac{1}{2}x$.

Solution

(a)

x	−1	−0.5	0	0.5	1	1.5	2	2.5
y	−0.5	0.2	1	1.9	3	4.3	6	8.2

(b) The graph is plotted as shown below:

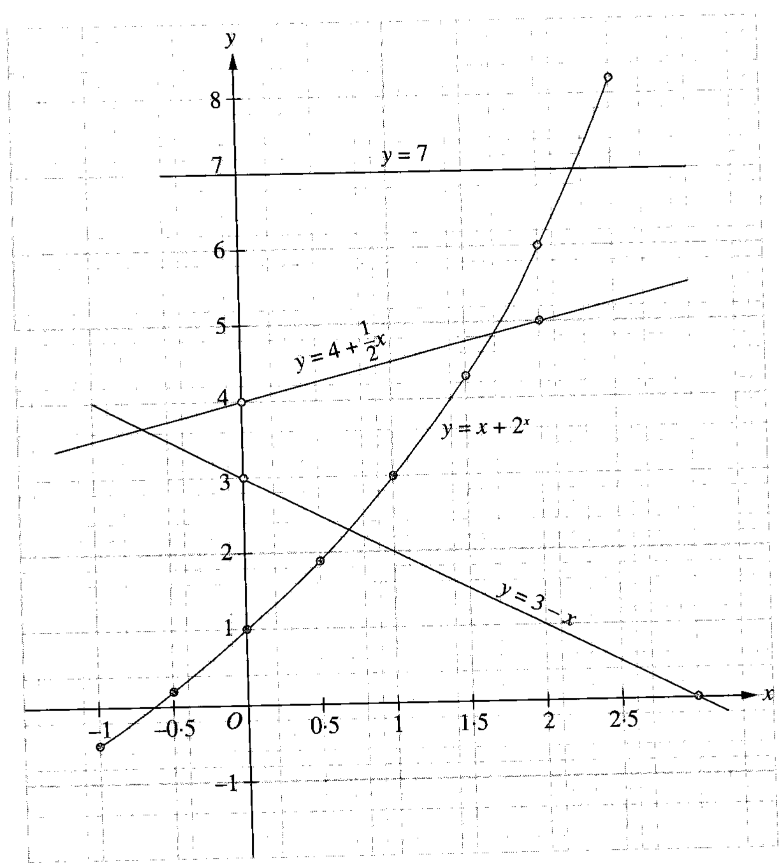


Fig. 8.13

(c) (i) To solve $x + 2^x = 7$, we draw the line $y = 7$. From Fig. 8.13, $y = 7$ intersects the graph at $x = 2.25$, i.e., the solution to the equation $x + 2^x = 7$ is $x = 2.25$.

(ii) To solve $x + 2^x = 3 - x$, we draw the line $y = 3 - x$. The point of intersection of the line and the curve is (0.7, 2.3), i.e., the solution to the equation $x + 2^x = 3 - x$ is $x = 0.7$.

(iii) To solve $2^x = 4 - \frac{1}{2}x$, we write

$$2^x + x = 4 - \frac{1}{2}x + x, \text{ i.e., } x + 2^x = 4 + \frac{1}{2}x,$$

then plot the graph of $y = 4 + \frac{1}{2}x$ and find the points of intersection of this line and the curve.

Since the line $y = 4 + \frac{1}{2}x$ cuts the curve at (1.7, 4.8), the solution to the equation $2^x = 4 - \frac{1}{2}x$ is $x = 1.7$.

For Example 13(c)(i)–(iii), manipulation of the equations is required so that one side of the equation equals the main graph. In this case, the main graph is $y = x + 2^x$.

Example 14

A metal cube of side x cm has a hole of cross-sectional area 4 cm² drilled right through it in the direction perpendicular to one of the faces. The volume of the resulting block is V cm³.

Show that $V = x(x - 2)(x + 2)$.

Copy and complete the following table of values. Hence plot the graph of V against x.

x	3	4	4.5	5	6
V			73		

The block is now melted and made into a solid rectangular block whose sides are 3 cm, 3 cm and (15 – x) cm. Estimate the value of x by drawing a suitable straight line on the same axes.

Solution

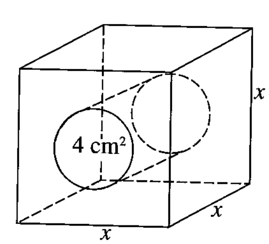
Total volume of cube = x^3 cm³

Volume of the cylindrical hole = $4x$ cm³

∴ the volume of the resulting block is $V = (x^3 - 4x)$,
i.e. $V = x(x - 2)(x + 2)$

The table of values is as shown below:

x	3	4	4.5	5	6
V	15	48	73	105	192



The graph is then plotted as in Fig. 8.14.

The volume of the solid rectangular block is

$$V_1 = 3 \times 3 \times (15 - x)$$
$$= (135 - 9x) \text{ cm}^3$$

To find the value of x, the straight line function $V_1 = 135 - 9x$ is plotted on the same axes. The table of values is:

x	3	4	5
$V_1 = 135 - 9x$	108	99	90

From the graph, the curve and the line intersect at point A where x is approximately 4.8.

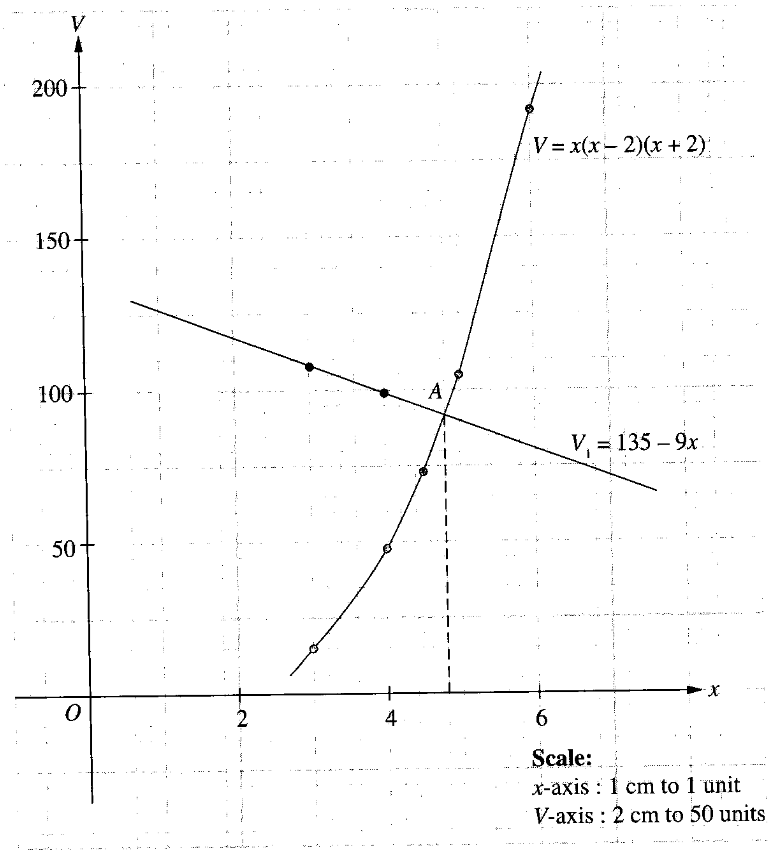


Fig. 8.14

Exercise 8d

1. By drawing graphs of $y = x^3$ and $y = 5x + 2$ for values of x between –3 and 3, solve the equation $x^3 = 5x + 2$.

2. Draw the graphs of $y = \frac{1}{2}(x + 1)(4 - x)$ and $y = \frac{4}{x}$ for $-2 \leqslant x \leqslant 3$. Hence, find the roots of the equation $x(x + 1)(4 - x) = 8$ in the given range of values of x.

3. Taking 2 cm to represent 1 unit on both axes, draw $y = 3 + \frac{2}{x}$ for $\frac{1}{2} \leqslant x \leqslant 4$. Using the same axes, draw the graph of $y = x$. Hence, solve the equation $x^2 - 3x - 2 = 0$.

4. Use a graphical method to solve each of the following equations.

(a) $x^3 + x^2 - 6x - 7 = 0$	**(b)** $3x^3 - 6x^2 + 9x - 1 = 0$	**(c)** $4x^3 + x^2 - 10 = 0$
(d) $x^3 + 3x^2 + x + 6 = 0$	**(e)** $x^3 - 6x + 5 = 0$	**(f)** $(2x - 5)(x^2 - 3x + 2) = 0$

5. Draw the graph of $y = \frac{2}{3x}$ for $-6 \leqslant x \leqslant 6$ and use it to solve the following equations:

(a) $\frac{2}{3x} = x - 1$ **(b)** $3x^2 = 2 - 5x$

6. Draw the graph of $y = x^3$ for values of x from –3 to 3. Using the same scales, draw the graph of $y = 18 - 5x$. From your graph, find one root of the equation $x^3 + 5x - 18 = 0$. Hence find, correct to one decimal place, one root of the equation $x^3 + 6x + 10 = 0$.

7. Given that $y = 30 - 3x - \frac{60}{x}$, copy and complete the following table:

x	2	$2\frac{1}{2}$	3	4	5	6	7	8	9	10
y	–6	$-1\frac{1}{2}$		3			0.43		–3.67	

Hence, plot the graph $y = 30 - 3x - \frac{60}{x}$ and read off the greatest value of y. Which value of x does this value of y correspond to?

8. Given that $y = x^2 - 5x + 6$, copy and complete the following table:

x	0	1	2	3	4	5
y	6		0			

Hence, draw $y = x^2 - 5x + 6$ for values of x from 0 to 5, taking 2 cm to represent one unit on both axes. Use your graph to find

(a) the least value of y; **(b)** the roots of the equation $x^2 - 5x + 6 = 0$;
(c) the values of x when $y = 1$; **(d)** the roots of the equation $x^2 - 5x + 5 = 2$.

9. Draw the graph of $y = 2x^3 - 6x^2 + 3x - 10$ and use it to solve the following equations.

(a) $2x^3 - 6x^2 + 3x - 10 = 0$ **(b)** $2x^3 - 6x^2 + 3x + 5 = 0$ **(c)** $2x^3 - 6x^2 + x + 8 = 0$
(d) $2x^3 - 6x^2 + 5x - 12 = 0$ **(e)** $2x^3 - 6x^2 = 0$

10. Given that $y = x^2(2x + 3)$, copy and complete the following table:

x	–2	$-\frac{3}{2}$	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
y		0	1	$\frac{1}{2}$		1	

Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 2 units on the y-axis, draw the graph of $y = x^2(2x + 3)$ for values of x from –2 to 1. From your graph, solve the equation $x^2(2x + 3) = -\frac{1}{2}$. On the same axes, draw the graph $y = 1 - x$. Use your graph to solve the equation $x^2(2x + 3) = 1 - x$.

11. **(a)** Given that $y = 2^x - 1$, copy and complete the following table of values:

x	–1	–0.5	0	0.5	1	1.5	2	2.5
y	–0.5	–0.3	0	0.4	1		3	

(b) Using a scale of 4 cm to represent 1 unit on the x-axis, and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 2^x - 1$ for $-1 \leqslant x \leqslant 2.5$.

(c) Use your graph to solve the following equations.

(i) $2^x - 1 = 4$ **(ii)** $2^x = 3$ **(iii)** $2^x = 4 - x$ **(iv)** $2^x + 2x = 5$

12. Given that $y = \frac{1}{10}x(12 - x^2)$,

(a) copy and complete the following table of values for $y = \frac{1}{10}x(12 - x^2)$.

x	–2	–1	0	1	2	3	4	5
y	–1.6	–1.1	0	1.1	1.6	0.9		

(b) Using a scale of 2 cm to represent 1 unit on each axis, draw the graph of $y = \frac{1}{10}x(12 - x^2)$ for $-2 \leqslant x \leqslant 5$.

(c) Use your graph to solve the following equations:

(i) $\frac{1}{10}x(12 - x^2) = 0$ **(ii)** $\frac{1}{10}x(12 - x^2) = 1$ **(iii)** $x(12 - x^2) = -5$

13. The radius of an open cylindrical water tank is x metres. The capacity of the water tank is 2.5 m^3. If the total surface area of the tank is y m^2, show that

$$y = \frac{5}{x} + \pi x^2.$$

The following is a table of values for $y = \frac{5}{x} + \pi x^2$.

x	0.5	1.0	1.5	2.0	2.5	3.0	3.5
y	10.8	8.1	10.4	15.1	21.6	29.9	39.9

Using scales of 4 cm for 1 m on the x-axis and 2 cm for 5 m^2 on the y-axis, draw the graph of $y = \frac{5}{x} + \pi x^2$ for $0.5 \leqslant x \leqslant 3.5$.

From your graph, estimate the radius of the tank when the surface area of the tank is the least.

14. $\triangle ABC$ is right-angled at B. The length of $AB = 6$ cm and that of $BC = 3$ cm. A line PQ is drawn parallel to AB, meeting CA at Q and CB at P. Given that $CP = x$ cm, show that

(a) $PQ = 2x$ cm; **(b)** the area (y cm^2) of $\triangle PQB$ is given by $y = x(3 - x)$.

Draw the graph of $y = x(3 - x)$, using convenient scales for values of x between 0 and 3, and plotting at least five points. From your graph, determine the range of values of x for which y exceeds 1.5.

Summary

1. The general form of a **cubic** graph is $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$. Normally, this type of graph has two bends.

2. The general form of a **reciprocal** graph is $y = \frac{a}{x}$, where $a \neq 0$, $x \neq 0$. It consists of two separate curves.

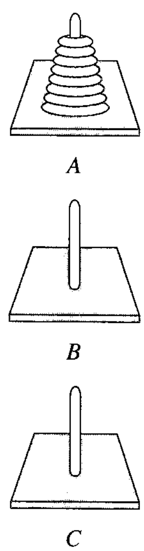
3. The general form of an **exponential** graph is $y = a^x$, where $a > 0$. The entire graph lies above the x-axis.

4. The solution to a pair of simultaneous equations can be found by drawing their graphs. The **point of intersection** of these graphs gives the **solution**.

5. To find the solution to an equation $f(x) = 0$, plot the graph of $y = f(x)$. The points of intersection of the graph $y = f(x)$ and the x-axis, or $y = 0$, give the roots of the equation $f(x) = 0$.

In a game called Tower of Hanoi, there are a number of discs of different diameters but with the same thickness on a rod A. You are to transfer all the discs from rod A to either rod B or C in the minimum number of moves using the following rules.

(1) Move only one disc at a time.
(2) A larger disc must never be on top of a smaller disc at any time.



How many moves are needed when rod A consists of (a) 2 discs (b) 3 discs (c) 4 discs (d) 10 discs (e) n discs?

Can you find a rule for the minimum number of moves when rod A consists of n discs?

Review Questions 8

1. **(a)** Copy and complete the following table which gives values of $y = x^3 - 3x - 10$, and draw the graph.

x	−3	−2	−1	0	1	2	3	4
y	−28			−10			8	42

   (b) Use your graph to find
   (i) the value of y when $x = 1.8$;
   (ii) the value of x when $y = 10$.

2. **(a)** Copy and complete the following table which gives values of $y = x(x - 2)(x + 2)$.

x	−3	−2	−1	0	1	2	3
y	−15	0		0	−3		

   Using 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = x(x - 2)(x + 2)$.

   (b) Use your graph to find
   (i) the value of y when $x = 1.4$;
   (ii) the value of x when $y = 4.5$;
   (iii) the solution to the equation $x(x - 2)(x + 2) = 0$.

3. **(a)** Copy and complete the following table which gives values of $y = 1 - 2x - \frac{1}{x}$.

x	−4	−3	−2	−1	−0.5	−0.25
y	9.3			4		5.5

Using 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 1 - 2x - \frac{1}{x}$,

(b) Use your graph to find

(i) the values of x when $y = 7$ and 9;

(ii) the values of y when $x = -0.75$, -2.5 and -3.75.

4. Copy and complete the following table which gives values of $y = 3^x - 2$.

x	−1.5	−1	−0.5	0	0.5	1	1.5	2
y	−1.8			−1			3.2	7

Give answers for y correct to 1 decimal place.

(a) By using a suitable scale, draw the graph of $y = 3^x - 2$ for $-1.5 \leqslant x \leqslant 2$.

(b) Use your graph to find the values of

(i) y when $x = -0.2$ and 1.2, correct to 1 decimal place;

(ii) x when $y = 0$ and 5, correct to 1 decimal place.

5. Copy and complete the following table which gives values of $y = 3 - 2^x$.

x	−1	−0.5	0	0.5	1	1.5	2	3
y	2.5		2	1.6	1		−1	−5

Give answers for y correct to 1 decimal place.

(a) Using a scale of 4 cm for 1 unit on the x-axis and 2 cm for 1 unit on the y-axis, draw the graph of $y = 3 - 2^x$ for $-1 \leqslant x \leqslant 3$.

(b) Use your graph to determine, correct to 1 decimal place,

(i) the values of y when $x = 2.3$ and 0.3;

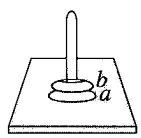
(ii) the values of x when $y = -3$ and 1.8.

6. Obtain, graphically, the solutions to each of the following pairs of simultaneous linear equations.

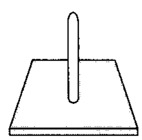
(a) $3x + y = 4$, $x - 2y = 6$ **(b)** $2x + 7y = 6$, $x - 5y = 3$ **(c)** $2x + y = 10$, $4x - 2y = 4$

We can actually play this game to find out the number of moves needed. We shall begin to play the game with 2 discs on rod A and then proceed to find the minimum number of moves when rod A has 3, 4, 5 discs. Then, we can try to find a pattern to discover the number of moves for the case when rod A consists of 10 discs and n discs. You can get a ready-made set or you may make one yourself and play the game with your friends.

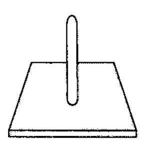
When rod A consists of 2 discs a and b, the steps taken are:



A



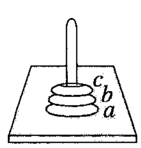
B



C

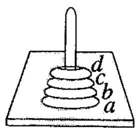
From A, move b to B, a to C and b to C. Thus, there are 3 moves.

When rod A consists of 3 discs a, b and c, the steps are as follows:



c to B, b to C, c to C, a to B, c to A, b to B and c to B. Thus, we need a total of 7 moves.

When rod A consists of 4 discs a, b, c and d, the steps taken could be d to B, c to C, d to C, b to B, d to A, c to B, d to B, a to C, d to C, c to A, d to A, b to C, d to B, c to C and d to C. Thus we need a minimum of 15 moves.



ACT IT OUT to find the solution. Many other problems may be solved by using such a similar method.

In conclusion, we see that

Number of discs in rod A	*Minimum number of moves*
2	*3*
3	*7*
4	*15*
⋮	⋮

Can you locate a pattern? Can you find the minimum number of moves for rod A with 10 discs? Write down a formula for the case when rod A consists of n discs.

7. Draw the graph of $y = x^2$ for $-4 \leqslant x \leqslant 4$. By plotting suitable straight lines on the graph, solve the following equations if real roots exist.

(a) $x^2 - 5 = 0$ **(b)** $x^2 = 3x - 2$ **(c)** $x^2 - 2x = 5$
(d) $2x^2 = 4x - 5$ **(e)** $3x^2 - 6x = 0$ **(f)** $4x - 3x^2 - 7 = 0$

8. Draw the graph of $y = 3 - x - x^2$ for values of x from -4 to 4 by using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis. Use your graph to solve the following equations where they exist in the given range.

(a) $3 - x - x^2 = 0$ **(b)** $5 - x - x^2 = 0$
(c) $3 - x - x^2 = 2 + x$ **(d)** $3 - x - x^2 = 4 - 2x$
(e) $2x^2 + 2x = 9$ **(f)** $x^2 + 2x = 4$

9. Copy and complete the following table for $1 \leqslant x \leqslant 6$.

x	1	1.5	2	2.5	3	4	5	6
$2x + \frac{16}{x} - 11$		2.67		0.4				3.67

Hence, draw the graph of $y = 2x + \frac{16}{x} - 11$ for values of x from 1 to 6, taking scales of 2 cm to represent one unit on both axes. With the same scale, draw the graph of $y = 3 - \frac{x}{2}$.

From your graph,

(a) solve the equation $2x + \frac{16}{x} - 11 = 3 - \frac{x}{2}$;

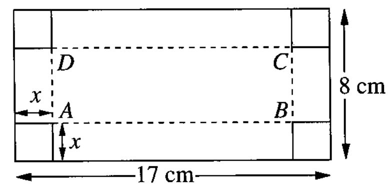
(b) give the range of values of x between $x = 1$ and $x = 6$ for which $2x + \frac{16}{x} - 11$ is greater than $3 - \frac{x}{2}$.

10. Given that $y = 3x - 2x^2$, copy and complete the following table:

x	-2	$-1\frac{1}{2}$	-1	0	1	2	$2\frac{1}{2}$	3
y		-9		0		-2		

Using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3x - 2x^2$ for $-2 \leqslant x \leqslant 3$. Using the same axes, draw the graph of $y = \frac{3}{2}(x - 5)$ and write down the values of x at the points where the two graphs meet. Write down, also, the equation of which these values are the solutions and rearrange this equation in the form $ax^2 + bx + c = 0$, where a, b and c are all whole numbers.

11. *(Answer this equation on a sheet of graph paper.)*



(a) The diagram above shows a rectangular sheet of cardboard measuring 17 cm by 8 cm. A square of side x cm is cut from each corner, and the cardboard is folded along the dotted lines to form an open rectangular box with base $ABCD$ and height x cm.
Show that the volume of the box is $2x(17 - 2x)(4 - x)$ cm^3.

(b) Given that $y = 2x(17 - 2x)(4 - x)$ and that corresponding values of x and y are shown in the table below, calculate the value of a and the value of b.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	0	56	90	105	104	90	66	a	b

(c) Using a scale of 2 cm to represent 0.5 unit on the x-axis and a scale of 2 cm to represent 20 units on the y-axis, draw the graph of $y = 2x(17 - 2x)(4 - x)$ for values of x in the range $0 \leqslant x \leqslant 4$.

(d) Use your graph to estimate the volume of the box whose height is 2.3 cm.

(e) Use your graph to estimate the greatest possible volume of the box.

(f) Use your graph to estimate the volume of the box whose length is 15.2 cm.

12. *(Answer this question on a sheet of graph paper.)*
A stone is thrown from the top of a vertical cliff. Its position during its flight is represented by the equation $y = 56 + 10x - x^2$, where y metres is the height of the stone above the sea and x metres is its horizontal distance from the cliff.

(a) **(i)** Solve the equation $0 = 56 + 10x - x^2$.
(ii) Explain briefly what the positive solution of this equation represents.

Some corresponding values of x and y are given in the following table:

x	0	2	4	6	8	10
y	56	72	80	80	72	56

(b) **(i)** By considering the symmetry of the y values in the table, state the value of x at which the stone reaches its greatest height.
(ii) Use this value of x in the given equation to calculate the greatest height reached.

(c) Taking 2 cm to represent 1 metre on the x-axis and 2 cm to represent 5 metres on the y-axis, draw the graph of $y = 56 + 10x - x^2$ for values of x in the range $0 \leqslant x \leqslant 10$ and values of y in the range $55 \leqslant y \leqslant 90$.

(d) Use your graph to find how far the stone travels horizontally while its height is more than 76 metres. *(C)*

Problem Solving

Example 15

In Fig. 8.15(a), ABCD is a parallelogram and AP is a line segment. If BP = 4BC and the area of ABCD is 35 cm², find the area of △ABP.

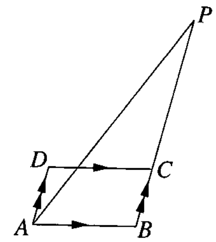


Fig. 8.15(a)

Solution

Method 1: Use a diagram

Draw a big parallelogram $ABPQ$. Since $BP = 4BC$, we mark points U and S on BP such that $BC = CU = US = SP$. Then, draw lines QP, RS and TU parallel to AB to get the diagram as shown in Fig. 8.15(b).

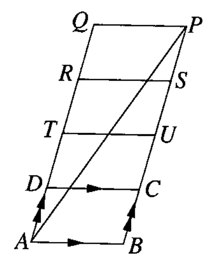


Fig. 8.15(b)

Now, the area of $ABPQ = 4 \times 35 = 140\text{ cm}^2$

Thus, the area of $\triangle ABP = \frac{1}{2} \times 140 = 70\text{ cm}^2$

Method 2:

Produce PB and drop a perpendicular from A to PB. Call this point K. (See Fig. 8.15(c).)

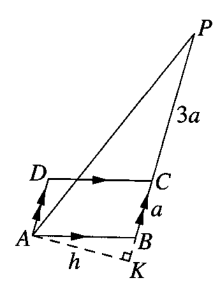


Fig. 8.15(c)

Let $AK = h$ and $BC = a$.

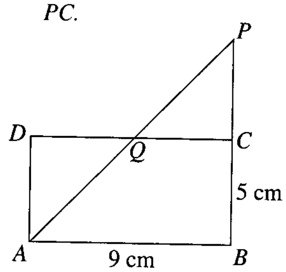
$\therefore \quad PC = 3a$ and $PB = 4a$.

Now, area of parallelogram $ABCD = ah = 35\text{ cm}^2$.

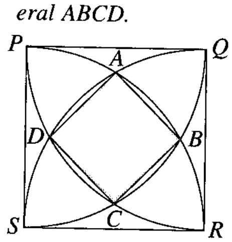
$$\text{Thus, area of } \triangle ABP = \frac{1}{2}(4a)h = 2ah$$
$$= 2(35)\text{ cm}^2$$
$$= 70\text{ cm}^2$$

Problems

1. The figure shows a rectangle ABCD and triangles ADQ and PQC. Given that the area of △PQC is greater than the area of △ADQ by 9 cm², calculate the length of PC.



2. PQRS is a square whose side is 6 units. With centres P, Q, R and S, arcs of circles of radius 6 units are drawn as shown in the diagram below. Find the area of quadrilateral ABCD.



Example 16

We know that 11 + 7 + 2 = 20, but do you know that

$$\begin{array}{rcccccc} & E & L & E & V & E & N \\ & & S & E & V & E & N \\ + & & & & T & W & O \\ \hline & T & W & E & N & T & Y \end{array}$$

is also true when each alphabet represents a digit?

Solution

The opening hint is in the thousand-digit, whereby $E + E$ gives E. It is clear that E cannot be 1, 2, 3, 4, 5, 6, 7. E in the hundred thousand place cannot be 9, as T cannot be 0. Thus E has to be 8 with 2 carried forward from the hundred placing. Thus T has to be 9.

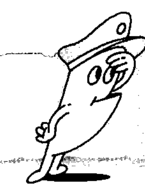
From the tenth placing, E is 8 and W can be either 2 or 3.

If $W = 2$, then $L + S = 11$, and $L + S$ can be 5 + 6 or 4 + 7. Only 5 + 6 is possible. Thus we can have

$$\begin{array}{rr} & 858784 \\ & 68784 \\ + & 923 \\ \hline & 928491 \end{array} \quad \text{or} \quad \begin{array}{rr} & 868784 \\ & 58784 \\ + & 923 \\ \hline & 928491 \end{array}$$

If $W = 3$, we have

$$\begin{array}{rr} & 858682 \\ & 78682 \\ + & 930 \\ \hline & 938294 \end{array} \quad \text{or} \quad \begin{array}{rr} & 878682 \\ & 58682 \\ + & 930 \\ \hline & 938294 \end{array}$$



Exploration

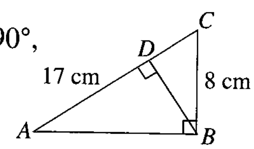
1. Draw the graphs of the equations $y = \sin 2x$ and $y = \cos x$ for $0° \leqslant x \leqslant 90°$. Hence, find the solution of the equation $\sin 2x = \cos x$ in this range.
2. The graph of $y = 2x^3 + ax^2 + bx + 23$ passes through the points (–1, 30), (2, 3) and (0, c).
   - **(a)** Show that $a = -3$ and $b = -12$, and find the value of c.
   - **(b)** Draw the graph of $y = 2x^3 - 3x^2 - 12x + 23$ for values of x in the range $-2.5 \leqslant x \leqslant 3$.
   - **(c)** From your graph, find the range of values of x for which y is decreasing.
   - **(d)** Using your graph, estimate
     - **(i)** the area, in square units, between the graphs, the x-axis, the y-axis and the line $x = -2.5$;
     - **(ii)** the area, in square units, between the graph, the x-axis and the lines $x = 1$ and $x = 3$.

     In each case, do you expect the estimated area to be bigger or smaller than the actual area? Why? Discuss how the estimates can be improved.
   - **(e)** Add an appropriate straight line to your graph to find the solution to the equation $2x^3 - 3x^2 - 17x + 13 = 0$ in the range $-2.5 \leqslant x \leqslant 3$.

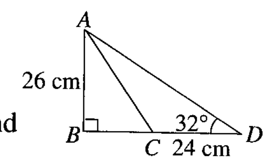
Revision Exercise II No. 1

1. (a) $ABCDE$ is a pentagon. If $\hat{A} = \hat{B} = \hat{C} = x°$ and $\hat{D} = \hat{E} = 120°$, find x.
   (b) The angles of a pentagon are $(x + 10)°$, $(x + 50)°$, $(2x - 10)°$, $2x°$ and $(2x + 10)°$. Find x.

2. (a) In the figure, $A\hat{B}C = B\hat{D}A = 90°$, $BC = 8$ cm and $AC = 17$ cm. Calculate the area of $\triangle ABC$ and find the length of BD.



   (b) In the figure, $A\hat{B}C = 90°$, $B\hat{D}A = 32°$, $AB = 26$ cm and $CD = 24$ cm. Calculate $A\hat{C}B$.



3. (a) Given that y varies inversely as the square root of x, and that $y = 4$ when $x = 9$, calculate the value of y when $x = 81$.
   (b) Express $\frac{3}{3x - 1} - \frac{2}{4x + 3}$ as a fraction in its lowest terms.

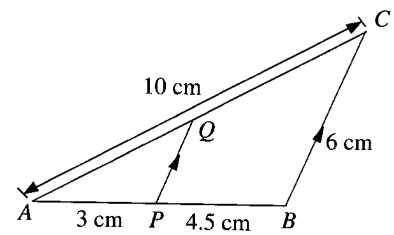
4. Two rectangles have the same area of 480 cm². The difference between their lengths is 10 cm and the difference between their widths is 4 cm. Calculate the dimensions of the rectangles.

5. Simplify the following algebraic fractions:
   (a) $\frac{3}{x} + \frac{4}{3x} - \frac{1}{5x}$
   (b) $\frac{x^2 - 64}{x^2 - 9} \div \frac{x - 8}{x + 3}$
   (c) $\frac{9x^2 + y^2}{9x^2 - y^2} - \frac{3x - y}{3x + y}$

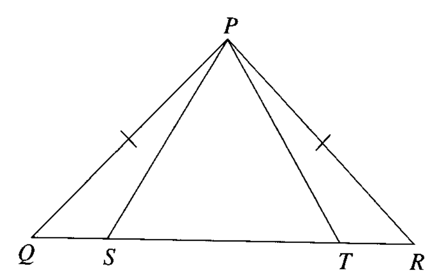
6. $PQRS$ is a square whose vertices are $P(1, 6)$, $Q(2, 1)$, $R(7, 2)$ and $S(h, k)$.
   (a) Find the values of h and k.
   (b) Find the equation of the line QR.
   (c) Calculate the area of $PQRS$.

7. Set up a table of values for $y = 2^x - 5$, such that $-1.5 \leqslant x \leqslant 2.5$. Draw the graph of $y = 2^x - 5$ using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis. Use your graph to find
   (a) the value of y when $x = 0.8$;
   (b) the value of x when $y = -1.5$.

☆8. In the figure below, $AP = 3$ cm, $PB = 4.5$ cm, $BC = 6$ cm and $AC = 10$ cm. Find the lengths of AQ and PQ.



9. In the following figure, $PQ = PR$ and $P\hat{S}T = P\hat{T}S$.



Is $\triangle PQS$ congruent to $\triangle PRT$? State your reasons clearly.

10. The volumes of two similar kettles are 2 160 cm³ and 640 cm³. What is the ratio of their bases? If the smaller kettle has a height of 16 cm, what is the height of the larger kettle?

Revision Exercise II No. 2

1. Each interior angle of a regular polygon exceeds the exterior angle by 160°. Find the number of sides of the polygon.

2. From the top of an observation tower, T, 20 m high, the angle of depression of a ship, A, which is due east of T, is 25.4°. Meanwhile, another ship, B, which is due west of T, finds the angle of elevation of T to be 54.7°. Calculate the distance between A and B.

3. Simplify the following expressions:

   (a) $\frac{2}{5x} + \frac{1}{4x}$

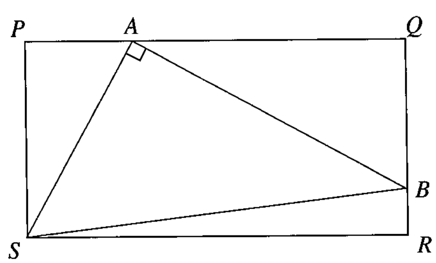
   (b) $\frac{3}{x-1} - \frac{7}{(x-1)^2}$

   (c) $\frac{2}{x-2} + \frac{3}{(x-1)(x-2)}$

   (d) $\frac{1}{x+3} - \frac{4}{x^2-9}$

4. In the diagram below, $PQRS$ is a rectangle. The point A is on PQ and the point B is on QR, such that $S\hat{A}B = 90°$.

   Given that $PQ = 8$ cm, $QR = 4$ cm, $QB = 3$ cm and $PA = y$ cm, write down expressions in terms of y for $(AS)^2$ and $(AB)^2$.



   Form an equation in y and show that it simplifies to $y^2 - 8y + 12 = 0$.

   Solve this equation and, hence, find the two possible values of the area of $\triangle ABS$.

5. Two similar jars have capacities 250 cm³ and 432 cm³. If the smaller jar has a surface area of 245 cm², calculate the surface area of the larger jar.

6. The line $3x + 4y = 24$ cuts the x and y axes at the points A and B respectively. Find

   (a) the gradient of AB;

   (b) the area of $\triangle OAB$, where O is the origin.

7. Given that $1 \leqslant x \leqslant 3$ and that $2 \leqslant y \leqslant 4$, find

   (a) the largest possible value of $y^2 - x$;

   (b) the smallest possible value of

   (i) $(y - x)^2$; **(ii)** $\frac{x}{y}$.

8. **(a)** Given that y varies inversely as x^2 and that $y = 12$ when $x = 2$,

   (i) express y in terms of x;

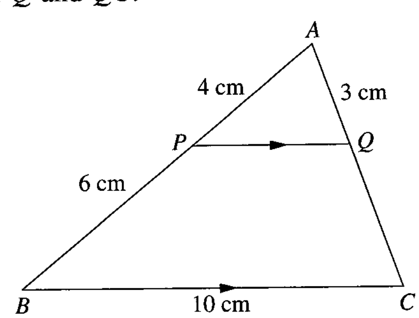
   (ii) calculate the value of y when $x = 3$;

   (iii) calculate the value of x when $y = 3$.

   (b) Evaluate each of the following without the use of a calculator.

   (i) $\left(1\frac{9}{16}\right)^{-\frac{1}{2}}$ **(ii)** $(0.01)^{-\frac{3}{2}}$ **(iii)** $\left(\frac{1}{3}\right)^{-3}$

*9. For the figure below, find the lengths of PQ and QC.



10. The following is a table of values for the function $y = x^2 - 2x + 1$.

x	−2	−1	0	1	2	3	4
y	a	4	1	0	b	4	9

   Calculate the values of a and b. Using a scale of 2 cm for 1 unit for both axes, draw the graph of $y = x^2 - 2x + 1$.

   (a) From your graph, find

   (i) the value of x when $y = 3$;

   (ii) the value of y when $x = 1.5$.

(b) By drawing the line $y = 1 - x$ on the same graph, find the x-coordinates at the points of intersection.

Revision Exercise III No. 3

1. The surface areas of two similar solids are 522 cm^2 and 58 cm^2 respectively. If the smaller solid has a height of 4 cm, find the height of the larger solid.

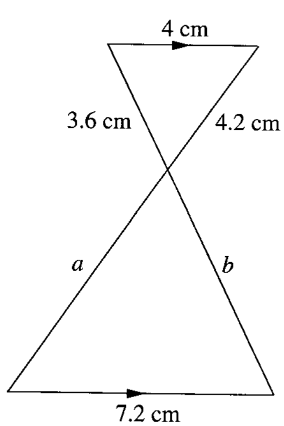
2. Factorise the following:
   (a) $x(x + 2) - x - 2$
   (b) $2 - x(7 - 5x)$
   (c) $(3x - 1)^2 - x^2$
   (d) $x^3 - 5x^2 + x - 5$

3. The wheel of a bicycle has a diameter of 70 cm. Find the number of revolutions the wheel will make in a distance of 22 km. $\left(\text{Take } \pi = \frac{22}{7}.\right)$

4. (a) Simplify the following, giving your answers with positive indices only.
   (i) $p^5 \div p^{-2}$
   (ii) $p^4 \div \sqrt{p} \times p^{-7}$
   (iii) $\left(\frac{x^{-3}y}{x^2y^{-2}}\right)^{-5}$

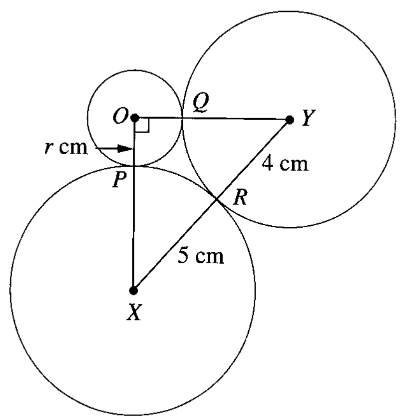
   (b) Given that y is directly proportional to $(2x + 5)$ and that $y = 21$ when $x = 1$, express y in terms of x.

5. Calculate the lengths of a and b in the given figure.



6. A cone, P, of base radius x cm and height h cm, has a volume of 120 cm^3.
   (a) Calculate the volume of another cone similar to P, but with base radius $3x$ cm.
   ☆(b) Calculate the volume of a cone Q which has a base radius $2x$ cm and height $3h$ cm.

7. The following diagram shows three circles: P, Q and R are the points of contact and O, X and Y are the centres of the circles. Given that $XR = 5$ cm, $YR = 4$ cm, $OP = r$ cm and $X\hat{O}Y = 90°$, write down an equation in r and show that it simplifies to $r^2 + 9r - 20 = 0$. Solve this equation and write down the lengths of OX and OY.



8. When 4 is added to a certain positive number, the result is 60 times the reciprocal of the number. Find the number.

9. A string, 15 cm long, is used to form a rectangle. If one side of the rectangle is x cm, write down the length of the other side in terms of x. If the area of the rectangle is y cm^2, show that $y = \frac{1}{2}(15 - 2x)x$.
   Plot the graph of $y = \frac{1}{2}(15 - 2x)x$ for values of x from 0.5 to 6.5, both inclusive. From your graph, find the maximum value of y and the corresponding value of x at this point. Hence, write down the dimensions of the rectangle when the area is a maximum.

10. The following is an incomplete table of values for the graph of $y = (x + 2)(4 - x)$:

x	-2	-1	0	1	2	3	4
y	0			9			0

(a) Calculate the missing values of y.

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = (x + 2)(4 - x)$ for $-2 \leqslant x \leqslant 4$.

(c) Use your graph to find two values of x which satisfy the equation $(x + 2)(4 - x) = 3$.

(d) Use your graph to estimate the biggest value of y.

(e) State the range of values of x for which $(x + 2)(4 - x)$ is greater than 8.

Mid-year Examination Specimen Paper 1

Time: $2\frac{1}{2}$ h

*Answer **all** the questions in **Section A** and any 5 questions in **Section B**.*

Section A (50 marks)

*Answer **all** the questions in this section. Calculators are not allowed.*

1. The KK Women and Children's Hospital set a world record in 1966 having delivered a total of 39 835 babies. Calculate the number of babies that were born in a day, giving you answer correct to the nearest whole number.
(Take 1 year to be 365 days.) [2]

2. **(a)** Solve the equation

$$\frac{2(3x-2)}{3} - \frac{3(x+3)}{4} = 1. \quad [2]$$

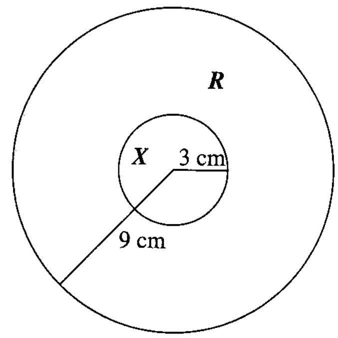
(b) Given that $5l - 3(m + 4) = \frac{m}{l}$, express m in terms of l. [2]

3. A straight line passes through the points (0, 3) and (2, 11). Find
(a) its gradient;
(b) its equation. [2]

4. Evaluate each of the following:
(a) $3^{\frac{1}{3}} \times 12^{\frac{2}{3}} \times 4^{\frac{1}{3}}$
(b) $16^{-\frac{3}{4}} \div 4^{\frac{1}{2}} \times 8^0$ [4]

5. **(a)** Calculate the exact value of $\sqrt{0.04 \times 144}$. [1]
(b) Estimate $\frac{3.98^2 - 1.98 \times 2.02}{2.01}$ to the nearest whole number. [1]

6. The bull's-eye, ***X***, and the shaded outer ring, ***R***, of a target are formed by two concentric circles of radii 3 cm and 9 cm.
(a) Express, in terms of π, the area of ***X***.
(b) Find the numerical value of the ratio $\frac{\text{area of } R}{\text{area of } X}$. [2]



7. The volumes of two similar bags are 108 cm^3 and 500 cm^3. Find the ratio of their
(a) heights; [2]
(b) curved surface areas. [2]

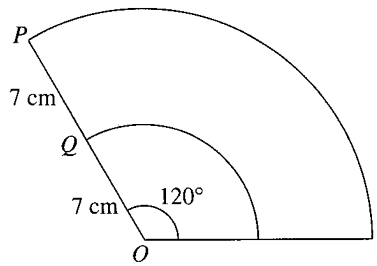
8. Given that y varies directly as the square root of x and that $y = 4$ when $x = 9$, calculate
(a) the value of y when $x = 64$; [2]
(b) the value of x when $y = 7$. [2]

9. The simple interest for 15 months on a loan of \$72 000 is \$5 400. Calculate the rate of interest per annum. [2]

10. The lowest temperatures recorded at the South Pole and in Singapore were –89.2°C and 19.4°C respectively.
(a) Find the difference in the temperatures recorded. [1]
(b) On the day the Singapore expedition team arrived at the South Pole, the temperature was mid-way between the two extreme temperatures. Find this temperature. [2]

11. The windscreen wiper of a car sweeps through an angle of 120°. The shaded region in the given diagram represents the area of glass swept clean by the wiper.

Given that $OQ = PQ = 7$ cm, find the area of glass swept clean.
$\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$



[4]

12. **(a)** Find the volume of a pyramid with height 5 cm and square base of sides 2 cm each. [2]

(b) If the volume of a sphere of radius r is $\frac{4\pi r^3}{3}$, find the volume of another sphere of radius $\frac{r}{4}$. [2]

13. **(a)** Factorise completely $4x^2 - y^2$. [1]

(b) Solve the equations:

(i) $(x - 5)^2 = 9$ [2]

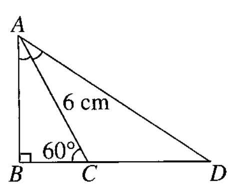
(ii) $x^2 + 4x = 0$ [1]

14. In $\triangle ABD$, $A\hat{B}D = 90°$. AC bisects $B\hat{A}D$. Given that $ACB = 6$ cm and $A\hat{C}B = 60°$, calculate

(a) AB;

(b) the length AD. [4]

(Take sin 30° = 0.5, cos 30° = 0.87, tan 30° = 0.58, sin 60° = 0.87, cos 60° = 0.5, tan 60° = 1.73)



15. Simplify

(a) $\frac{3}{x-2} + \frac{1}{x+2} - \frac{2x-5}{x^2-4}$ [3]

(b) $\frac{2x}{x+7} - \frac{5x}{x-3}$ [2]

Section B (50 marks)

*Answer any **5** questions in this section.*

16. The following is an incomplete table of values for the graph of $y = 3x - 2x^2$:

x	-2	$-1\frac{1}{2}$	-1	0	1	2	$2\frac{1}{2}$	3
y		-9		0		-2	-5	

(a) Fill in the missing values of y.

(b) Using scales of 2 cm to 1 unit on the x-axis and 1 cm to 1 unit on the y-axis, draw the graph of $y = 3x - 2x^2$, for $-2 \leqslant x \leqslant 3$.

(c) Use your graph to find two values of x for which $y = -1$.

(d) By drawing $y = x - 5$ on the same axes, find the two values of x which satisfy the equation $3x - 2x^2 = x - 5$.

17. **(a)** The world's largest fountain situated at Suntec City has a total base area of 1 683.07 m^2. How many people can stand on the base if each person occupies a standing area of 420 cm^2? Give your answer correct to the nearest whole number.

(b) Factorise completely

(i) $12x^2 - 27x$

(ii) $4x + xz - 3yz - 12y$

(c) Solve the equation $\frac{6}{x} = 1 + \frac{5}{x+2}$.

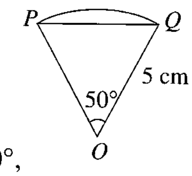
18. **(a)** A solid cylinder has a height of 10 cm and a base diameter of 14 cm.

(i) Find the total surface area of the cylinder.

(ii) Given that the cylinder is made from a material of density 8 g/cm^3, calculate its mass.

$\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$

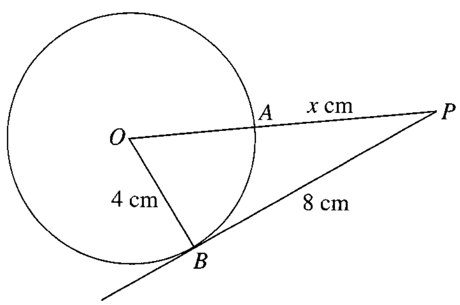
(b) In the diagram, OPQ is a sector of a circle with centre O and radius 5 cm. Given that $P\hat{O}Q = 50°$, calculate



(i) the perimeter of the sector;
(ii) the area of the sector;
(iii) the area of $\triangle OPQ$.
(Take π to be 3.14.)

19. (a) The straight line $2x + 3y = k$ passes through the point (1, 3). Find the value of k. Given that this line is also parallel to the line $6x - ly = 87$, find the value of l.

(b) In the diagram, O is the centre of the circle and PB is its tangent at B. OP intersects the circle at A. Given $OB = 4$ cm, $AP = x$ cm, $O\hat{B}P = 90°$ and $PB = 8$ cm, calculate x correct to 3 significant figures.



20. (a) Solve the equation $2x^2 + 7x - 5 = 0$, giving your answer correct to 3 significant figures.

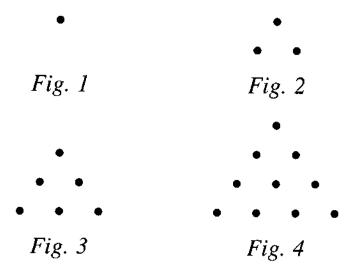
(b) A tank can be filled by two pipes in one hour and twenty minutes. If the two pipes are to fill the tank individually, the smaller pipe will take two hours more than the bigger pipe to do so. Find the time each pipe will take to fill the tank separately.

21. (a) Make x the subject of the formula

$$\frac{a}{b} = \sqrt{\frac{2x-3}{3x+5}}.$$

(b) Simplify $(x^{\frac{1}{2}} + 2y^{\frac{1}{2}})(x^{\frac{1}{2}} - 2y^{\frac{1}{2}})$.

(c)



Study the above pattern carefully. Then answer the following questions:

(i) Draw the next two figures and label them as Fig. 5 and Fig. 6.
(ii) How many beads do you need for the seventh figure?
(iii) Which figure will use up 210 beads?

Mid-year Examination Specimen Paper 2

Time: $2\frac{1}{2}$ h

*Answer **all** the questions in **Section A** and any 5 questions in **Section B**.*

Section A (50 marks)

*Answer **all** the questions in this section. Calculators are not allowed.*

1. (a) Given that $p(q - t) = q^2(p - t)$, find the value of t when $p = 2$ and $q = -2$. [2]

(b) If $p = 2 \times 10^{-3}$ and $q = 7.5 \times 10^{-4}$, express the product pq in the standard form $A \times 10^n$, where $1 \leqslant A < 10$ and n is an integer. [2]

2. Simplify **(a)** $\dfrac{4xy^3}{12x^2yz}$ [1]

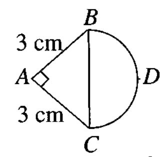
(b) $\dfrac{m^2 + 2m - 3}{m^2 + 8m + 15}$ [2]

3. (a) The cost price of an article is \$12 and the gain is $22\frac{1}{2}\%$. Find its selling price. [2]

(b) In 1997, a bookseller sold 33 000 books. In 1998, he sold 39 600 books. Calculate the percentage increase in his sales. [2]

4. Jack, Jill and John share some marbles in the ratio of 2 : 3 : 8 respectively. Given that Jack and John have 300 marbles altogether, calculate Jill's share. [2]

5. Calculate the area of the semicircle BDC in the diagram. (Take π to be 3.14.)



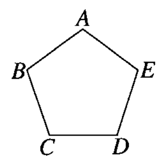
[3]

6. The following diagram shows a regular pentagon $ABCDE$.

(a) State the order of rotational symmetry of the pentagon. [1]

(b) Calculate $A\hat{B}C$. [1]

(c) Can a tessellation be rotated with tiles consisting of regular pentagons of the same size? If so, sketch the tessellation. If not, briefly explain why. [2]



7. A straight line is drawn through the points (0, 3) and (6, 6). Find

(a) the gradient of the line; [1]

(b) the equation of the line. [2]

8. Factorise completely:

(a) $6a^2 - 3a - 30$ [1]

(b) $16x^4 - 81y^8$ [2]

9. The volume of a square pyramid is 70 cm^3. If each side of the base is 5 cm long, find the height of the pyramid. [2]

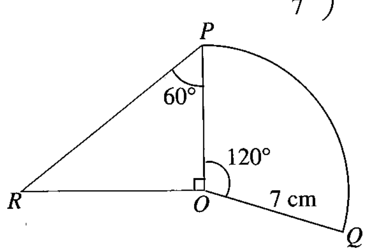
10. In the diagram below, POQ is the sector of a circle, with centre O and radius 7 cm.

Given that $R\hat{P}O = 60°$, $P\hat{O}Q = 120°$ and $P\hat{O}R = 90°$, calculate

(a) OR; **(b)** PR; [2]

(c) the area of sector POQ. [2]

$\left(\textit{sin } 60° = 0.87, \textit{ cos } 60° = 0.5, \textit{ tan } 60° = 1.73. \textit{ Take } \pi \textit{ to be } \frac{22}{7}. \right)$



11. In 1999, the price of a handphone is \$480. In the year 2000, the price of the same handphone is reduced by 15%. Find its new price in 2000.

The price of \$480 was acutally a decrease of 25% over the price in 1998. Calculate the initial price in 1998. [4]

12. (a) If y is directly proportional to x^2, and if $y = 4$ when $x = 4$, find y when $x = 3$. [2]

(b) Given that $F = \frac{9}{5}C + 32$, find the value of

(i) F when $C = 35$;

(ii) C when $F = 104$. [2]

13. The surface areas of two spheres, A and B, are in the ratio of 25 : 144. Find

(a) the radius of A if the radius of B is 15 cm; [2]

(b) the volume of A if the volume of B is 864 cm^3. [2]

14. Evaluate **(a)** $\left(\frac{3}{4}\right)^{-2}$ and **(b)** $16^{-\frac{3}{4}} + \left(\frac{1}{27}\right)^{\frac{2}{3}}$. [3]

15. Consider the following number patterns

$2^3 - 2 = 6 = 1 \times 2 \times 3$

$3^3 - 3 = 24 = 2 \times 3 \times 4$

$4^3 - 4 = 60 = 3 \times 4 \times 5$

⋮ ⋮ ⋮

(a) Write down the 5th line of the sequence.

(b) Express $19^3 - 19$ as a product of 3 consecutive numbers.

(c) Express $x^3 - x$, where x is a whole number, as a product of 3 consecutive numbers. [3]

Section B (50 marks)

Answer any 5 questions in this section.

16. A man walked a distance of 20 km at an average speed of v km/h. Write down an expression for the time, in hours, he took for the journey.

He returned by the same route but his average speed was 1 km/h less. Write down an expression for the time, in hours, he took for the return journey.

Given that the difference between these two times was 45 minutes, form an equation in v and show that it simplifies to $3v^2 - 3v - 80 = 0$. Solve this equation, giving your answer correct to 1 decimal place. Hence, find the time taken, correct to the nearest minute, for the whole journey.

17. **(a)** Given that $\frac{4x}{5} - \frac{3}{10} \leqslant x - 2\frac{1}{4}$, state the smallest possible value of x when

(i) x is a prime number;

(ii) x is an integer;

(iii) x is a rational number.

(b) Two rectangles have an area of 630 cm^2 each. The difference in the lengths of the rectangles is 5 cm while that in their widths is 3 cm. Find the dimensions of the two rectangles.

18. **(a)** A solid cylinder with base diameter of 14 cm has a height of 20 cm. It is made of iron and has a density of 7 g/cm^3. Find

(i) the area of its curved surface;

(ii) its weight.

(b) The lower part of a toy is a hemisphere of radius 3.5 cm and its upper part is a cone of the same radius and a height of 10 cm. Find its volume.

(c) A conical monument is 16 m high and has a base diameter of 24 m. How many litres of paint will be required to paint the monument if 1 litre of paint is needed for an area of 100 m^2?

$\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$

19. **(a)** Solve the following equations:

(i) $\frac{2}{3x} = \frac{x-2}{4(x+3)}$

(ii) $\frac{x-3}{x+5} = \frac{x+7}{x-1}$

(iii) $\sqrt[3]{x} = \frac{1}{2}$

(b) If y varies inversely as $2x^2 + 5$, and if $y = 7$ when $x = 2$, find the value of y when $x = -3$.

20. **(a)** Make h the subject of the formula

$$t = 2\pi\sqrt{\frac{h^2 + g^2}{h^2}}.$$

(b) Simplify:

(i) $\frac{x^2 + x - 6}{x^2 - x - 12} \div \frac{3x - 6}{x^2 + 2x - 24}$

(ii) $\frac{2x + 3}{x^2 - 1} - \frac{x}{x + 1} - \frac{5}{x - 1}$

(c) The coordinates of A, B and C are (1, 2), (3, 10) and (7, 1) respectively. Given that M is the mid-point of AB, calculate the length of CM.

21. **(a)** A man buys n articles for \$$x$ each. He then sells k of them at \$$(5x - 2)$ for 3 and the remaining at \$$(x - 1)$ each. Find his profit in \$, giving your answer in terms of n, x and k.

(b) Two cyclists, 10 km apart, are moving towards one another. One is going at 3 km/h faster than the other. If they pass each other after 20 minutes, find the speed of each cyclist.

(c) Solve the equation $x^2 + 5x + 2 = 0$, giving your answer correct to 1 decimal place.

Mid-year Examination Specimen Paper 3

Time: $2\frac{1}{2}$ h

*Answer **all** the questions in **Section A** and any 5 questions in **Section B**.*

Section A (50 marks)

*Answer **all** the questions in this section. Calculators are not allowed.*

1. Solve the equations:

(a) $6x^2 - x = 2$ **(b)** $\frac{x^2}{2x + 3} - \frac{x}{2} = 1$ [4]

2. The point $(p, 3p)$ lies on the line $2x + 5y = 51$. Calculate the value of p. [2]

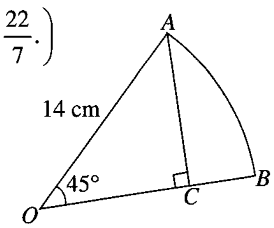
3. Calculate the cost of a rectangular sheet of plywood measuring 80 cm by 70 cm if the wood costs \$7 per m^2. [2]

4. Find the largest integer k which satisfies the inequality $2(k + 4) > 3(k + 1)$. [2]

5. Evaluate **(a)** $81^{\frac{3}{4}}$;
   (b) 7^{-2};
   (c) $16^{-\frac{3}{4}} + 27^{\frac{2}{3}}$. [4]

6. A map is drawn to a scale of 1 : 50 000.
   (a) Calculate the distance between two points on the map which are 85 km apart.
   (b) On the map, the area of a lake is 12 cm^2. Calculate the actual area of this lake, giving your answer in m^2. [2]

7. In the following diagram, $A\hat{O}B = 45°$ and $A\hat{C}O = 90°$. Given that AB is an arc of a circle with centre O and radius 14 cm, calculate
   (a) the area of $\triangle OAC$; [2]
   (b) the area of the shaded portion. [2]
   $\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$



8. **(a)** Given that $n = \frac{2m + l}{m}$, express m in terms of l and n. [2]
   (b) Express, as a single fraction in its simplest form,
   $$\frac{1}{2x + 3} + \frac{6}{(2x - 3)(2x + 3)}.$$ [2]

9. The surface areas of two similar trophies are in the ratio 25 : 81. If the smaller trophy has a height of 14 cm and a capacity of 420 cm^3, find the height and capacity of the larger trophy. [4]

10. Given that y varies inversely as x^2 and that $y = 12$ when $x = \frac{1}{2}$, calculate the value of y when $x = 3$. [2]

11. A solid cylinder has a radius of 5 cm and a height of 8 cm. Taking π to be 3.142, calculate the
   (a) curved surface area of the cylinder; [2]
   (b) volume of the cylinder. [2]

12. Factorise completely
   (a) $3x^2 - 12y^2$ [2]
   (b) $3bx + 6ax - 4ay - 2by$ [2]

13. **(a)** The line $y = ax + b$ is parallel to the line $3y - 4x = 125$ and passes through the point (1, 3). Find the values of a and b. [2]
   (b) The points (1, 1), (2, 4) and (3, k) are collinear. Find the equation of the line and the value of k. [2]

14. **(a)** A clock gains t seconds every one hour. How many minutes will it gain in w weeks? Give your answer in terms of t and w. [2]
   (b) Mr Lim paid \$616 000 for his flat after the developer increased its price by 12%. Find the initial price of the flat before the increase. [3]

15. A straight line passes through the points (0, 3) and (6, 6). Find the
   (a) gradient of the line; [1]
   (b) equation of the line. [2]

Section B (50 marks)

*Answer any **5** questions in this section.*

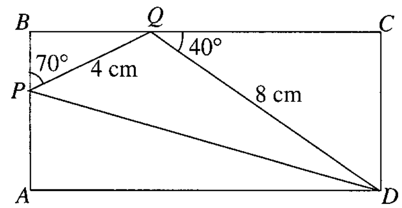
16. **(a)** Solve the equation $2 - 3x - 7x^2 = 0$, giving your answer correct to 3 significant figures.
   (b) The distance between P and Q is 330 km. A train, A, travelling from P to Q at a speed of x km/h will take half an hour less than another train, B, travelling from Q to P at a speed of $(x - 5)$ km/h. Form an equation in x and then find the time taken for both trains, A and B, to travel between P and Q.

17. **(a)** Solve the equation $\frac{x}{1+x} + \frac{x+1}{1-3x} = \frac{1}{4}$.

(b) Make y the subject of the formula $x^2y^2 + x^2 + y^2 = 2$.

(c) Solve the equation $8^x = 2^4 \div 4^2$.

18. In the diagram, $ABCD$ is a rectangle, $PQ = 4$ cm, $QD = 8$ cm, $B\hat{P}Q = 70°$ and $C\hat{Q}D = 40°$.
Calculate **(a)** CD; **(b)** $A\hat{P}D$; **(c)** AD.



19. The following is an incomplete table of values for the graph of $y = x^2 + \frac{10}{x} - 8$:

x	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4
y	$12\frac{1}{4}$				$4\frac{1}{3}$	10.5

(a) Fill in the missing values of y.

(b) Using scales of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of

$$y = x^2 + \frac{10}{x} - 8, \quad \text{for } \frac{1}{2} \leqslant x \leqslant 4.$$

(c) Use your graph to find
(i) the minimum value of y;
(ii) the two values of x for which $y = 2$.

(d) By drawing a suitable straight line, find the solution to the equation

$$x^2 + \frac{10}{x} = 16 - 2x.$$

20. **(a)** A solid metal cylindrical rod of radius 5 cm and length 30 cm is melted and then recast into equal cones, each of radius 1 cm and vertical height 2 cm. How many such cones can be made from the rod?

(b) A cylindrical jar of diameter 14 cm and depth 20 cm is half-filled with water. When 300 lead shots of the same size are dropped into the jar, the level of water rises by 2.8 cm. Find the diameter of each lead shot.

(c) A metallic sphere of radius $10\frac{1}{2}$ cm is melted down and then recast into small cones of radius $3\frac{1}{2}$ cm and height 3 cm each. How many of such cones can be made? $\left(\text{Take } \pi \text{ to be } \frac{22}{7}.\right)$

21. **(a)** At the beginning of 1996, Mr Chin bought a used car for \$56 000. During the whole of 1996 and 1997, he used the car to travel a total distance of 68 000 km. The road tax for the car was \$1 058 per year, insurance costs were \$856 per year and maintenance charges were \$86 for every three months. On average, the fuel consumption of the car was 1 litre of petrol for every 11 kilometres and 1 litre of engine oil for every 5 000 km. Each litre of petrol costs 108 cents and each litre of engine oil costs \$6.70. Calculate the total expenditure incurred by Mr Chin in purchasing and running his car. At the end of 1997, Mr Chin sold his car for 80% of the original price that he paid for. If Mr Chin's company compensated him 45 cents for every kilometre travelled, calculate the gain/loss he made in purchasing, operating and selling his car during the two-year period.

(b) **(i)** Find the integral value of x which satisfies the inequality $x < 5\sqrt{0.4} < x + 1$.
(ii) List the pairs (x, y) for which x and y are positive integers such that $5x + 3y = 35$.

CHAPTER 9

Further Graphs and Graphs Applied to Kinematics

In this chapter, you will learn how to

- △ draw distance-time and speed-time graphs;
- △ solve problems involving distance-time and speed-time graphs;
- △ find the distance covered by a particle using the speed-time graph.

Preliminary Problem

When we travel from one place to another, we often talk about time, speed and distance. When two motor car enthusiasts talk they will usually bring up terms like acceleration and the time needed to reach a maximum speed. The study of moving objects in relation to distance, time, speed and acceleration without consideration of their causes is known as kinematics.

Distance-Time Graphs

We have studied various forms of graphs and their applications to problems. We shall now look at some simple graphs as applied to kinematic problems involving distance-time and speed-time.

Example

A cyclist starts a 50-km journey at 08 00. The table below is the distance-time chart of his journey.

Time (h)	*08 00*	*08 30*	*09 00*	*09 30*	*10 00*	*1030*
Distance (km)	*0*	*10*	*20*	*20*	*35*	*50*

Plot a graph with the distance on the vertical axis and the time on the horizontal axis. Join the points by straight lines. From the graph, find the gradient of the different parts of the graph and hence deduce the different speeds at which the cyclist is moving. Find the average speed of the cyclist over the whole journey.

Solution

Fig. 9.1 shows the graph of the cyclist's journey between 08 00 and 10 30. The graph can be divided into three parts: from 08 00 to 09 00; from 09 00 to 09 30 and from 09 30 to 10 30.

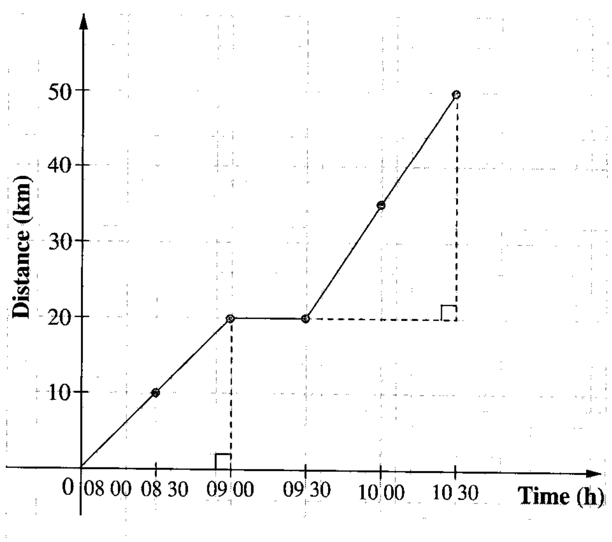


Figure 9.1

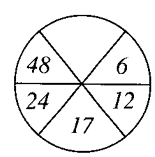
CHECK THIS OUT

Study the way in which you travel to school. Draw an approximate distance-time graph of your journey and compare it with that of your classmates.

Are You Game Enough?

In each of the following case fill in the missing numbers.

(a)



(b) 325, [27], 476

123, [23], 971

359, [ ], 684

(c) 1, 2, 9, 28, [ ]

(d) 5, 8, 14, 26, [ ]

(e) (15) (14) (25)

(7) (5) (11)

(16) (18) ( )

Gradient of the first part of graph = $\frac{20 \text{ km}}{1\text{h}}$ = 20 km/h.

The graph is a horizontal line for the second part of the journey and the gradient is, therefore, **zero**.

Gradient of the third part = $\frac{30 \text{ km}}{1\text{h}}$ = 30 km/h.

The gradient of the first part of the graph is 20 km/h. This is in fact the speed of the cyclist. Thus the gradient of a distance-time graph measures the speed of a moving body, since **speed is a measure of the rate of change of distance over time** of a body.

The second part of the graph shows the cyclist remaining at the same place, i.e., 20 km from the starting point. He is actually *at rest* from 09 00 to 09 30. In a distance-time graph, a horizontal line signifies that the body is at rest.

The gradient of the third part of the graph is 30 km/h, i.e., the cyclist is moving at a speed of 30 km/h.

The average speed of the whole motion is defined as $\frac{\text{total distance travelled}}{\text{total time taken}}$.

$$\therefore \quad \text{average speed} = \frac{50 \text{ km}}{2\frac{1}{2} \text{ h}} = 20 \text{ km/h}$$

NB: Do not mistake the average speed to be $\frac{20 + 0 + 30}{3}$ km/h.

Example 1 shows that when the graph is a straight line, the body is moving at a constant speed or is at rest. What happens when the graph is a curve? The study of this usually involves Calculus. We shall not go into great detail here but will just discuss this briefly.

Back in Time

Gottfried Wilhelm Leibniz a German philosopher and mathematician (1646–1716), published his famous theory of calculus in 1684.

Example 2

(a) Convert the speed 36 m/s into kilometres per hour.
(b) Convert the speed 72 km/h into metres per second.
(c) A wheel of radius 14 cm is turning at 30 revolutions per minute. Find the speed of a point on the rim of the wheel, giving your answer in m/s.
$\left(\text{Take } \pi = \frac{22}{7}\right)$

Solution

(a) $36 \text{ m/s} = \frac{36 \text{ m}}{1 \text{ s}}$

$$= \frac{\frac{36}{1\,000} \text{ km}}{\frac{1}{3\,600} \text{ h}}$$

$= 129.6$ km/h

(b) $72 \text{ km/h} = \frac{72 \text{ km}}{1 \text{ h}}$

$$= \frac{72 \times 1\,000 \text{ m}}{60 \times 60 \text{ s}}$$

$= 20$ m/s

(c) In 1 minute, a point on the rim moves $30 \times 2 \times \pi \times 14$ cm.

Hence, its speed $= \dfrac{30 \times 2 \times \frac{22}{7} \times 14 \text{ cm}}{60 \text{ s}} = 44$ cm/s $= 0.44$ m/s.

Gradient of a Curve

If a straight line and a circle (which is a curve) have only one point of contact, then that straight line is called a **tangent**. Thus, a tangent is a *straight line drawn just touching a curve at a particular point.* The gradient of the tangent is also known as the gradient of the curve at this point.

CHECK THIS OUT

The gradient of a curve at a point can be found by using calculus. Find out how this is done and use it to counter-check your answers.

Example 3

The variables x and y are connected by the equation $y = \frac{1}{2}(5x - x^2)$.

Corresponding values of x and y are given in the table below:

x	-1	0	1	2	$2\frac{1}{2}$	3	4	5
y	a	0	2	3	b	3	2	0

(a) Calculate the values of a and b.

(b) Taking 2 cm to represent 1 unit on both axes, draw the graph of $y = \frac{1}{2}(5x - x^2)$ *for* $-1 \leqslant x \leqslant 5$.

(c) By drawing a tangent, find the gradient of the curve at the point $x = 1$.

(d) Use your graph to estimate, correct to one decimal place, the values of x satisfying the equation $5x - x^2 - 1 = 0$.

(e) By drawing the line $2y = x + 2$ *on the same graph, find the range of values of x for which* $5x - x^2 > x + 2$.

Solution

(a) When $x = -1$, $y = \frac{1}{2}[5(-1) - (-1)^2] = -3$

$\therefore \quad a = -3$

When $x = 2\frac{1}{2}$, $y = \frac{1}{2}\left[5\left(2\frac{1}{2}\right) - \left(2\frac{1}{2}\right)^2\right] = 3\frac{1}{8}$

$\therefore \quad b = 3\frac{1}{8}$

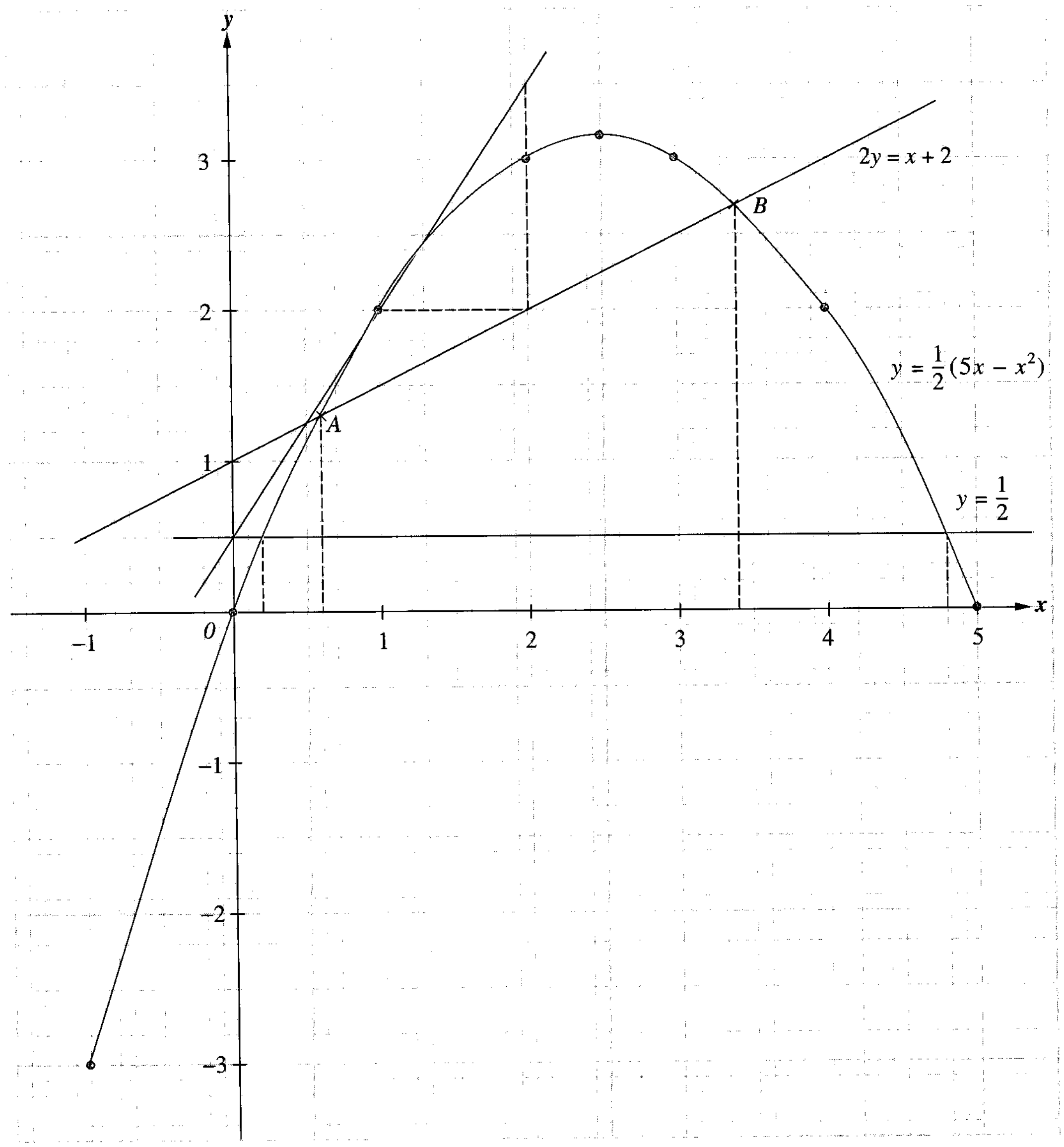
Are You Game Enough?

Mr Tan made a car journey from Singapore to Kuala Lumpur at an average speed of 80 km/h. On his return journey, his average speed was 90 km/h.

Mr Lee made the same journey from Singapore to Kuala Lumpur and back at an average speed of 85 km/h throughout.

Assume they travelled using the same route, would the time taken by Mr Lee and Mr Tan be the same? If not, who would have taken a longer time?

(b) The graph of $y = \frac{1}{2}(5x - x^2)$, for $-1 \leqslant x \leqslant 5$, is shown below:



(c) A tangent is drawn to the curve at the point $x = 1$. From the graph, the gradient $\approx \frac{1.5}{1} = 1.5$.

(d) $5x - x^2 - 1 = 0$

$5x - x^2 = 1$

$\frac{1}{2}(5x - x^2) = \frac{1}{2}$

The solution is given by the points of intersection of the curve and the line $y = \frac{1}{2}$.

Find the graph, $x \approx 0.2$ or 4.8.

(e) The line $2y = x + 2$ intersects the curve at A and B.

$$5x - x^2 > x + 2$$

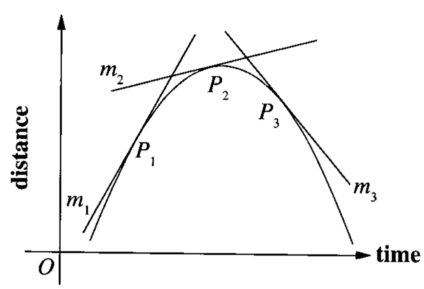
$$\frac{1}{2}(5x - x^2) > \frac{1}{2}(x + 2)$$

Thus the values of x for which $\frac{1}{2}(5x - x^2) > \frac{1}{2}(x + 2)$ occur in the region between A and B, where the curve is above the line. Hence, the range is approximately $0.6 < x < 3.4$.

Gradients of a Distance-Time Curve

We shall now study the applications of the *gradients of tangents* in kinematics.

When the distance-time graph is *a straight line*, the gradient measured is of that between any two points on the line. When the distance-time graph is *a curve*, the gradient at a point P is defined as the gradient of the tangent to the curve at the point P. The following figure shows the gradients of a curve at different parts of the curve.



Thus, the gradient of the tangent m_1 gives the gradient of the curve at P_1; the gradient of the tangent m_2 gives the gradient of the curve at P_2; and so on. Notice that the gradient of P_1 is greater than that at P_2. This implies that at P_1 *the body is moving faster* than at P_2. At P_3, the gradient is negative. This implies that *the body is moving in the opposite direction, i.e., back to its starting point.*

Example

A train started from A and travelled to a point 8 km from A. The following readings were taken of the time (in minutes) since leaving A and the distance (in km) from A.

Time (min)	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Distance (km)	*0.3*	*1.1*	*2.3*	*4.8*	*6.8*	*7.4*	*7.8*	*8.0*

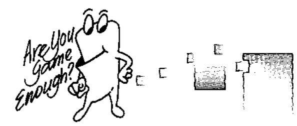
Draw a graph of these values, using a scale of 2 cm for 2 minutes on the horizontal axis and a scale of 2 cm for 2 km on the vertical axis.

Notes

The gradient of the tangent at a point on a distance-time graph gives the speed at that particular point. It is also called the ***instantaneous speed****.*

Use your graph

(*a*) *to estimate the time taken to travel the first 4 km;*
(*b*) *to find, by drawing a tangent, the speed of the train in km/h, 5 minutes after leaving A.*



Mrs Lim has decided to go to Johor Bahru for a shopping spree next Sunday. She can either take a train from Tanjong Pagar which is near to her house, or she can drive to Johor Bahru. Draw a graph for a train journey and a separate graph for a car journey. What would be the most significant difference in the two travel graphs?

Solution

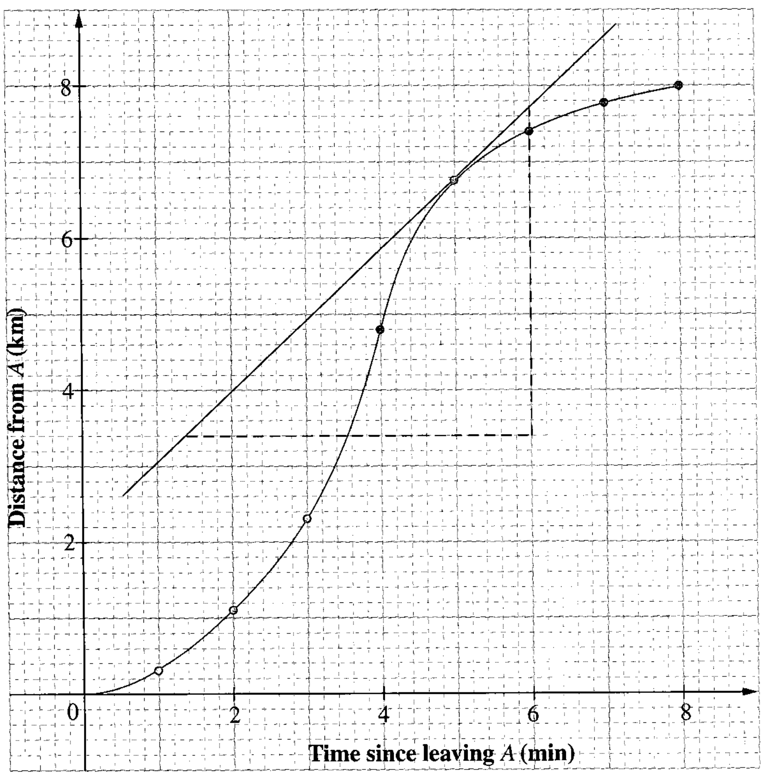


Fig. 9.2

The graph is shown in Fig. 9.2.

Note: The points on the graph are joined by a smooth curve instead of line segments as we assume that the train is not travelling at a constant speed.

(a) From the graph, the train takes approximately 3.8 minutes to travel the first 4 km.
(b) A tangent is drawn at the point 5 minutes after leaving A. The gradient of the tangent

$$\approx \frac{4.4 \text{ km}}{4 \text{ min}} = \left(4.4 \div \frac{4}{60}\right) \frac{\text{km}}{\text{h}}$$

$= 66$ km/h, i.e. the speed of the train at this point.

NB: The graphical method of finding gradients yields only approximate results. A slight change in the drawing will give very different results.

Exercise 9a

1. Convert the following speeds into km/h:

   (a) 24 m/s (b) 18 m/s (c) 40 m/s (d) a m/s

2. Convert the following speeds into m/s:

   (a) 16 km/h (b) 32 km/h (c) 45 km/h (d) b km/h

3. The Sjoormen submarine in the Singapore Navy is capable of attaining a speed of 16 nautical miles an hour. Given that one nautical mile is equivalent to 1.852 km, express the speed of the Sjoormen submarine in

   (a) km/h (b) m/s

4. A cyclist set out at 09 00 for a destination 40 km away. He cycled at a speed of 15 km/h until 10 30. Then, he rested for half an hour before completing his journey at a speed of 20 km/h. Draw the distance-time graph to represent the journey and use your graph to estimate the time at which the cyclist reached his destination.

5. The BIONIX Infantry Fighting Vehicle developed by the Singapore Army is capable of attaining a maximum speed of 70 km/h. Express this speed in m/s.

6. A motorist travelled for 2 hours on a muddy road at an average speed of 30 km/h; then he travelled another 168 km in 3 hours on a dual-carriage way; and finally he travelled 114 km on an expressway at an average speed of 76 km/h. Calculate the

   (a) distance he travelled on the muddy road; (b) average speed on the dual-carriage way;
   (c) time he spent on the expressway; (d) average speed for the whole journey.

7. A bicycle wheel of diameter 70 cm is turning at 5 revolutions per second. Find the speed of a point on the rim of the wheel, giving your answers in km/h. $\left(\text{Take } \pi = 3\frac{1}{7}.\right)$

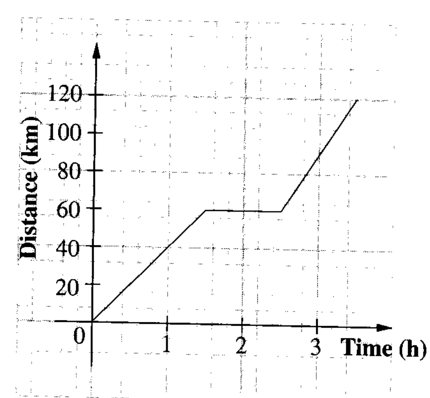
8. A wheel of radius 40 cm is turning about a fixed axis through the centre of the wheel at a rate of 2 revolutions per minute. Calculate the

   (a) angle through which the wheel turns in 10 seconds;
   (b) distance moved by a point on the rim in 10 seconds.

9. A cyclist starts a 30-km journey at 09 00. He maintains an average speed of 20 km/h for the first three-quarter hour and then stops. Subsequently, he continues his journey at an average speed of 30 km/h, finally arriving at his destination at 11 20. Draw the distance-time graph and state, in minutes, the duration of his stop.

10. Use the travel graph for a train to find

    (a) the time interval during which the train remained stationary;
    (b) the average speed for the first two hours;
    (c) the average speed for the last two hours.



11. A motorist left town X at 08 00 for town Y, situated 120 km away, travelling at a constant speed of v km/h, so as to arrive at Y at 10 40. But, after travelling for 80 minutes, his car developed engine trouble and he had to stop for 30 minutes to repair it. Then he continued his journey at a speed of u km/h, so as to arrive at Y at 10 40.

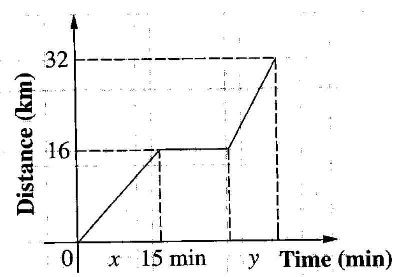
(a) Sketch the distance-time graph of the car. **(b)** Calculate the values of v and u.

12. A motorist set off from town A at 08 00 for town B 70 km away. He drove at a speed of 45 km/h until his car broke down 1 hour later and he had to spend 30 minutes repairing it. Then he completed his journey at a speed of 20 km/h. Draw the distance-time graph to represent the journey and use your graph to estimate the motorist's time of arrival at town B.

13. Two men start moving towards each other at the same time. If they are originally 32 km apart and one is cycling at 20 km/h while the other walking at 7 km/h, how long will it take them to pass each other? At what time will they be 5 km apart? Illustrate your answer using a distance-time graph.

14. The diagram shows the distance-time graph for a car which travels a distance of 16 km in x minutes at an average speed of v km/h. Then, after a 15-minute stop, the car goes a further 16 km in y minutes at an average speed of $2v$ km/h. The total time for the journey is 75 minutes. Find

(a) the ratio $\frac{x}{y}$, **(b)** x. (C)



15. At 09 00, A cycles to meet B, who is staying 20 km away. The cyclist travels at a steady speed of 18 km/h for half an hour. A is delayed for 20 minutes and then continues his journey at 8 km/h. At 09 00, B sets off from home on the same road to meet A and travels at 7 km/h. Draw the distance-time graph for the above information and find the time at which A and B meet. How far away are they from B's home?

16. The variables x and y are connected by the equation $y = (x + 2)(4 - x)$. Corresponding values of x and y are given in the table below:

x	-2	-1	0	1	2	3	4
y	0	p	8	9	q	r	0

(a) Find the values of p, q and r.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = (x + 2)(4 - x)$ for $-2 \leqslant x \leqslant 4$.

(c) Use your graph to find the gradient of the curve at the point where $x = -1$ by drawing a tangent.

(d) By drawing a suitable straight line on your graph, find the solution of the equation $(x + 2)(4 - x) = x + 3$.

(e) Find the range of values of x for which

$$\frac{1}{2}(x + 2)(4 - x) > 4 - \frac{1}{2}x.$$

17. The variables x and y are connected by the equation $y = 12 + 10x - 3x^2$. Copy and complete the table of values for $y = 12 + 10x - 3x^2$.

x	–2	–1	0	1	2	3	4	5
y	–20		12	19	20			

(a) Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = 12 + 10x - 3x^2$ for $-2 \leqslant x \leqslant 5$.

(b) Find the gradient of the curve at $x = 0$ and $x = 4$ by drawing two appropriate tangents.

(c) By drawing a suitable straight line on your graph, find the range of values of x for which $14 + 9x - 3x^2 \geqslant 0$.

18. *(Answer the whole of this question on a sheet of graph paper.)*

The variables x and y are connected by the equation $y = \frac{1}{10}\left(120 - x^2 - \frac{96}{x}\right)$.

Some corresponding values of x and y are given in the table below:

x	1	2	3	4	6	8	10	12
y	2.3	6.8	7.9	8.0	6.8	4.4	1.0	a

(a) Calculate the value of a.

(b) Taking 1 cm to represent 1 unit on each axis, draw the x and y axes for $0 \leqslant x \leqslant 12$ and $-4 \leqslant y \leqslant 10$. Draw the graph of $y = \frac{1}{10}\left(120 - x^2 - \frac{96}{x}\right)$ for values of x in the range $1 \leqslant x \leqslant 12$.

(c) **(i)** Use your graph to find the x-coordinates of the points on the curve for which $y = 4$.

(ii) Write down, but do not simplify, an equation in x which is satisfied by these values of x.

(d) By drawing another line on your graph, find the solution of equation $x = \frac{1}{10}\left(120 - x^2 - \frac{96}{x}\right)$, which lies between $x = 4$, and $x = 8$.

(e) By drawing a tangent, find the gradient of the curve $y = \frac{1}{10}\left(120 - x^2 - \frac{96}{x}\right)$ at the point (8, 4.4).

19. A lift moves up from ground level to a 60-m level in 10 seconds, stops for 10 seconds and then descends to the ground in 10 seconds. The table shows the height of the lift on the upward and downward journeys, t seconds after leaving ground level:

t (s)	0	2	4	6	8	10	20	22	24	26	28	30
H (m)	0	3	16	44	57	60	60	57	44	16	3	0

(a) Plot a graph of H (m) against t (s) for values of t from 0 to 30. (Let 2 cm represent 5 seconds on the horizontal axis.)

(b) Find the gradient of the graph at $t = 8$ and explain briefly what this gradient represents.

(c) A man, waiting at the 40-m level, starts to go downstairs at $t = 15$. He moves at a steady speed of 2 m/s. From the graph, find the height at which the lift passes him.

20. A train left A for B, 7 km away. The table gives the time since leaving A and the distance from A.

Time (min)	0	2	4	6	8	10	12
Distance (km)	0	0.25	1.15	2.83	5.40	6.65	7.0

Plot the graph of the values. Take 1 cm to represent 1 minute horizontally and 2 cm to represent 1 km vertically. Estimate, from the graph, the speed of the train in km per minute when the train moved 6 km. Another train passed through B two minutes after the first train left A. It travelled towards A at a uniform speed of 60 km/h. On the same axes, plot the graph of this journey and, hence, determine the distance from A at which the trains pass each other.

21. A train started from A and travelled to a point B, 3 km away. The table below gives the time after the train left A and the distance from A.

Time (min)	0	1	2	3	4	5	6
Distance (km)	0	0.2	0.7	1.8	2.5	2.9	3.0

Using a scale of 2 cm to represent 1 minute on the horizontal axis and a scale of 4 cm to represent 1 km on the vertical axis, plot a graph using the given values. From your graph, find the

(a) approximate time taken to travel 1 km;

(b) gradient of the graph at a time of $1\frac{1}{2}$ minutes and explain briefly what this value represents;

(c) time taken to travel the last 1 km.

Speed-Time Graphs

We shall now look at another commonly used graph which is the **speed-time graph**. On a speed-time graph, the *speed* is represented on the *vertical* axis and the *time*, on the *horizontal* axis. Fig. 9.3 is a simple example of a body moving at constant speed.

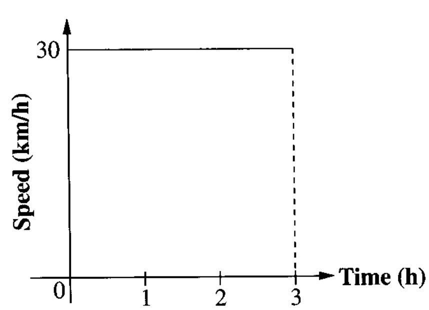


Fig. 9.3

Notes

1. *If the direction in which a body moves is taken into consideration then the graph obtained is called a* ***velocity-time graph****.*
2. *In a speed-time graph, the numerical value of the distance covered is equal to the numerical value of the area under the graph. So, it is just for convenience that we write area under the graph = x km.*

Notice that when a speed-time graph is a horizontal straight line, the body is moving at constant speed. (Do you remember what a horizontal line in a distance-time graph tells us?)

Fig. 9.3 shows a body moving at 30 km/h for 3 hours. The total distance travelled in this case = speed × time = 30 × 3 = 90 km.

Now look at the area under the graph for the interval. The horizontal line represents 3 h and the vertical represents 30 km/h.

$$\therefore \quad \text{area} = 30 \ \frac{\text{km}}{\text{h}} \times 3 \text{ h} = 90 \text{ km}$$

NB: The unit of time (h) cancels out in the multiplication and the result gives the distance covered, i.e., 90 km.

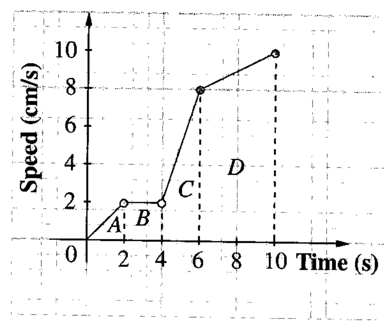
This is no mere coincidence. In fact, it can be proven that in all speed-time graphs, the distance covered is usually represented by the area under the graph.

The rate of change of speed is called the **acceleration** and this is the measure of how fast the speed is increasing or decreasing over time. It is the gradient of the speed-time graph. If the acceleration is negative, then it is called **retardation** or **deceleration**, with the body tending to slow down.

Example 5

The graph shows the speed of a body over a period of 10 seconds. Find

(a) the rate of change of speed during the first 2 seconds;
(b) the total distance travelled during the first 2 seconds;
(c) the average speed during the 10 seconds.



Solution

(a) During the first 2 seconds,

$$\text{acceleration} = \frac{2-0}{2-0} = 1 \ \frac{\text{cm/s}}{\text{s}} = 1 \text{ cm/s}^2$$

NB: 1 cm/s^2 is read as '1 cm per second squared.'

(b) Distance travelled during the first 2 seconds is the area under the graph from $t = 0$ to $t = 2$.

$$\therefore \quad \text{distance travelled} = \frac{1}{2} \times 2 \text{ cm/s} \times 2 \text{ s} = 2 \text{ cm}$$

(c) Distance travelled during the 10 seconds is the area under the graph for the 10 seconds.

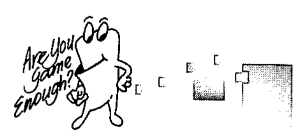
$\therefore$ distance = **(Area of *A*) + (Area of *B*) + (Area of *C*) + (Area of *D*)**

Area of *A* $= \frac{1}{2} \times 2 \times 2 = 2$ cm **Area of *B*** $= 2 \times 2 = 4$ cm

Area of *C* $= \frac{1}{2}(2 + 8) \times 2 = 10$ cm **Area of *D*** $= \frac{1}{2}(8 + 10) \times 4 = 36$ cm

$\therefore$ total distance travelled $= 2 + 4 + 10 + 36 = 52$ cm

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{52}{10} = 5.2 \text{ cm/s}$$



Suppose a car moves up the slope of a hill at an average speed of 30 km/h. At what speed must the car go back down the slope in order to achieve an average speed of 60 km/h for the whole journey?

Example 6

A particle moves along a straight line AB so that, after t seconds, its speed, v m/s, in the direction AB is given by

$$v = 3t^2 - 15t + 20.$$

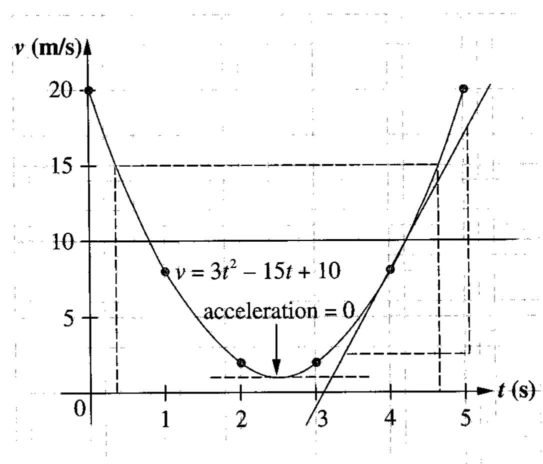
The corresponding values of t and v are given in the table below:

t	0	1	2	3	4	5
v	20	8	2	k	8	h

Calculate the values of k and h.
Taking 1 cm to represent 1 second on the horizontal axis and 1 cm to represent 5 m/s on the vertical axis, draw the graph of $v = 3t^2 - 15t + 20$ for the range $0 \leqslant t \leqslant 5$. Use your graph to estimate

(a) the value of t when the speed is 10 m/s;
(b) the time at which the acceleration is zero;
(c) the gradient at t = 4, and explain what this value represents;
(d) the time interval when the speed is less than 15 m/s.

Solution



When $t = 3$, $v = 3(3)^2 - 15(3) + 20 = 2$. $\therefore\ k = 2$
When $t = 5$, $v = 3(5)^2 - 15(5) + 20 = 20$. $\therefore\ h = 20$

(a) From the graph, $t \approx 0.8$ or 4.2 when speed = 10.

(b) The acceleration is zero when the gradient of the curve is zero. From the graph, the acceleration is zero at $t \approx 2.5$.

(c) A tangent is drawn at the point $t = 4$. From the graph,

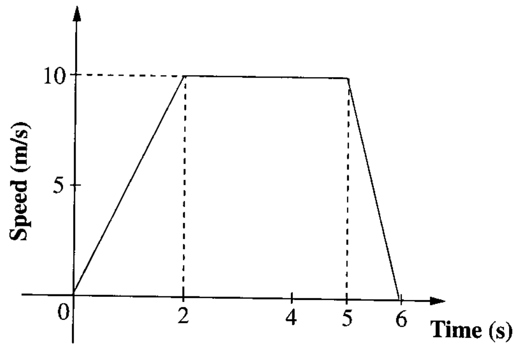
$$\text{gradient of the tangent} = \frac{15.0 \text{ m/s}}{1.7 \text{ s}} = 8.8 \text{ m/s}^2$$

(The gradient of the tangent of the speed-time graph gives the acceleration of the particle at that instant.)

(d) The time interval for which the speed is less than 15 m/s is $0.35 < t < 4.65$.

Exercise 9b

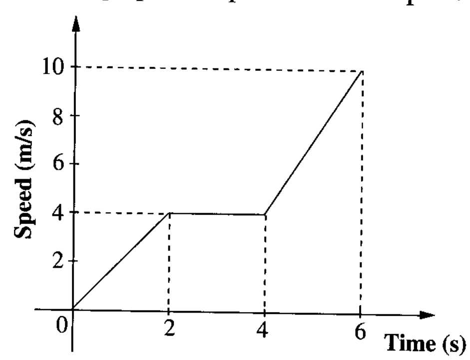
1. The diagram shows the speed-time graph of a car.



Calculate

(a) its acceleration during the first 2 seconds;
(b) the distance travelled during the first 4 seconds;
(c) the average speed for the whole journey.

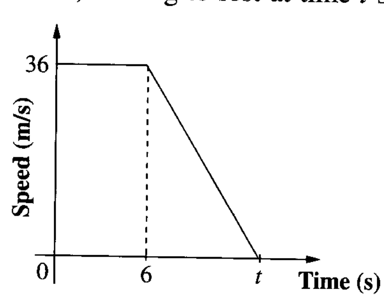
2. The diagram shows the speed-time graph of a particle over a period of 6 seconds.



Calculate

(a) its acceleration during the first two seconds;
(b) the greatest acceleration;
(c) the total distance moved;
(d) the average speed of the particle.

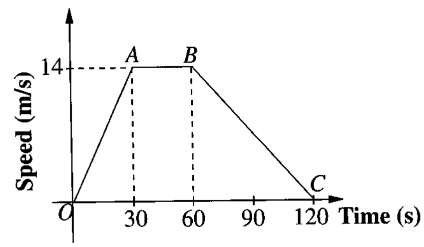
3. The diagram is the speed time graph of a particle which travels at a constant speed of 36 m/s and then slows down at a rate of 12 m/s^2, coming to rest at time t seconds.



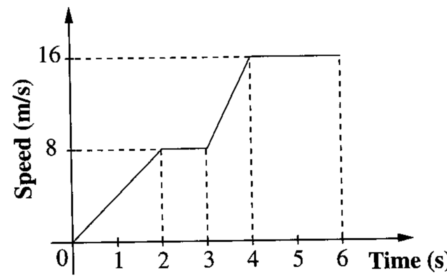
Calculate the

(a) value of t;
(b) average speed for the whole journey.

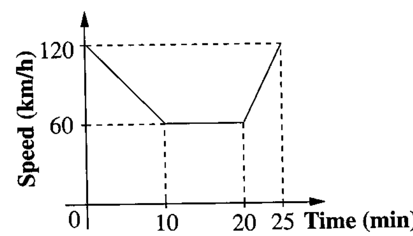
4. The graph shows the speed v m/s of a car after t s.
   (a) What does the gradient of OA represent?
   (b) What is the speed of the car when $t = 15$?
   (c) How far does the car travel between $t = 30$ and $t = 60$?



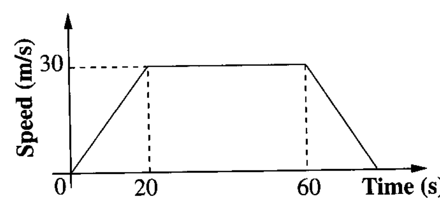
5. The graph shows the speed of a body during a period of 6 seconds. Find
   (a) its acceleration during the first 2 seconds;
   (b) the total distance travelled during the 6 seconds;
   (c) the average speed during the 6 seconds.



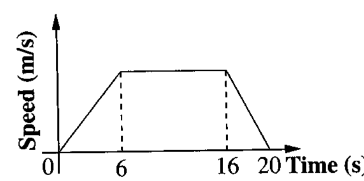
6. The graph illustrates the speed of a car in kilometres per hour during a period of 25 minutes. Find
   (a) the total distance travelled, in kilometres, over the 25 minutes;
   (b) the average speed in, km/h, during the 25 minutes.



7. The diagram shows the speed-time graph of a train.
   (a) Calculate the acceleration of the train during the first 20 seconds.
   (b) Calculate the distance the train travels from rest before it begins to decelerate.
   (c) Given that the train decelerates at 0.75 m/s^2, calculate the time taken for the whole journey.



8. The graph shows the speed of a motor car during a period of 20 seconds. The distance covered in the first 6 seconds is 60 metres. Calculate the
   (a) maximum speed;
   (b) average speed during the 20 seconds.



9. A particle travels at a constant speed of 6 m/s for 24 seconds and then slows down to rest in the next 3 seconds. Sketch the speed-time graph for the motion and calculate the
   (a) retardation in the last 3 seconds; (b) total distance covered in the whole duration.

10. An electric train accelerates uniformly from rest to reach 14 m/s in 10 seconds. Sketch a speed-time graph for the motion. What is the total distance covered?

11. A train starts from rest and accelerates at a uniform rate for 45 seconds, at the end of which it is travelling at a speed of 30 m/s. It then travels at this constant speed. Sketch a speed-time graph and calculate the
    (a) speed after 10 seconds; (b) total distance covered in the first 2 minutes.

12. A car travels 63 km in $1\frac{1}{2}$ hours and then travels at a constant speed of 36 km/h for the next $1\frac{1}{2}$ hours. It is then brought to rest uniformly for a further $\frac{1}{2}$ hour. Calculate the

(a) average speed for the first $1\frac{1}{2}$ hours; **(b)** total distance covered;
(c) average speed for the whole journey.

13. A car is retarded uniformly from a speed of 32 m/s to a speed of 24 m/s in a time of 24 seconds. It is then brought to rest uniformly after a further 4 seconds. Sketch the speed-time graph for the motion of the car. Hence, or otherwise, calculate the

(a) retardation of the car during the first 24 seconds; **(b)** total distance covered;
(c) average speed for the whole journey; **(d)** speed of the car after 25 seconds.

14. The speed of an object, v m/s, at time t seconds from the start is given by $v = 6 + 2t$. Sketch the speed-time graph and calculate the

(a) speed when $t = 3$; **(b)** acceleration;
(c) average speed during the first 4 seconds; **(d)** average speed during the fourth second.

15. The speed of a body, v m/s, after time t seconds is given in the following table:

t (s)	0	2	4	6	8	10	12
v (m/s)	0	2	7	12	19	28	42

Using a scale of 1 cm to represent 1 second on the horizontal axis and 1 cm to represent 5 m/s on the vertical axis, plot the graph of v against t for values for t from 0 to 12.

(a) By drawing two tangents, find the accleration of the body when $t = 4$ and $t = 10$.
(b) Estimate the speed of the body at $t = 5$ and $t = 11$.

16. A body starts from a point A and moves towards a point Q, which it reaches after 7 seconds. The velocity v cm/s after t seconds is given in the table below:

t (s)	0	1	2	3	4	5	6	7
v (cm/s)	0	3	7	15	15	7	3	0

Using a scale of 2 cm to represent 1 second on the horizontal axis and a scale of 2 cm to represent 2 cm/s on the vertical axis, draw the graph of v against t for $0 \leqslant t \leqslant 7$.

(a) By drawing a tangent, find the acceleration of the body at $t = 2$ and $t = 6$.
(b) Estimate the time at which the acceleration of the body is zero.

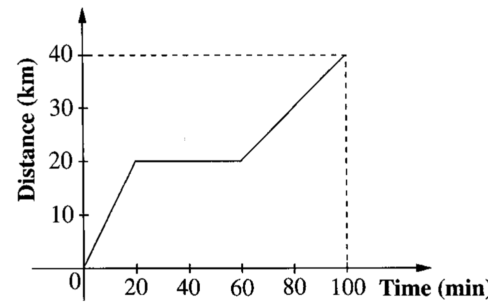
Summary

1. Average speed = $\dfrac{\text{Total distance travelled}}{\text{Total time taken}}$.
2. The gradient of a distance-time graph gives the instantaneous speed of a particle.
3. The gradient of a speed-time graph gives the instantaneous acceleration of a particle.
4. The area under a speed-time graph gives the distance travelled.

Review Questions 9

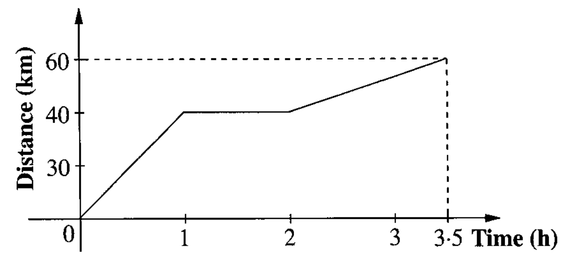
1. The diagram shows the distance-time graph of a motorist after leaving a starting point. Using the graph, find

(a) its speed during the first 20 minutes;
(b) its speed during the last 40 minutes;
(c) the average speed for the whole journey.



2. The diagram shows the travel graph of a train. Find

(a) the time interval during which the train is stationary;
(b) the greatest speed at which it travelled;
(c) the average speed for the whole journey.



3. A wheel of radius 28 cm is turning at 40 revolutions per minute. Calculate the speed of a point on the rim of the wheel, giving your answer in metres per minute. $\left(\text{Take } \pi = 3\frac{1}{7}.\right)$

4. The F-16 Falcon aircraft is a major defence force in the Singapore Airforce. The advanced aircraft is capable of attaining a speed equivalent to twice the speed of sound (i.e. 2 Mac). If the speed of sound is 330 m/s, express the speed of the F-16 Falcon in km/h.

5. A car travelled from X to Y in 20 minutes at a constant speed of 45 km/h. Then, after stopping for 30 minutes, the car returned from Y to X at a constant speed of 60 km/h. Draw the distance-time graph of the car. Find, from your graph,

(a) the distance between X and Y; **(b)** the time taken for the return journey.

6. A bus leaves a town P for another town Q at time 13 00 and travels at 60 km/h. A car travelling at 80 km/h leaves P by the same route an hour later. Using a horizontal scale of 2 cm for 1 hour and a vertical scale of 2 cm for 100 km, draw the distance-time graphs and find, graphically, when the car overtakes the bus. If the bus arrives at Q half an hour later than the car, find, graphically or otherwise, the distance from P to Q.

7. A motorist, travelling at a constant speed, leaves A at 11 00, intending to arrive at B, 100 km away, at 13 00. Half an hour later, one of the tyres of his vehicle has a puncture and the motorist is delayed for 18 minutes. How fast must he then proceed in order to reach B on time? At what time will he meet a cyclist who leaves B at 11 45 for A, travelling at a constant speed of 20 km/h? Illustrate your answer by using a distance-time graph.

8. An object travels along a straight line AB such that at time t seconds, the speed v m/s in the direction AB is given by $v = 50 + 9t - 2t^2$.

Corresponding values of t and v are given in the table below.

t	0	1	2	3	4	5	6	7
v	50	57	60	59	a	b	32	15

Calculate the values of a and b.

Using 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 10 m/s on the vertical axis, draw the graph of $v = 50 + 9t - 2t^2$ for $0 \leqslant t \leqslant 7$.

Use your graph to find the

(a) value of t when $v = 20$;
(b) value of t when the acceleration is zero;
(c) acceleration when $t = 1$.

9. The variables x and y are connected by the equation $y = x - 2 + \frac{3}{x}$. Copy and complete the table of values for this equation.

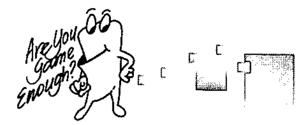
x	0.5	1	1.5	2	2.5	3	3.5	4
y	4.5	2		1.5	1.7		2.36	2.75

(a) Using a scale of 4 cm to represent 1 unit on both axes, draw the graph of $y = x - 2 + \frac{3}{x}$ for $0.5 \leqslant x \leqslant 4$.
(b) State the minimum value of y and the corresponding value of x.
(c) Find the range of values of x for which $y < 2.2$.
(d) By drawing a tangent, find the gradient of the graph at the point $x = 3$.
(e) By drawing a straight line, find the value of x for which $2x + \frac{3}{x} = 8$.

10. *(Answer this question on a sheet of graph paper.)*
The number of bacteria in a colony doubles every hour. The colony starts with 50 bacteria. The table below shows the number of bacteria in the colony after time t.

Time in hours (t)	0	1	2	3	4	5	6	7
Number of bacteria (n)	50	100	200	400	800	1 600	3 200	6 400

(a) Using a horizontal scale of 2 cm to represent 1 hour and a vertical scale of 2 cm to represent 1 000 bacteria, draw the graph of n against t.

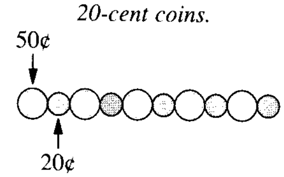


1. A man with a dog, a rabbit and a piece of lettuce arrives at a river crossing and wants to cross the river. There is a small boat for him to use. However the boat can only take him and either the dog, the rabbit or the lettuce.

He does not dare leave the dog alone with the rabbit or the rabbit alone with the lettuce for the dog would attack the rabbit and the rabbit would eat the lettuce.

Can you help the man to arrange a way to transport himself and all his belongings safely across the river?

2. The diagram below shows a row of ten coins, comprising five 50-cent coins and five 20-cent coins.



You are to get all the 50-cent coins to the right end and all the 20-cent coins to the left end. If you are allowed only to move the coins by interchanging the positions of any two neighbouring coins, how many moves will you have to make?

(b) Use your graph to find the value of n when $t = 6.5$.

(c) (i) By drawing a tangent, find the gradient of the graph when $t = 5.5$.

(ii) State briefly what this gradient represents.

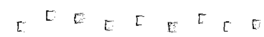
(d) The number of bacteria in another colony is given by the equation $n = 4\,000 - 500t$.

(i) On the same axes, draw a graph to represent the number of bacteria in this colony.

(ii) Find the value of t when the numbers in the colonies are equal.

(e) Given that the equation of the first graph is $n = k2^t$, find the value of k. (*C*)

How many moves do you have to make if you start with ten 50-cent coins and ten 20-cent coins arranged in the same way?



11. The following is a table of values for the graph of $y = 2x + \frac{12}{x} - 8$:

x	1	1.5	2	2.5	3	4	5	6
y	6	3	2	1.8	2	a	b	6

(a) Calculate the values of a and b.

(b) Using a scale of 2 cm to represent 1 unit on both axes, plot the graph of $y = 2x + \frac{12}{x} - 8$ for $1 \leqslant x \leqslant 6$.

(c) Use your graph to solve the equation $2x + \frac{12}{x} = 12$.

(d) Find the range of values of x for which $2x + \frac{12}{x} \leqslant 10 + \frac{1}{2}x$.

(e) By drawing a tangent, find the gradient of the graph at the point where $x = 4$.

12. A particle moves along a straight line AB so that, after t seconds, the speed v m/s in the direction AB is given by $v = 2t^2 - 8t + 8$.

Corresponding values of t and v are given in the table below:

t	0	1	2	3	4	5	6	7
v	8	2	h	2	8	18	32	k

Calculate the values of h and k.

Taking 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 10 m/s on the vertical axis, draw the graph of $v = 2t^2 - 8t + 8$ for $0 \leqslant t \leqslant 7$.

Use your graph to find approximately, the

(a) values of t when the speed is 10 m/s;

(b) time at which the acceleration is zero;

(c) gradient of the curve at $t = 1$ and $t = 5$. What does this gradient represent?

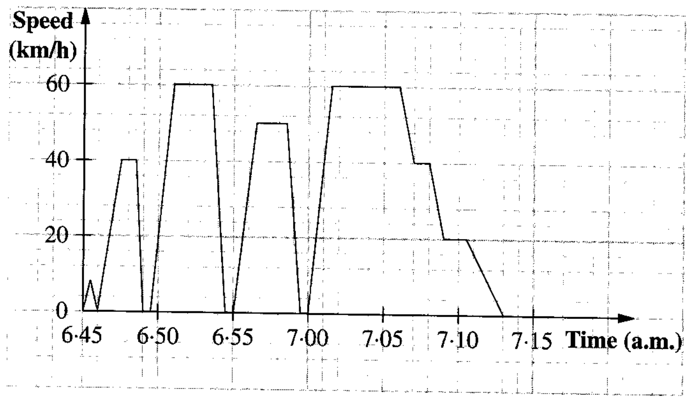
13. Car A travelling at a steady speed of 10 metres per second passes a stationary car, B. Two seconds later, car B starts to accelerate uniformly for 6 seconds (that is, its speed increases at a uniform rate during the interval of 6 seconds) and reaches a speed of 15 metres per second. It then continues with this speed until it overtakes car A.

On the same axes, using a horizontal scale of 1 cm for 1 second and a vertical scale of 2 cm for 5 m/s, draw the speed-time graphs for both cars A and B. Find the

(a) acceleration of car B;

(b) distance travelled by car B when it reaches the same speed as a car A.

14.



The graph shows the speed of Mrs Ong's car on her way to school. Use the graph to answer the following questions:

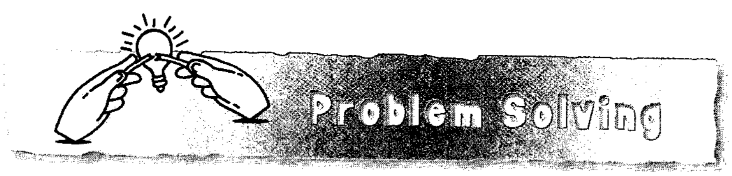
(a) How fast was Mrs Ong driving at 7.05 a.m.?

(b) Where do you think Mrs Ong was, from 6.45 a.m. to 6.46 a.m.? Why did Mrs Ong drive so slowly at that time?

(c) At what time did Mrs Ong reach her school?

(d) How many times did Mrs Ong stop on her way to school? Why do you think she made those stops?

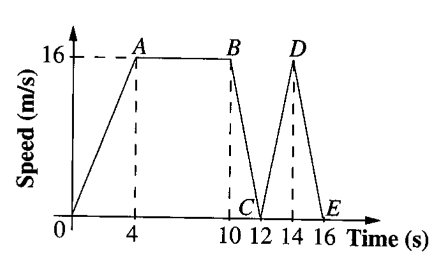
(e) Why do you think Mrs Ong slowed down from 7.06 a.m. onwards?



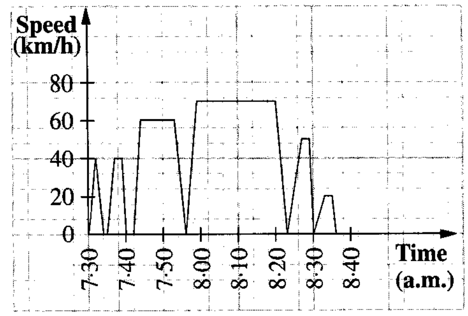
1. A cyclist starting from rest accelerates uniformly to his maximum speed of 12 m/s, which he then maintains for the next 3.6 km. He then applies his brakes and decelerates to rest at a rate numerically equal to three times his acceleration. Sketch a speed-time graph and find the total time the cyclist is in motion if the total distance moved is 3 744 m.

2. A particle is projected vertically upwards from the ground with a speed of 80 m/s. Sketch the speed-time graph and find the total time of flight. (Take the acceleration due to gravity to be 10 m/s^2.)

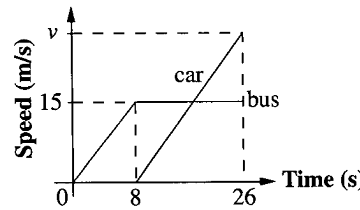
3. The diagram shows a speed-time graph of a particle moving in a straight line. Describe the motion of the particle as it passes through O, A, B, C, D and E.



4. The diagram shows the speed-time graph of a bus ferrying workers from their quarters to a factory. Based on the information given by the graph, make up a story about the journey and share it with your class.



5. The diagram shows the speed-time graph of a bus and a car. The bus accelerates from rest to a speed of 15 m/s in a time of 8 seconds. It then continues to travel at this constant speed.



   (a) Find the speed of the bus after 6 seconds.
   (b) Find the distance travelled by the bus in 26 seconds.
   (c) A car starts from the same place as the bus but 8 seconds later. It accelerates uniformly until it overtakes the bus, after the bus has travelled for 26 seconds. Calculate the speed of the car at this instant.

CHAPTER 10

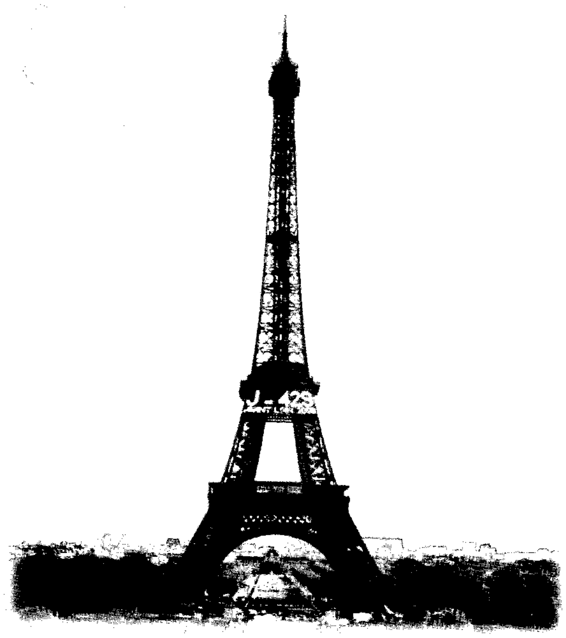
Trigonometry

In this chapter, you will learn how to

- △ determine the trigonometrical values of obtuse angles;
- △ solve any scalene triangle given: two sides and one angle or two angles and one side or three sides;
- △ find the area of complicated triangles;
- △ find the angle between a line and a plane;
- △ solve simple practical problems using bearing and trigonometry and those involving distance and height by simplification.

Preliminary Problem

Many measurements in this world are difficult or impossible to obtain directly. For instance, the height of the school flagpole or the height of Mount Everest. We can, however, obtain the height of, say, the Eiffel Tower with the help of Trigonometry.



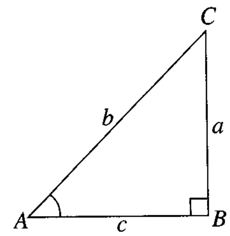
Trigonometrical Ratios

We learned in Book 2 that for a right-angled triangle ABC, the trigonometrical ratios of an angle are defined as:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{b},$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{c}{b},$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{c}.$$



We shall now define the trigonometrical ratios for obtuse angles.

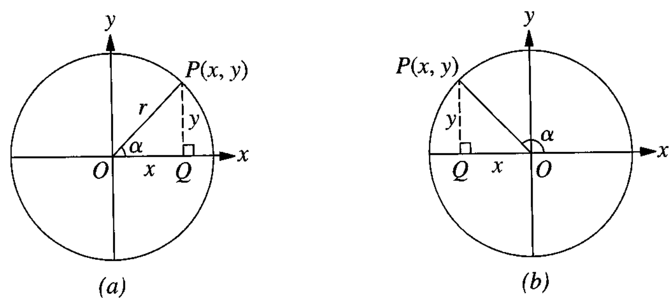


Fig. 10.1

Back in Time

The study of astronomy led to vast improvements in the area of trigonometry. In the 16th century, Copernicus' challenge of the theory of an earth-centred universe indirectly resulted in the extension of trigonometric tables which were used to obtain more accurate calculations. Today, trigonometry is widely used to find the heights of mountains, distances across lakes and countries, and so on.

Notes

Here, we only study trigonometrical ratios for acute and obtuse angles. We will study trigonometrical ratios for other angles later.

Figure 10.1 shows two circles, each with radius r units and point P making an angle α with the positive x-axis. In Fig. 10.1(a) the angle α is acute and in Fig. 10.1(b), the angle α is obtuse.

For any acute and obtuse angle α, we define the *sine*, *cosine* and *tangent* ratios as follows:

$$\sin \alpha = \frac{y}{r}, \cos \alpha = \frac{x}{r}, \tan \alpha = \frac{y}{x}$$

where x and y are the coordinates of point P (x, y) on the usual coordinate plane. Their values depend on their position on the plane.

Consider the trigonometrical ratios of α in Fig. 10.2. In this case, $OQ = 3$ units, $PQ = 4$ units and $OP^2 = 3^2 + 4^2$ by Pythagoras' theorem.

$\therefore$ $OP = 5$ units, i.e., a circle with centre O and radius 5 units will pass through point P.

Hence, $\sin \alpha = \frac{4}{5}$, $\cos \alpha = \frac{3}{5}$, $\tan \alpha = \frac{4}{3}$.

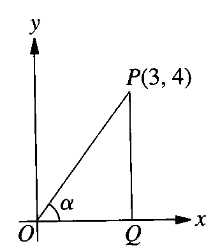


Fig. 10.2

Problems

Can x be equal to 0 in $\tan a = \frac{y}{x}$, where a is an acute or obtuse angle?

NB: This is in agreement with what we have learned in Book 2.

Consider the trigonometrical ratios of A in Fig. 10.3(a) and (b).

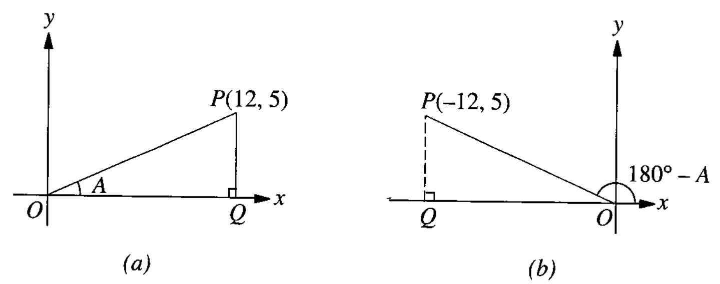


Fig. 10.3

Notes

Hence, we can define the ***trigonometrical ratio*** *of an acute angle in terms of the sides of a right-angled triangle, or in terms of the coordinates.*

The magnitude of $OQ = 12$ and that of $PQ = 5$. Hence, by Pythagoras' theorem,

$$OP^2 = 12^2 + 5^2.$$

$\therefore$ $OP = 13$, i.e., a circle with centre O and radius 13 units will pass through the point P for both Fig. 10.3(a) and (b).

For your information

Trignometry is the study of the relationships between the sides and angles of a triangle in terms of the trigonometric functions of angle (ie, ***sine, cosine*** *and* ***tangent****.)*

The word 'trigonometry' simply means 'triangle measure'.

Hence, in Fig. 10.3(a),

$$\left.\begin{aligned} \sin A &= \frac{5}{13} \\ \cos A &= \frac{12}{13} \\ \tan A &= \frac{5}{12} \end{aligned}\right\} \quad (1)$$

In Fig. 10.3(b),

$$\left.\begin{aligned} \sin (180° - A) &= \frac{PQ}{OP} = \frac{5}{13} \\ \cos (180° - A) &= \frac{OQ}{OP} = -\frac{12}{13} \\ \tan (180° - A) &= \frac{PQ}{OQ} = -\frac{5}{12} \end{aligned}\right\} \quad (2)$$

NB: Comparing (1) and (2), we have, for any acute angle A,

$$\sin A = \sin (180° - A)$$
$$\cos A = -\cos (180° - A)$$
$$\tan A = -\tan (180° - A)$$

The above is also true for any obtuse angle A as shown below:

$$\sin 110° = \sin (180° - 110°) = \sin 70°$$
$$\cos 120° = -\cos (180° - 120°) = -\cos 60°$$
$$\cos 132° = -\tan (180° - 132°) = -\tan 48°$$

Example 1

Express the following in terms of trigonometrical ratios of acute angles and, hence, find their values.

(a) sin 125° *(b) cos 136°* *(c) tan 147°*

Solution

(a) $\sin 125° = \sin(180° - 125°) = \sin 55°$
$= 0.819\,2$

(b) $\cos 136° = -\cos(180° - 136°) = -\cos 44°$
$= -0.719\,3$

(c) $\tan 147° = -\tan(180° - 147°) = -\tan 33°$
$= -0.649\,4$

Back in Time

The Greeks, in their analysis of the arcs of circles, were the first to establish the relationships or ratios between the sides and the angles of a right-angled triangle. The Chinese also recognised the ratios of the sides of a right-angled triangle and some survey problems involving such ratios were quoted in 'Zhou Bi Suan Jing'.

Example 2

Given that $0° \leqslant x \leqslant 180°$, find the possible values of x if

(a) $\sin x = 0.5$; *(b) $\cos x = -0.866$;* *(c) $\tan x = -1.191\,8$.*

Solution

(a) $\sin x = 0.5$
$= \sin 30°$ or $\sin 150°$
$\therefore\ x = \sin^{-1} 0.5$
$= 30°$ or $150°$

(b) $\cos x = -0.866$
$= -\cos 30°$
$\therefore\ x = \cos^{-1}(-0.866)$
$= 180° - 30° = 150°$

(c) $\tan x = -1.191\,8 = -\tan 50°$
$\therefore\ x = \tan^{-1}(-1.191\,8) = 180° - 50° = 130°$

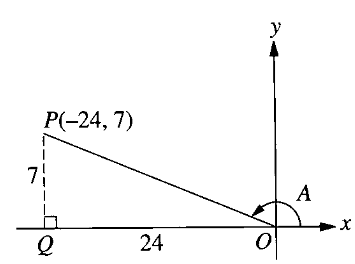
Example 3

If A is an obtuse angle and $\tan A = -\frac{7}{24}$, find, without using a table or calculator, the values of (a) sin A; (b) cos A.

Solution

The figure shows the point P making an obtuse angle A with the positive x-axis.

Given that $\tan A = -\frac{7}{24}$, we have $PQ = 7$ units and $OQ = -24$ units.



By Pythagoras' Theorem,

$$OP^2 = 7^2 + (-24)^2$$
$$OP = 25 \text{ units}$$

Hence, (a) $\sin A = \frac{7}{25}$ and (b) $\cos A = -\frac{24}{25}$.

Trigonometrical Ratios of Special Angles: 30°, 60°, 45°

Consider an equilateral $\triangle ABC$, with sides 2 units each. The line AD, perpendicular to BC, bisects $B\hat{A}C$.
Using the right-angled $\triangle BAD$, we have:

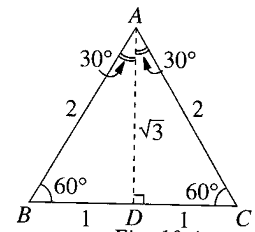


Fig. 10.4

$$\sin 30° = \frac{1}{2} \qquad \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2} \quad \text{AND} \quad \cos 60° = \frac{1}{2}$$

$$\tan 30° = \frac{1}{\sqrt{3}} \qquad \tan 60° = \sqrt{3}$$

Now, $A\hat{B}C = 60°$, $B\hat{A}D = 30°$, $BD = 1$ unit and $AD = \sqrt{3}$ units (by Pythagoras' theorem).
To find the trigonometrical ratio of 45°, consider a right-angled, isosceles $\triangle ABC$ with $A\hat{C}B = 90°$ and $AC = BC = 1$ unit.

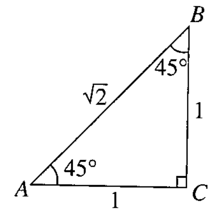


Fig. 10.5

Now, $A\hat{B}C = B\hat{A}C = 45°$ and
$AB = \sqrt{2}$ (by Pythagoras' theorem).

Thus, $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, $\tan 45° = 1$.

Exercise 10a

For Your Information

The trigonometrical ratios of 30°, 60° and 45° are often used in Physics and other branches of science. Their values are often expressed in surd form for convenience. One way of remembering these values is to learn how to draw the two triangles (Fig. 10.4 and Fig. 10.5) as the values can be easily derived from these triangles.

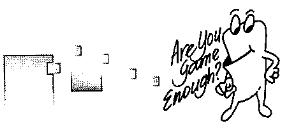
1. Express the following in terms of trigonometrical ratios of acute angles. Hence, find their values.
   (a) sin 110° **(b)** sin 176° **(c)** sin 98° **(d)** cos 99°
   (e) cos 107° **(f)** cos 175° **(g)** cos 132° **(h)** cos 156°
   (i) tan 93° **(j)** tan 118° **(k)** tan 175° **(l)** tan 143°

2. Find an acute angle whose *sine* is
   (a) 0.52; **(b)** 0.75; **(c)** 0.875; **(d)** 0.345 6.

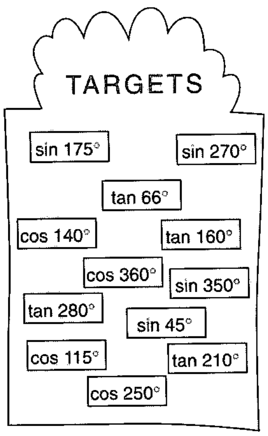
3. Find an acute whose *cosine* is
   (a) 0.67; **(b)** 0.756; **(c)** 0.5; **(d)** 0.985.

4. Find an acute angle whose *tangent* is
   (a) 0.123; **(b)** 1.23; **(c)** 2.75; **(d)** 0.125 6.

5. Given that $0° < x < 180°$, find the angle x for each of the following:
   (a) $\sin x = 0.752\ 6$ **(b)** $\sin x = 0.951\ 5$
   (c) $\sin x = 0.471\ 2$ **(d)** $\sin x = 0.187\ 2$
   (e) $\cos x = -0.237\ 5$ **(f)** $\cos x = -0.782\ 6$
   (g) $\cos x = -0.523\ 6$ **(h)** $\tan x = -0.111\ 1$
   (i) $\tan x = -0.781\ 4$ **(j)** $\tan x = -3.782$
   (k) $\tan x = -1.12$ **(l)** $\cos x = -0.456\ 2$



To win a shooting competition, Norman is required to hit targets with positive trigonometric ratios. Identify the correct targets and shade them in the diagram given below.



6. Taking sin 45° and cos 45° to be 0.7 each, find the value of
(a) 2 cos 45° + 3 sin 135°; (b) 3 cos 135° + 4 sin 135°;
(c) cos 135° – 2 sin 45°.

7. If $\sin x° = \sin 27°$, such that $0° < x < 180°$, write down the possible values of x.

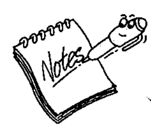
8. If A is an acute angle and $\sin A = \frac{24}{25}$, find, without using a trigonometrical table or a calculator, the values of **(a)** $\tan A$; **(b)** $\cos A$.

9. If A is an obtuse angle and $\cos A = -\frac{3}{5}$, find, without using a table or a calculator, the values of **(a)** $\sin A$; **(b)** $\tan A$.

10. If $\tan A = -\frac{8}{15}$, such that $0° < A < 180°$, find, without using a table or a calculator, the values of **(a)** $\sin A$; **(b)** $\cos A$.

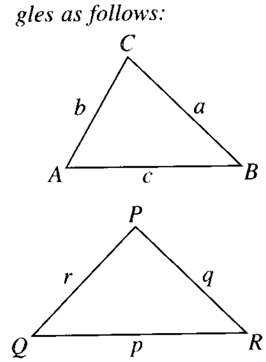
11. If $\sin A = \frac{7}{25}$, such that $90° < A < 180°$, find, without using a table or a calculator, the values of **(a)** $\cos A$; **(b)** $\tan A$.

12. Express the following in terms of trigonometrical ratios of acute angles. Hence, find their values without the use of a table or calculator.
(a) sin 120° **(b)** cos 120° **(c)** tan 120° **(d)** sin 135°
(e) cos 135° **(f)** tan 135° **(g)** sin 150° **(h)** cos 150°
(i) tan 150°



In this chapter and the subsequent chapters on trigonometry, we shall use small letters to denote the sides facing the angles, which are correspondingly denoted by capital letters.

Thus, we label two triangles as follows:



Area of a Triangle

In lower secondary, we learnt that the area of a triangle is given by the formula

$$Area = \frac{1}{2} \times base\ (b) \times height\ (h)$$

Consider Fig. 10.6, where $\angle C$ is an acute angle,

Area of $\triangle ABC = \frac{1}{2} \times b \times h$ ——— (1)

In $\triangle BCD$, $\sin C = \frac{h}{a}$

$\therefore\ h = a \sin C$ ——— (2)

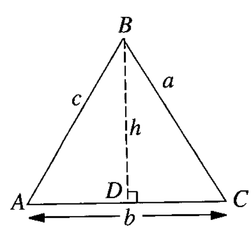


Fig. 10.6

Substituting (2) into (1), we have

$$Area = \frac{1}{2}ab \sin C$$

Consider Fig. 10.7, where $\angle C$ is an obtuse angle.

Area of $\triangle ABC = \frac{1}{2} \times b \times h$ ——— (1)

In $\triangle BCD$, $\sin C = \frac{h}{a}$

$\therefore \quad h = a \sin C$ ——— (2)

Substituting (2) into (1), we have

$$\text{Area} = \frac{1}{2}ab \sin C$$

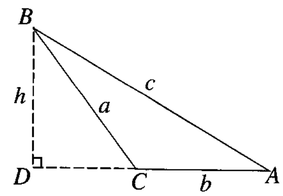


Fig. 10.7

By considering sin A and sin B in a similar way, we can prove that

$$Area = \frac{1}{2}bc \sin A \text{ and } Area = \frac{1}{2}ac \sin B \text{ respectively.}$$

Therefore we have the following:

$$\textbf{area of } \triangle ABC = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B.$$

Back in Time

*An interesting aspect about sound waves is that they are related to the **sine curve**. This is the essence behind the development in electronic musical instruments today and was discovered by a French mathematician known as Joseph Fourier.*

Example 4

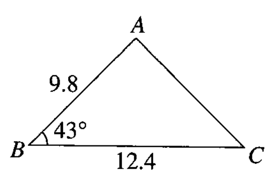
Find the area of $\triangle ABC$ if a = 12.4 cm, $\hat{B}$ = 43° and c = 9.8 cm, giving your answer correct to 2 decimal places.

Solution

Area of $\triangle ABC = \frac{1}{2}ac \sin B$

$= \frac{1}{2} \times 12.4 \times 9.8 \times \sin 43°$

$= 41.44 \text{ cm}^2$



Example 5

In $\triangle PQR$, $\hat{P}$ = 104° and PQ = 2PR. If the area of $\triangle PQR$ is 12.5 cm², find the length of PR, giving your answer correct to 3 significant figures.

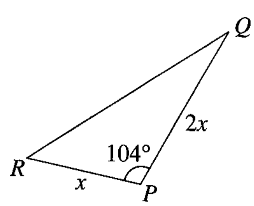
Solution

Let the length of PR be x cm,

$\therefore \quad PQ = 2x$ cm, i.e., $q = x$ and $r = 2x$

Area of $\triangle PQR = \frac{1}{2}qr \sin P$

i.e. $\quad 12.5 = \frac{1}{2}(x)(2x) \sin 104°$



$$\therefore \quad x^2 = \frac{12.5}{\sin 104}$$

$$x = \pm\sqrt{\frac{12.5}{\sin 104°}} = \pm 3.59 \text{ cm.}$$

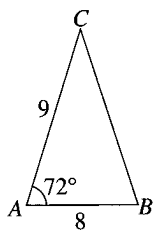
$\therefore \quad PR = 3.59$ cm.

NB: The negative length should be ignored.

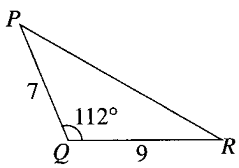
Exercise 10b

1. Find the area of each of the following figures. *(All lengths are in cm.)*

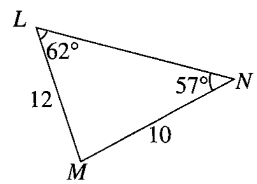
(a)



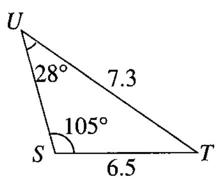
(b)



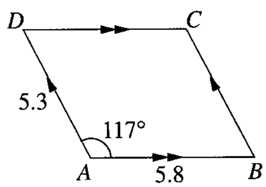
(c)



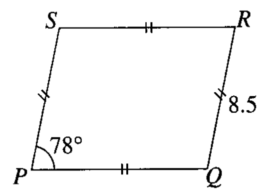
(d)



(e)



(f)



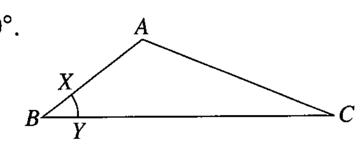
2. In $\triangle ABC$, $\hat{A} = 45°$, $b = 15$ cm and $c = 22$ cm. Find the area of the triangle.

3. In $\triangle PQR$, $\hat{P} = 72°$, $q = 152$ cm and $r = 125$ cm. What is the area of triangle?

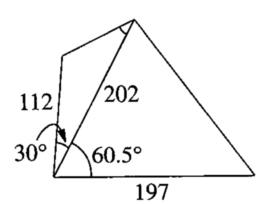
4. In $\triangle ABC$, the side $a = 43$ cm, $\hat{B} = 67°$ and $c = 32$ cm. Calculate the area of the triangle and the perpendicular height from A to BC.

5. Given that the area of a rhombus is 40 cm^2 and that each side is of length 15 cm, calculate the angles of the rhombus.

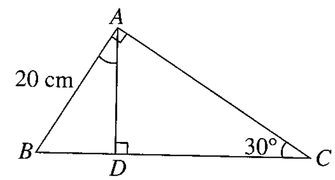
6. ABC is a triangle in which $AB = 40$ cm, $BC = 80$ cm and $A\hat{B}C = 30°$. XY is an arc of a circle, centre B and radius 10 cm. Calculate
(a) the area of $\triangle ABC$; **(b)** the area of the shaded portion.



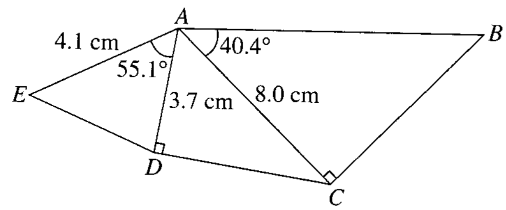
7. The figure shows the plan of an estate, the given lengths being in metres. Calculate the area of the estate.



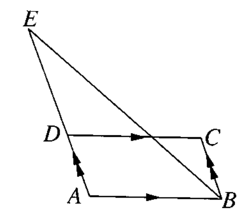
8. In the figure, $AB = 20$ cm, $B\hat{A}C = 90°$, $A\hat{C}B = 30°$ and AD is perpendicular to BC. Find
   (a) $B\hat{A}D$;
   (b) BD;
   (c) the area of $\triangle ABC$.



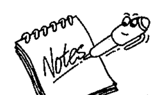
9. In the figure, $A\hat{D}C = A\hat{C}B = 90°$, $E\hat{A}D = 55.1$, $C\hat{A}B = 40.4°$, $AE = 4.1$ cm, $AD = 3.7$ cm and $AC = 8.0$ cm.
   Find (a) $A\hat{C}D$; (b) AB; (c) the area of $\triangle AED$.



★ 10. In the figure, $ABCD$ is a parallelogram and BE is a line segment. If $DE = 2AD$, and the area of $ABCD$ is 20 cm², find the area of $\triangle ABE$.



11. The diagonals of a parallelogram are 15.6 cm and 17.2 cm. They intersect at an angle of 120°. Find the area of the parallelogram.



We can use the ***Sine Rule*** *to solve any problem involving triangles when at least either of the following is known:*

(a) two angles and a side;
(b) two sides and an angle opposite a given side.

12. In a quadrilateral $ABCD$, $AB = 3.2$ cm, $BC = 5.1$ cm, the diagonal $BD = 7.5$ cm and $C\hat{B}D = 34.4°$. The area of $\triangle ABD$ is 11.62 cm² and $A\hat{B}D$ is obtuse. Calculate
    (a) the area of $\triangle BCD$;
    (b) the size of the obtuse $\angle ABD$.

The Sine Rule

For any triangle ABC, we learnt that $\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$.

Dividing each side by $\frac{1}{2}abc$, we have $\frac{\frac{1}{2}bc \sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac \sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}ab \sin C}{\frac{1}{2}abc}$

$$\therefore \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

We can also write this rule as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Example 6 (Given two angles and one side)

Solve $\triangle ABC$ in which $\hat{A} = 67.6°$, $\hat{B} = 45.5°$ and $c = 7.6$ cm.

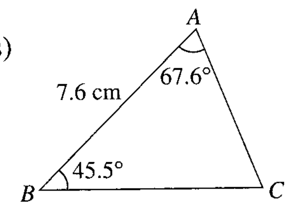
Solution

$$\hat{C} = 180° - (45.5° + 67.6°) = 66.9°$$

Now $\dfrac{7.6}{\sin 66.9°} = \dfrac{a}{\sin 67.6°}$

$a = \dfrac{7.6 \sin 67.6°}{\sin 66.9°}$

$= 7.64$ cm (to 2 dec. places)



and $\dfrac{7.6}{\sin 66.9°} = \dfrac{b}{\sin 45.5°}$

$b = \dfrac{7.6 \sin 45.5°}{\sin 66.9°}$

$= 5.89$ cm

$\therefore$ $\hat{C} = 66.9°$, $a = 7.64$ cm, $b = 5.89$ cm.

For Your Information

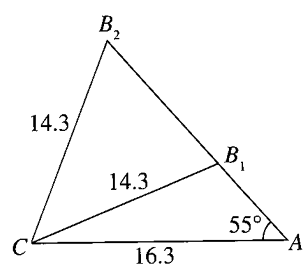
(a) In any triangle, the sum of the three angles is 180°.
(b) A triangle can be constructed if and only if the sum of the lengths of the two shorter sides is greater than the third side.
(c) In any triangle, the ***shortest*** *side is opposite the* ***smallest*** *angle and the* ***longest*** *side is opposite the* ***largest*** *angle.*
(d) No definite solution can be obtained if ***only*** *3 angles are given.*
(e) 'Solve $\triangle ABC$' means to find the unknown lengths and angles of the triangle.

Example 7 (Given two sides and one angle)

Solve $\triangle ABC$ in which $\hat{A} = 55°$, $b = 16.3$ cm and $a = 14.3$ cm.

Solution

Using the information above, we notice that it is possible to construct two different triangles, namely $\triangle AB_1C$ and $\triangle AB_2C$, both having measurements agreeing with the given data.



A set of given information that would give two possible sets of solutions is said to be **ambiguous**.

Thus, for ambiguous cases, **two** solution sets will be obtained.

From the above example, the given angle is *acute* and the side facing the given angle *is less* the other given side. The two sets of answers are obtained using the Sine Rule.

Now $\dfrac{14.3}{\sin 55°} = \dfrac{16.3}{\sin B}$

$\therefore$ $\sin B = \dfrac{16.3 \sin 55°}{14.3} = 0.933\ 7$

$\hat{B} = \sin^{-1} 0.933\ 7$

$= 69.0°$ or $111.0°$ (to 1 dec. place)

When $\hat{B} = 69°$, $\hat{C} = 180° - (69° + 55°) = 56°$.

For Your Information

We can calculate the remaining sides of a right-angled triangle and its angles when either:

(a) any two sides are given

OR

(b) one side and one acute angle are given.

Now, $\dfrac{16.3}{\sin 69°} = \dfrac{c}{\sin 56°}$

$\therefore \quad c = \dfrac{16.3 \sin 56°}{\sin 69°} = 14.47$ cm (to 2 dec. places)

When $\hat{B} = 111°$, $\hat{C} = 180° - (111° + 55°) = 14°$.

Now, $\dfrac{16.3}{\sin 69°} = \dfrac{c}{\sin 14°}$

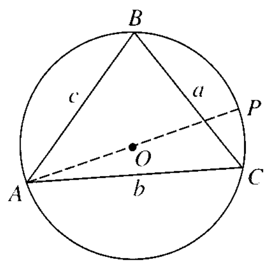
$\therefore \quad c = \dfrac{16.3 \sin 14°}{\sin 69°} = 4.22$ cm (to 2 dec. places)

$\therefore$ the two sets of solutions are: $\hat{B}_1 = 111°$, $\hat{C}_1 = 14°$, $c_1 = 4.22$ cm and $\hat{B}_2 = 69°$, $\hat{C}_2 = 56°$, $c_2 = 14.47$ cm.

CHECK THIS OUT

Did you know that there are other versions of proof for the Sine Rule?

Below is a circle of radius R, with the vertices of a triangle, A, B and C on its circumference.



By joining A to O and extending it to cut the circle at P, prove that

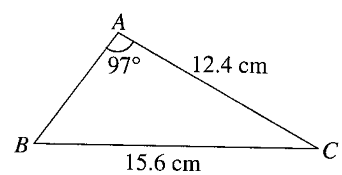
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$

Investigate, also, another proof of the Sine Rule by joining B to O, and then C to O.

Example 8 (Given two sides and an obtuse angle)

Solve $\triangle ABC$ in which $\hat{A} = 97°$, $b = 12.4$ cm and $a = 15.6$ cm.

Solution



$\dfrac{\sin 97°}{15.6} = \dfrac{\sin B}{12.4}$

$\sin B = \dfrac{12.4 \sin 97°}{15.6} = 0.788\ 9$

$\hat{B} = \sin^{-1} 0.788\ 9$

$= 52.1°$ (to 1 dec. place)

NB: B cannot be an obtuse angle, otherwise the sum of angles in the triangle will exceed 180°.

Now, $\hat{C} = 180° - (97° + 52.1°) = 30.9°$

and $\dfrac{\sin 97°}{15.6} = \dfrac{\sin 30.9°}{c}$

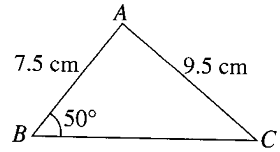
$c = \dfrac{15.6 \sin 30.9°}{\sin 97°} = 8.07$ cm (to 2 dec. places)

$\therefore \quad \hat{B} = 52.1°$, $\hat{C} = 30.9°$, $c = 8.07$ cm.

Example 9

Solve $\triangle ABC$ in which $\hat{B} = 50°$, $b = 9.5$ cm and $c = 7.5$ cm.

Solution



$\dfrac{\sin 50°}{9.5} = \dfrac{\sin C}{7.5}$

$\sin C = \dfrac{7.5 \sin 50°}{9.5} = 0.604\ 8$

$\hat{C} = \sin^{-1} 0.604\ 8$

$= 37.2°$ or $142.8°$ (to 1 dec. place)

But C cannot be 142.8° because $c < b$ and therefore, $\hat{C} = 37.2°$.

When $\hat{C} = 37.2°$, $\hat{A} = 180° - (50° + 37.2°) = 92.8°$

$$\frac{\sin 50°}{9.5} = \frac{\sin 92.8°}{a}$$

$$a = \frac{9.5 \sin 92.8°}{\sin 50°}$$

$$= 12.39 \text{ cm} \quad \text{(to 2 dec. places)}$$

$\therefore \quad a = 12.39$ cm, $\hat{A} = 92.8°$ and $\hat{C} = 37.2°$.

NB: From the four previous examples, an ambiguous case will occur if

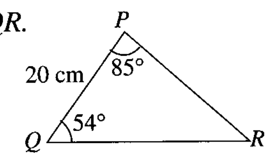
(a) two sides and a non-included angle are given,
(b) the given angle is acute and
(c) the side opposite the given angle is less than the other given side.

Conversely, no ambiguous case exists if

(a) the given angle is obtuse or
(b) the side opposite the given angle is greater than the other given side.

Example 10

In $\triangle PQR$, $\hat{P} = 85°$, $\hat{Q} = 54°$ and $r = 20$ cm. Calculate the area of $\triangle PQR$.



Solution

$$\hat{R} = 180° - (54° + 85°) = 41°$$

Now, $$\frac{\sin 41°}{20} = \frac{\sin 85°}{p}$$

$$p = \frac{20 \sin 85°}{\sin 41°} = 30.37 \text{ cm}$$

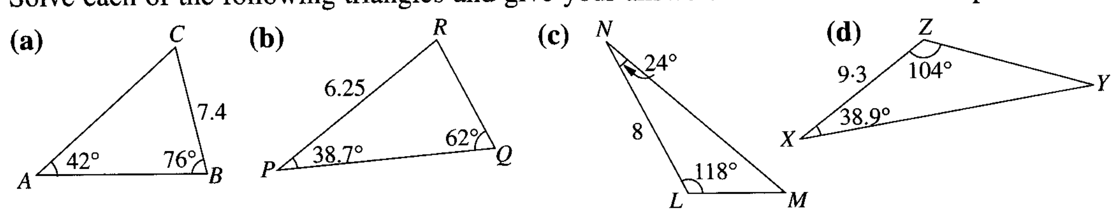
$$\text{Area of } \triangle PQR = \frac{1}{2} pr \sin Q$$

$$= \frac{1}{2} \times 30.37 \times 20 \times \sin 54°$$

$$= 245.7 \text{ cm}^2$$

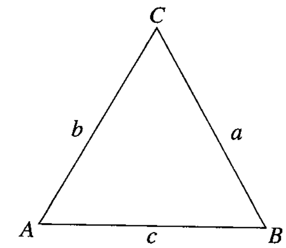
Exercise 10c

1. Solve each of the following triangles and give your answers correct to 1 decimal place:



2. Solve $\triangle ABC$ given that

(a) $\hat{A} = 72.3°$, $\hat{B} = 50.5°$ and $c = 5.6$ cm;
(b) $\hat{B} = 84.5°$, $\hat{C} = 46.8°$ and $a = 7.45$ cm;
(c) $\hat{A} = 53.4°$, $\hat{C} = 28.6°$ and $b = 5.93$ cm;
(d) $\hat{A} = 92°$, $b = 6.93$ cm and $a = 15.31$ cm;
(e) $\hat{B} = 98°$, $a = 14.5$ cm and $b = 17.4$ cm;
(f) $\hat{C} = 35°$, $b = 8.7$ cm and $c = 9.5$ cm;
(g) $\hat{A} = 125°$, $b = 3$ cm and $a = 12$ cm.



3. In $\triangle PQR$, $p = 7$ cm, $Q = 47°$ and $\hat{R} = 97°$. Find r.

4. Solve $\triangle ABC$ in which $BC = 4.9$ cm, $AB = 9.1$ cm and $\hat{B} = 90°$.

5. In $\triangle PQR$, $\hat{Q} = 60°$, $\hat{P} = 75°$ and $q = 14$ cm. What is the length of the longest side?

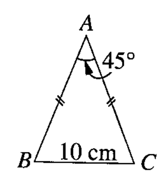
6. Examine the following data for the various triangles and state whether the *ambiguous* case is involved or not. State clearly your reasons.

(a) $\triangle ABC$, $\hat{A} = 92°$, $b = 7.5$ cm, $a = 8.5$ cm
(b) $\triangle PQR$, $\hat{P} = 47°$, $p = 75$ cm, $q = 80$ cm
(c) $\triangle DEF$, $d = 37$ cm, $e = 37$ cm, $\hat{D} = 58°$
(d) $\triangle MNO$, $m = 19$ cm, $n = 15$ cm, $\hat{N} = 39°$
(e) $\triangle ABC$, $b = 80$ cm, $c = 67$ cm, $\hat{C} = 43°$
(f) $\triangle PQR$, $p = 19$ cm, $q = 25$ cm, $\hat{Q} = 52°$
(g) $\triangle DEF$, $d = 45$ cm, $e = 75$ cm, $\hat{E} = 92°$

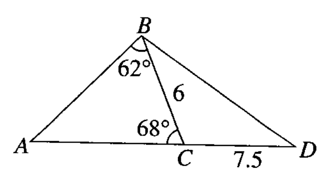
7. Solve $\triangle PQR$ in which $p = 8$ cm, $q = 12$ cm and $\hat{P} = 30°$.

8. Solve $\triangle ABC$ in which $\hat{A} = 58°$, $b = 15.4$ cm and $a = 14.0$ cm.

9. Find the area of $\triangle ABC$ if $BC = 10$ cm. $\hat{A} = 45°$ and $AB = AC$.



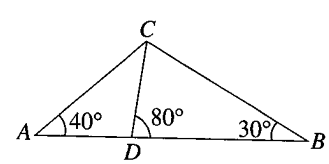
10. In the diagram, ACD is a straight line, $A\hat{B}C = 62°$, $A\hat{C}B = 68°$, $BC = 6$ cm and $CD = 7.5$ cm. Calculate **(a)** AC, **(b)** the area of $\triangle ABD$.



11. The angle A of $\triangle ABC$ is 35°. Given that $\sin B = \frac{4}{3} \sin A$, calculate two possible values of angle B. Given in addition that $BC = 5$ cm, calculate the length of AC.

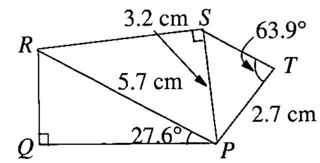
12. In $\triangle ABC$, $C\hat{A}B = 40°$ and $C\hat{B}A = 30°$. The point D on AB is such that $B\hat{D}C = 80°$ and $CD = 5$ cm. Calculate

(a) AD;
(b) DB;
(c) the distance of C from AB.

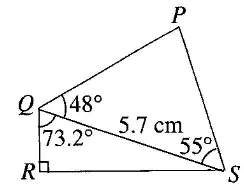


13. The base of a triangle is 3.57 cm and the angles adjacent to it are 51.8° and 87.7°. Solve the triangle.

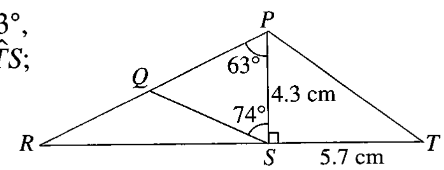
14. In the figure, $P\hat{Q}R = P\hat{S}R = 90°$, $Q\hat{P}R = 27.6°$, $P\hat{T}S = 63.9°$, $PR = 5.7$ cm, $PS = 3.2$ cm and $PT = 2.7$ cm. Calculate
(a) QR; (b) $S\hat{P}R$; (c) $P\hat{S}T$.



15. In the figure, $QS = 5.7$ cm, $Q\hat{R}S = 90°$, $S\hat{Q}R = 73.2°$, $P\hat{Q}S = 48°$ and $P\hat{S}Q = 55°$. Calculate (a) QR; (b) PQ.



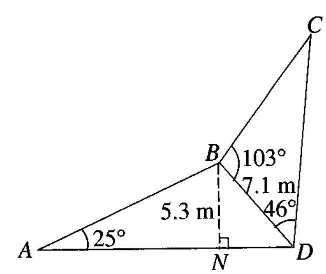
16. In the figure, RST is a straight line, $P\hat{S}T = 90°$, $S\hat{P}R = 63°$, $P\hat{S}Q = 74°$, $PS = 4.3$ cm and $ST = 5.7$ cm. Calculate (a) $P\hat{T}S$; (b) PR; (c) QS.



17. On a map whose scale is 8 cm to 1 km, an estate is shown as a quadrilateral $ABCD$. The length of the diagonal AC is 7 cm, $B\hat{A}C = 55°$, $B\hat{C}A = 77°$, $D\hat{A}C = 90°$ and $D\hat{C}A = 40°$. Calculate
(a) the length, in cm, of the side AB on the map;
(b) the length, in km, which is represented by AD;
(c) the area, in km², which is represented by $\triangle ADC$.

18. $PQRS$ is a quadrilateral in which the sides PQ and SR are parallel. Given that $PS = 5$ cm, $QS = 7$ cm, $P\hat{S}Q = 90°$ and $Q\hat{R}S = 40°$, calculate (a) PQ; (b) $P\hat{Q}S$; (c) QR.

19. The figure shows a framework in which AD is horizontal with $BD = 7.1$ m, $B\hat{A}D = 25°$, $B\hat{D}C = 46°$, $D\hat{B}C = 103°$ and the height of B above AD is 5.3 m.
Calculate (a) AB; (b) $D\hat{B}N$; (c) CD.



20. A helicopter is hovering at a point lying in the same vertical plane as two other points P and Q on the horizontal ground. Its distances from P and Q are 850 m and 1 200 m respectively. Given that the angle of elevation from Q is 43°, find PQ correct to 3 significant figures.

The Cosine Rule

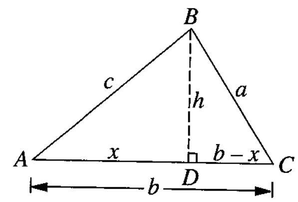
For any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

where a, b and c are the lengths of the sides facing the angles, A, B and C respectively. The proof is given below:

We first prove that $a^2 = b^2 + c^2 - 2bc \cos A$. There are three possible measures of A; an acute angle, an obtuse angle, and a right angle.

Case 1: $\hat{A}$ is acute



In $\triangle BCD$, $a^2 = h^2 + (b - x)^2$

$a^2 = h^2 + b^2 + x^2 - 2bx$ ——— (1)

In $\triangle BAD$, $\cos A = \frac{x}{c}$

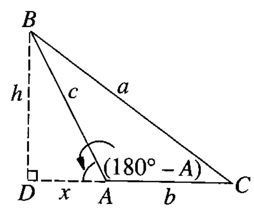
$\therefore \quad x = c \cos A$ ——— (2)

and $c^2 = x^2 + h^2$ ——— (3)

Substituting (2) and (3) into (1):

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Case 2: $\hat{A}$ is obtuse



In $\triangle BCD$, $a^2 = h^2 + (b + x)^2$

$a^2 = h^2 + b^2 + x^2 + 2bx$ ——— (1)

In $\triangle BAD$, $\cos (180° - A) = \frac{x}{c}$

$\therefore \quad x = c \cos (180° - A) = -c \cos A$ ——— (2)

and $c^2 = h^2 + x^2$ ——— (3)

Subsituting (2) and (3) into (1):

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Case 3: $\hat{A}$ is a right angle

Can you prove it by yourself?

In all the three cases, we obtained $a^2 = b^2 + c^2 - 2bc \cos A$.

By considering angles B and C in a similar manner, we can prove that

$$\mathbf{b^2 = a^2 + c^2 - 2ac \cos B \quad \text{and} \quad c^2 = a^2 + b^2 - 2ab \cos C}$$

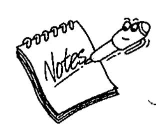
By rearranging the formula, we can express the *cosine of the angle* in terms of the lengths of the sides. Thus,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The Cosine Rule is sometimes also called the Cosine Formula.



*The **Cosine Rule** is used in the solution of triangles when two sides and an included angle are given*

OR

when three sides are given.

Example 11 (Given two sides and an included angle)

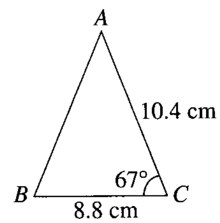
Solve $\triangle ABC$ in which a = 8.8 cm, b = 10.4 cm and $\hat{C} = 67°$.

Solution

By the Cosine Rule,

$$c^2 = 8.8^2 + 10.4^2 - 2 \times 8.8 \times 10.4 \times \cos 67°$$
$$= 114.08$$
$$\therefore \quad c = 10.68 \text{ cm}$$

Using the Sine Rule, we have



$$\frac{\sin 67°}{10.68} = \frac{\sin B}{10.4}$$

$$\sin B = \frac{10.4 \sin 67°}{10.68}$$

$$\therefore \quad \hat{B} = 63.7° \quad \text{(to 1 dec. place)}$$
$$\hat{A} = 180° - (63.7° + 67°)$$
$$= 49.3°$$

$\therefore \quad \hat{A} = 49.3°$, $\hat{B} = 63.7°$ and c = 10.68 cm.

For Your Information

*Although the Sine and the Cosine Rules can be used to solve triangles of all shapes and sizes, solutions to the right-angled and isosceles triangles are obtained with much greater ease using the Pythagoras' theorem and simple trigonometrical ratios. All isosceles triangles can be easily solved by dropping a **perpendicular** from the vertex to the base.*

Example 12 (Given three sides)

In $\triangle ABC$, a = 8 cm, b = 9 cm and c = 12 cm. Find the size of the smallest angle.

Solution

The smallest angle is the one facing the shortest side, i.e., A.

$$\cos A = \frac{9^2 + 12^2 - 8^2}{2 \times 9 \times 12} = 0.745\,4$$
$$\therefore \quad \hat{A} = 41.8°$$

Exercise 10d

1. In $\triangle PQR$, p = 5 cm, q = 7 cm and $\hat{R} = 60°$. Find r.
2. In $\triangle ABC$, a = 6 cm, b = 8 cm and $\hat{C} = 120°$. Find c.
3. In $\triangle PQR$, p = 9 cm, r = 7 cm and $\hat{Q} = 30°$. Find q.
4. In $\triangle PQR$, p = 72 cm, q = 57 cm and $\hat{R} = 96.7°$. Find r.
5. In $\triangle ABC$, a = 4.2 cm, b = 5.8 cm and $\hat{C} = 141.4°$. Find c.
6. In $\triangle ABC$, a = 3.8 cm, b = 5.3 cm and c = 6.7 cm. What is the size of the largest angle?

7. Given $\triangle DEF$, where $d = 7$ cm, $e = 5$ cm and $f = 3$ cm, find $\hat{D}$.

8. Solve $\triangle PQR$ in which $PQ = 7.8$ cm, $PR = 4.9$ cm and $QR = 9.1$ cm.

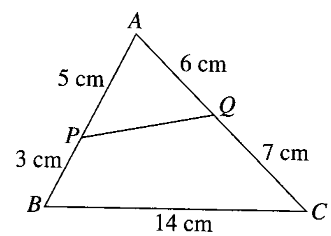
9. In $\triangle ABC$, $BC = 4$ cm, M is the mid-point of BC, $AM = 4$ cm and $A\hat{M}B = 120°$. Calculate
(a) AC; (b) AB; (c) angle ACB.

10. Solve $\triangle ABC$ in which $a = 7.8$ cm, $b = 10$ cm and $\hat{C} = 72°$.

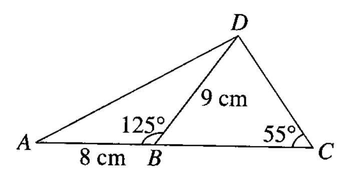
11. In $\triangle PQR$, $p = 7$ cm, $q = 8$ cm and $r = 9$ cm. Find the size of the smallest angle in the triangle.

12. XYZ is a triangle in which $XY = 9$ cm, $YZ = 13$ cm and $ZX = 8$ cm. Calculate $Y\hat{X}Z$.

13. In the figure, $AP = 5$ cm, $AQ = 6$ cm, $AB = 8$ cm, $AC = 13$ cm and $BC = 14$ cm. Find the value of cos A and hence use it to find the length of PQ.

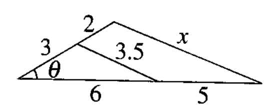


14. In the figure, ABC is a straight line and $AB = 8$ cm, $BD = 9$ cm, $A\hat{B}D = 125°$ and $B\hat{C}D = 55°$. Calculate
(a) CD; (b) AD

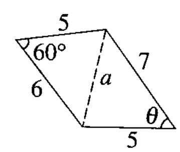


15. In a trapezium $ABCD$, AB and DC are the parallel sides and $AB = 4.5$ cm, $BC = 5$ cm, $CD = 7.5$ cm and $AD = 6$ cm. Draw BX // AD to cut CD at X. Find $B\hat{C}X$ and BD.

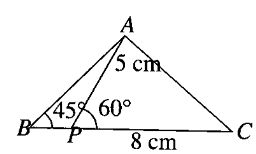
16. Calculate x in the figure. (*Hint:* Find cos θ.)



17. In the figure, find a and θ. (All lengths are in centimetres.)

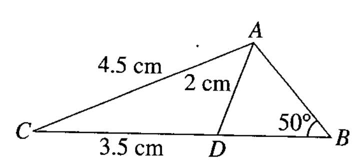


18. In the figure, $AP = 5$ cm, $PC = 8$ cm, $A\hat{P}C = 60°$ and $A\hat{B}C = 45°$. Find
(a) AB; (b) AC.

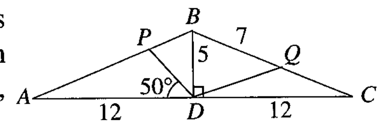


19. In $\triangle ABC$, $AB = 8$ cm, $BC = 5$ cm, and $CA = 6$ cm. BC is produced to R so that $CR = 3$ cm. Using the cosine formula, find cos $A\hat{C}B$. Hence calculate the length of AR.

20. In the diagram, CDB is a straight line. Given that $AD = 2$ cm, $AC = 4.5$ cm, $CD = 3.5$ cm and $A\hat{B}D = 50°$, calculate
(a) $A\hat{D}B$;
(b) the shortest distance from A to CDB;
(c) BD.



21. The diagram shows the supports of the roof of a building in which $BD = 5$ m, $AD = CD = 12$ m, $BQ = 7$ m and $P\hat{D}A = 50°$.



Calculate

(a) $B\hat{A}D$; **(b)** PD; **(c)** DQ.

When reading compass bearings, all directions are reckoned from either the north or the south. For instance, 070° is written as N70°E and 210° as S30°W. When reading true bearings, directions are given in terms of the angles measured clockwise from the north.

Bearings

Fig. 10.8 shows the position of four points A, B, C and D relative to an origin O. N, E, S, W represent north, east, south and west respectively.

The bearing of A from O is given by the acute angle NOA. It is an angle measured from the north, at O, in a clockwise direction and is always written as a three-figure number. Thus, the bearing of A from O is 070°.

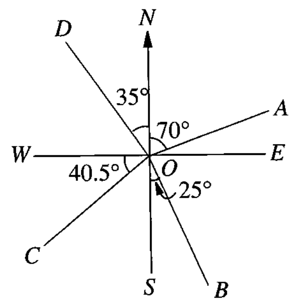


Fig. 10.8

The bearing of B from O is given by the obtuse angle NOB, which is equal to (180° – 25°). Thus, the bearing of B from O is 155°.

The bearing of C from O is given by the reflex angle NOC, which is equal to (270° – 40.5°). Thus, the bearing of C from O is 229.5°.

The bearing of N is taken as 000° or 360°. The bearing of E from O is 090°, that of S from O is 180° and the bearing of W from O is 270°.

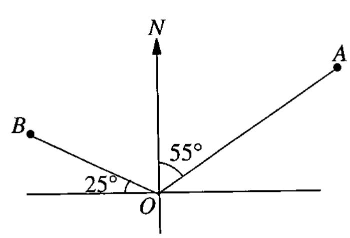
Can you give the bearing of D from O?

Decimal values of a compass bearing should be included. Thus, 007.5°, 046.2°, 125.63° are all possible compass bearings.

Example 13

State the bearing of

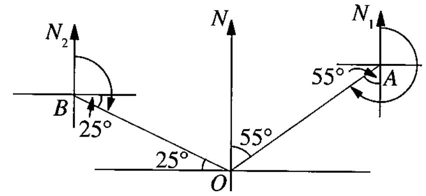
(a) A from O; *(b) B from O;*
(c) O from A; *(d) O from B.*



Solution

(a) The bearing of A from O is 055°.

(b) The bearing of B from O is given by the reflex angle NOB, which is $(270° + 25°)$. Thus, the bearing of B from O is 295°.



(c) The bearing of O from A is the reflex angle N_1AO. Thus, the bearing of O from A is $(180° + 55°)$. i.e., 235°.

(d) The bearing of O from B is the obtuse angle N_2BO. Thus, the bearing of O from B is $(90° + 25°)$, i.e., 115°.

Example 14

Three points A, B and C are on level ground such that B is due north of A, the bearing of C from A is 046° and the bearing of C from B is 125°. If the distance between A and B is 200 m, calculate the distance of C from A. Give your answer correct to 1 decimal place.

Solution

Since the bearing of C from B is 125°,

$A\hat{B}C = 55°$

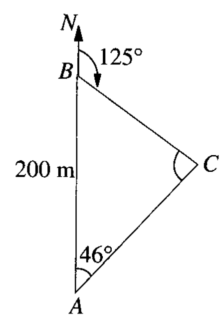
$\therefore \quad A\hat{C}B = 180° - 46° - 55° = 79°$

Using the Sine Rule, we have

$$\frac{200}{\sin 79°} = \frac{AC}{\sin 55°}$$

$$\therefore \quad AC = \frac{200 \sin 55°}{\sin 79°} = 166.9 \text{ m.}$$

i.e. the distance of C from A is 166.9 m.



Example 15

A boat sailed 20 km from a point P to an island Q, on a bearing of 150°. It then sailed another 30 km on a bearing of 50° to a lighthouse R. Calculate the distance of the lighthouse from P. Give your answer correct to 2 decimal places.

Solution

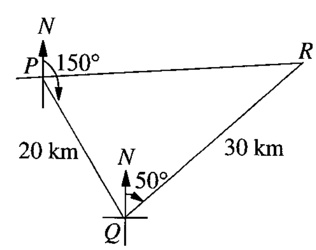
$P\hat{Q}R = 30° + 50° = 80°$

Using the Cosine Rule, we have

$$PR^2 = 20^2 + 30^2 - 2 \times 20 \times 30 \times \cos 80° = 1\,091.6$$

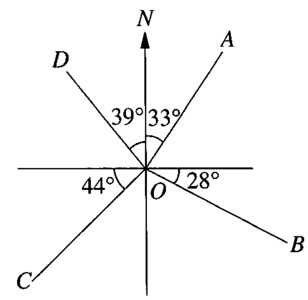
$\therefore \quad PR = 33.04$ km.

i.e. the distance of the lighthouse from P is 33.04 km.



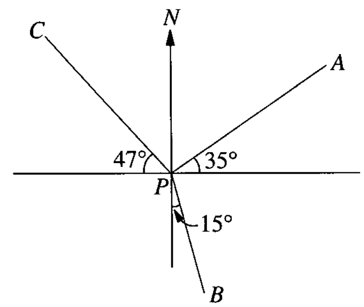
Exercise 10e

1. The figure shows the positions of O, A, B, C and D. State the bearing of



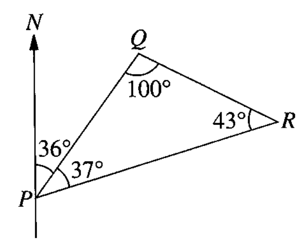
(a) A from O;
(b) B from O;
(c) C from O;
(d) D from O.

2. The figure shows the positions of P, A, B and C. State the bearing of



(a) A from P;
(b) B from P;
(c) C from P;
(d) P from A;
(e) P from B;
(f) P from C.

3. The figure shows the positions of P, Q and R. State the bearing of



(a) Q from P; **(b)** P from Q;
(c) R from P; **(d)** P from R;
(e) Q from R; **(f)** R from Q.

4. A, B, C and D are the four corners of a rectangular plot marked out on level ground. Given that the bearing of B from A is 040° and that the bearing of C from A is 090°, calculate the bearing of

(a) B from C; **(b)** A from C; **(c)** D from C.

5. A, B and C are three points on level ground. Given that the bearing of B from A is 122°, $C\hat{A}B = 32°$ and $A\hat{B}C = 86°$, calculate the possible bearing of C from B.

6. P, Q and R are three points on level ground. Given that the bearing of R from P is 135°, $P\hat{Q}R = 55°$ and $P\hat{R}Q = 48°$, calculate the bearing of

(a) P from R; **(b)** Q from R; **(c)** P from Q.

7. A point Q is 24 km away and at a bearing of 072° from P. From Q, a man walks, at a bearing of 320°, to a point R, located directly north of P. Calculate the distance of PR and QR.

8. A man swims 50 metres in the direction 045°, then 60 metres in the direction 145°. How far is he from the starting point?

9. The point B is 280 m due north of the point A. A man walks from A in the direction 050°. Calculate how far he has to walk before he is

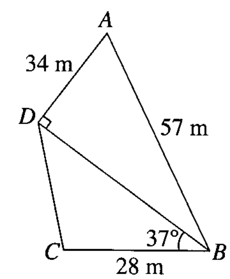
(a) equidistant from A and B; **(b)** as close as possible to B; **(c)** due east of B.

10. A point P is 12 kilometres due north of another point Q. The bearing of a lighthouse, R, from P is 135° and, from Q, it is 120°. Calculate the distance of PR.

11. Two ships P and Q leave a point at the same time. P sails at 10 km/h on a bearing of 030° and Q sails at 12 km/h on a bearing of 300°. Calculate their distance apart and the bearing of P from Q after 2 hours.

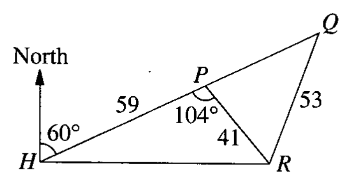
12. A man sails 30 km from a port P to a lighthouse, Q, on a bearing of 128° and then another 25 km to R on a bearing of 295°. Calculate the distance of PR.

13. A, B, C and D are four points on a field. A is due north of D, B is due east of D and $D\hat{B}C = 37°$. Given that $AD = 34$ m, $BA = 57$ m and $BC = 28$ m, calculate



(a) $B\hat{A}D$;
(b) the bearing of B from A;
(c) the area of $\triangle BCD$;
(d) CD.

14. The diagram represents a map showing a harbour H and three oil rigs, P, Q and R, where R is due east of H. HPQ is a straight line which lies on a bearing of 060° and the angle $HPR = 104°$.



It is given that $HP = 59$ km, $PR = 41$ km and $RQ = 53$ km.

(a) A supply ship leaves P at 10 45. It sails directly to R, where it stays for 50 minutes, then goes on to Q. When moving, it may be assumed that the ship travels at a constant speed of 12 km/h. At what time does it arrive at Q?
(b) Calculate the distance HR.
(c) Calculate the bearing of R from Q. (C)

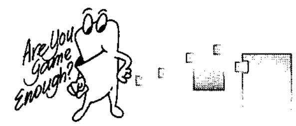
Three-Dimensional Problems

Lines and Planes in Space

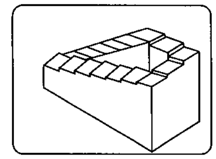
A **plane** is a flat surface like the floor or the surface of a blackboard. The walls of a room are other examples of planes. A plane is said to be two-dimensional since it has two dimensions, i.e. *length* and *breadth*. A point has zero dimensions, while a line has only one dimension, i.e. *length*. A solid or space has three dimensions, *length*, *breadth* and *height* or *depth*. It is said to be three-dimensional.

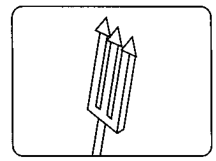
Normal to a Plane

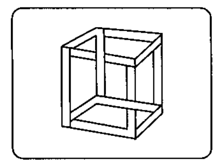
A line intersects a plane at a point. If this line is perpendicular to every line in the plane passing through that point, we call it a **normal**. The distance from a point to a plane is the length of the normal from the point to the plane.



Is it possible for you to construct each of the following 3-dimensional objects?


(a)


(b)


(c)

Draw a few lines which intersect at a point O on the plane and place a pencil perpendicular to the plane as shown in Fig. 10.9. Using the 90° vertex of a set square, you will find that the pencil is perpendicular to the lines drawn on the plane.

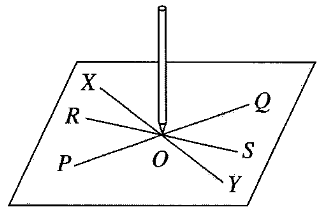


Fig. 10.9

NB: The pencil represents the normal to the plane at the point O on the plane.

Angles between Lines and Planes (Optional)

We have seen that a line may intersect a plane at a point. If the line is not *normal* to the plane, it will be inclined at an angle to the plane.

Fig. 10.10 shows the line PO intersecting a plane at the point O on the plane. How can the angle between this line and the plane be determined? Can we take it as the angle between the line and any line on the plane passing through the point O?

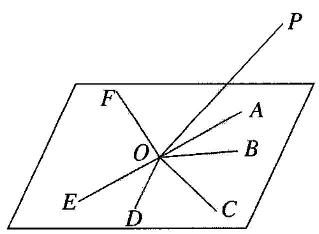


Fig. 10.10

On a piece of paper draw the lines OA, OB, OC, ... and so on as in Fig. 10.10. Hold a pencil with one end on the paper to represent OP. Then place another pencil on the plane in positions OA, OB, OC, ... and so on, in turn. Are the angles formed between the two pencils the same?

To avoid ambiguity, the angle between OP and the plane is always taken as the smallest angle between the pencils. This angle can be found by dropping a normal from P to meet the plane at Q and then joining OQ as shown in Fig. 10.11. $P\hat{O}Q$ will be the angle between the line OP and the plane.

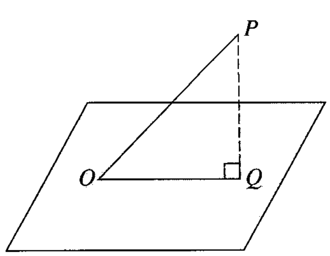


Fig. 10.11

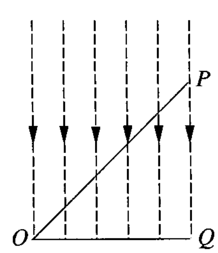
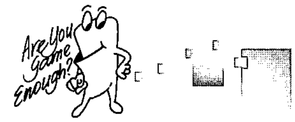


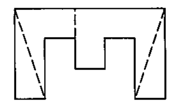
Fig. 10.12

Another way of looking at it is by using the idea of a 'shadow' or projection. Imagine that a lamp is vertically above the line OP as shown in Fig. 10.12. OQ will be the shadow of OP on a horizontal plane. This shadow OQ is known as the **projection** of OP onto the horizontal plane. Hence, the angle between OP and the plane is the angle between the line and its projection on the plane.

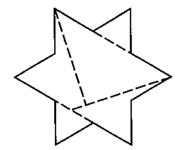
NB: The angle between a line and a plane is the angle between the line and its projection on the plane.



Two shapes are as shown below:



(a)

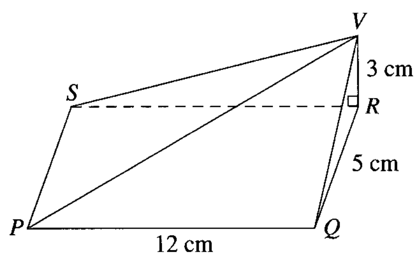


(b)

Cut each shape along the dotted lines and use the pieces to form two squares.

Example 16

The figure shows a pyramid with a rectangular base PQRS and vertex V vertically above R.



Given that $PQ = 12$ cm, $QR = 5$ cm and $VR = 3$ cm, calculate

(a) $V\hat{Q}R$,

(b) $V\hat{P}R$.

Solution

(a) In $\triangle VQR$, $\tan V\hat{Q}R = \frac{3}{5} = 0.6$

$\therefore \quad V\hat{Q}R = 31.0°$

(b) In $\triangle PQR$, $P\hat{Q}R = 90°$
Using Pythagoras' theorem, we have

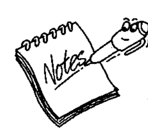
$$PR^2 = PQ^2 + QR^2 = 144 + 25 = 169$$

$$\therefore \quad PR = \sqrt{169} = 13 \text{ cm.}$$

In $\triangle VRP$, $V\hat{R}P = 90°$ (VR is the normal to the plane $PQRS$.)

$$\tan V\hat{P}R = \frac{3}{13} = 0.231$$

$$\therefore \quad V\hat{P}R = 13.0°.$$

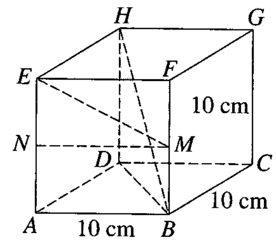


The basic technique in solving a three-dimensional problem is to reduce it to a problem in a plane.

Example 17

The diagram shows a cube of length 10 cm. M is the mid-point of BF. Calculate

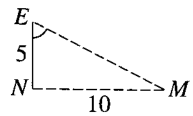
(a) $M\hat{E}N$,
(b) $E\hat{M}N$,
(c) $M\hat{E}F$,
(d) $H\hat{B}D$.



Solution

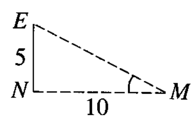
(a) $\tan M\hat{E}N = \frac{MN}{EN} = \frac{10}{5} = 2$

$\therefore \quad M\hat{E}N = 63.4°$.



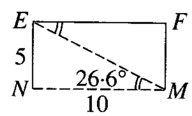
(b) $\tan E\hat{M}N = \frac{5}{10} = 0.5$

$\therefore \quad E\hat{M}N = 26.6°$.



(c) $M\hat{E}F = E\hat{M}N$

$\therefore \quad M\hat{E}F = 26.6°$.



$M\hat{E}F = E\hat{M}N$ due to the property of alternate angles and given that EF//NM.

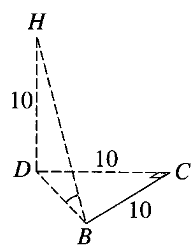
(d) In $\triangle BCD$, $B\hat{C}D = 90°$
Using Pythagoras' Theorem, we have

$$DB^2 = BC^2 + DC^2 = 10^2 + 10^2 = 100 + 100 = 200$$

$$DB = \sqrt{200} = 14.14 \text{ cm}$$

$$\tan H\hat{B}D = \frac{10}{14.14} = 0.7072$$

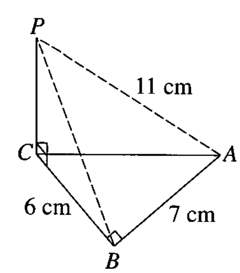
$\therefore \quad H\hat{B}D = 35.3°$.



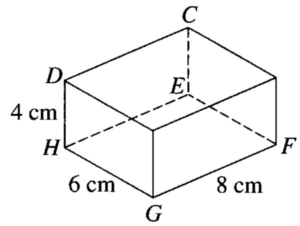
Exercise 10f

1. The figure shows $\triangle ABC$, right-angled at B and lying in a horizontal plane. P is a point vertically above C. Given that $AB = 7$ cm, $BC = 6$ cm and $AP = 11$ cm, calculate

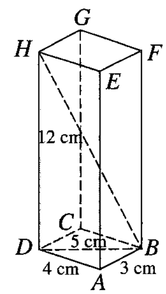
(a) PC,
(b) PAC,
(c) the angle of elevation of P from B.



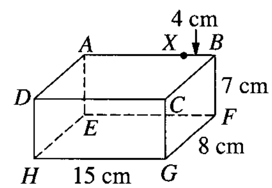
2. A rectangular box has a horizontal base $EFGH$. The corner D is vertically above H. Given that DH = 4 cm, HG = 6 cm and GF = 8 cm, calculate
   (a) DGH,
   (b) HF,
   (c) $D\hat{F}H$.



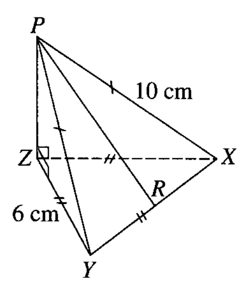
3. The diagram shows a rectangular box in which AB = 3 cm, AD = 4 cm, BD = 5 cm and DH = 12 cm. Calculate the length of the straight line BH and find
   (a) $B\hat{D}C$,
   (b) $H\hat{B}D$.



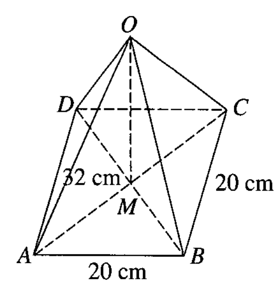
4. The diagram shows a rectangular box which has a horizontal base $EFGH$ where HG = 15 cm, GF = 8 cm and BF = 7 cm. X is a point on AB such that XB = 4 cm. Calculate $C\hat{E}G$ and $G\hat{X}F$.



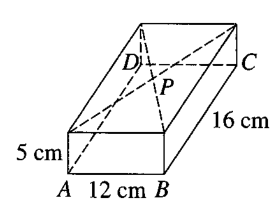
5. In the diagram, PZ is vertical and XYZ is an equilateral triangle of side 6 cm lying on a horizontal plane. PX = PY = 10 cm and R is the mid-point of XY. Calculate $P\hat{Y}Z$ and $P\hat{R}Z$.

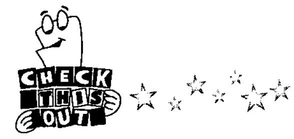


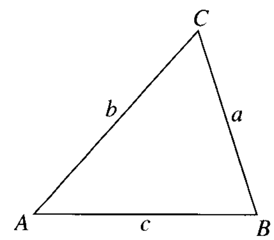
6. $OABCD$ is a pyramid. The square base $ABCD$ has sides of length 20 cm and lies in a horizontal plane. M is the point of intersection of the diagonals of the base, and O is vertically above M. The edge OA = 32 cm. Calculate
   (a) AM and hence the height of the pyramid,
   (b) $O\hat{A}M$.



7. P is the centre of the upper face of the rectangular block shown in the figure. Calculate
   (a) $P\hat{A}C$,
   (b) $P\hat{A}B$.







Heron of Alexandria (around AD 75) established a formula for finding the area of a triangle using its sides only.

The area of $\triangle ABC$ is given as Area = $\sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$.

Verify that the above formula is correct for each of the following cases:

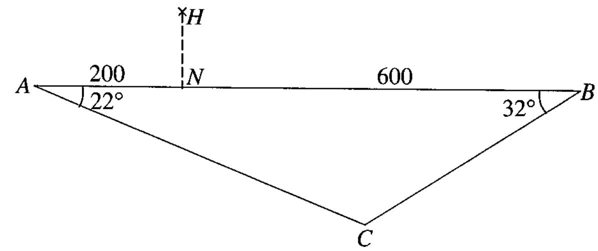
(a) a = 6 cm, b = 8 cm and c = 10 cm

(b) a = 8 cm, b = 9 cm and c = 10 cm

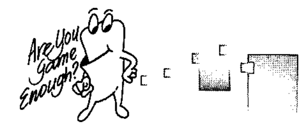
(c) a = 5 cm, b = 3 cm and c = 7 cm

Can you find out the proof for this formula?

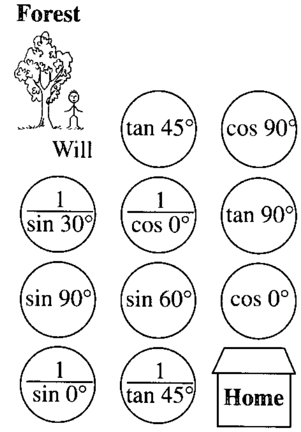
8. Three buoys, A, B and C, are positioned in a lake to provide a course for a yacht race. $AB = 800$ m, $A\hat{B}C = 32°$, $B\hat{A}C = 22°$ and N is the point on AB which is 200 m from A.



(a) Show that the distance AC is 524 m, correct to the nearest metre.
(b) Calculate the distance NC.
(c) A helicopter, H, is hovering at a point vertically above N.
  (i) The angle of elevation of the helicopter from A is 12°. Calculate the height of the helicopter.
  (ii) P is the point on AC which is nearest to the helicopter. Calculate the angle of elevation of the helicopter from P.
(*C*)



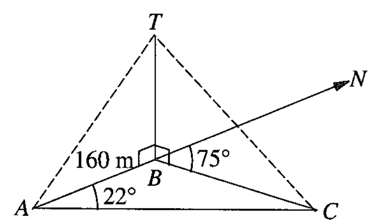
Will is out in a forest and wants to go home. However, he can only do so if he uses paths where the value of each trigonometric ratio is 1. Shade the paths he should follow in the diagram below:



Further Examples of Three-Dimensional Problems

Example 18

Three points A, B and C on level ground are such that B is due north of A, the bearing of C from A is 022° and the bearing of C from B is 075°. Given that AB is 160 m, calculate the distance of BC. A vertical mast BT stands at B such that $\tan T\hat{A}B = \frac{3}{16}$. *Find the angle of elevation of T from C.*



Solution

$$B\hat{C}A = 75° - 22° = 53°$$

$$\frac{160}{\sin 53°} = \frac{BC}{\sin 22°}$$

$$\therefore \quad BC = \frac{160 \sin 22°}{\sin 53°} = 75.1 \text{ m} \quad \text{(to 3 sig. figures)}$$

In $\triangle TAB$, $\quad \tan T\hat{A}B = \frac{3}{16} = \frac{BT}{160}$, $\therefore \quad BT = 30$ m.

In $\triangle TBC$, $\quad \tan T\hat{C}B = \frac{30}{75.05}$, $\quad \therefore \quad T\hat{C}B = 21.8°$.

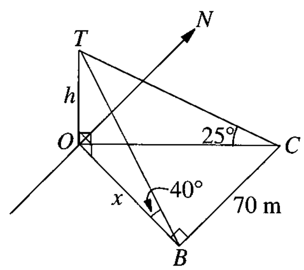
i.e. the angle of elevation of T from C is 21.8°.

$B\hat{C}A = 53°$ since ext. ∠ = sum of int. opp. ∠s.

BC is found by ***Sine Rule.***

Example 19

A man at B, due east of a vertical tower OT, observes the angle of elevation of the tower T to be 40°. He walks 70 m due north and finds that the angle of elevation of T from his new position C is 25°. Find the distance of OB and the height of the tower.



Check this out

Find out what a clinometer is and its purpose.

Solution

A diagram showing the relative positions of the tower, B and C is given. Let the height of the tower be h m and the distance of OB be x m.

In $\triangle OTB$, $\tan 40° = \frac{OT}{OB} = \frac{h}{x}$

$\therefore\ h = x \tan 40°$ ——— (1)

In $\triangle OTC$, $\tan 25° = \frac{h}{OC}$

$\therefore\ h = OC \tan 25°$ ——— (2)

In $\triangle OBC$, $OC^2 = OB^2 + BC^2$

$\therefore\ OC = \sqrt{x^2 + 70^2}$

OC is found by using Pythagoras' Theorem.

From (1) and (2): $x \tan 40° = OC \tan 25°$

$$\therefore\ OC = \frac{x \tan 40°}{\tan 25°} = \sqrt{x^2 + 70^2}$$

$$\left(\frac{x \tan 40°}{\tan 25°}\right)^2 = x^2 + 70^2$$

$$3.238x^2 = x^2 + 4\,900$$

$$x^2 = \frac{4\,900}{2.238} = 2\,189$$

$\therefore\ x = 46.79$ m
$= 46.8$ m (to 3 sig. figures)

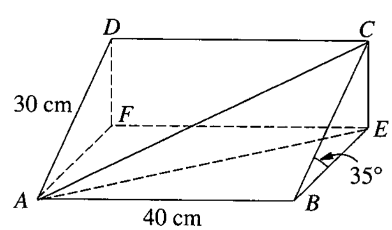
and $h = x \tan 40° = 46.79\ (0.839\ 1)$
$= 39.3$ m (to 3 sig. figures)

$\therefore\ OB = 46.8$ m and the height of the tower is 39.3 m.

Exercise 10g

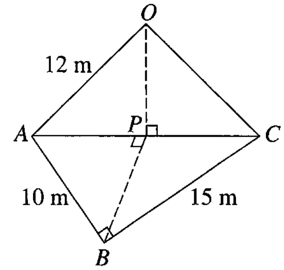
1. $ABCD$ represents the rectangular sloping surface of a desk. $ABEF$ is a horizontal rectangle and CE and DF are vertical lines. $AB = DC = FE = 40$ cm, $BC = AD = 30$ cm, and $C\hat{B}E = D\hat{A}F = 35°$. Calculate

   (a) AC, **(b)** CE, **(c)** $F\hat{A}E$. (C)

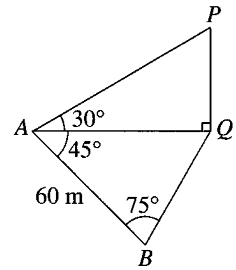


2. The figure shows three points A, B and C on horizontal ground where $A\hat{B}C$ is a right angle. AOC represents a vertical triangular wall with P as the foot of the perpendicular from O to AC. Given that $A\hat{P}B = 90°$, $AB = 10$ m, $BC = 15$ m and $OA = 12$ m, calculate

(a) $B\hat{A}C$, (b) AP,
(c) OP, (d) the angle of elevation of O from B.



3. In the given figure, the angle of elevation of the top of a vertical tower PQ from a point A is 30°. If Q, the foot of the tower, is on the same horizontal plane as A and B, and $AB = 60$ m, $B\hat{A}Q = 45°$ and $A\hat{B}Q = 75°$, find the height of the tower.



4. P, Q and R are three points on level ground with Q due east of P and R due south of P. A vertical mast PT stands at P and the angle of elevation of the top T, from Q is 3.43°. If $PQ = 1\ 000$ m and $PR = 750$ m, calculate

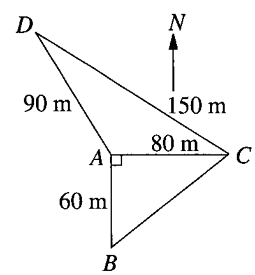
(a) the bearing of Q from R, (b) the height of the mast,
(c) the angle of elevation of T from R.

5. A man on a ship observes that the angle of elevation of a lighthouse is 12°. When he sails a further distance of 200 m away from the lighthouse, the angle of elevation becomes 10°. Find the height of the lighthouse.

6. A, B, C, and D are four points on horizontal ground. $AB = 60$ m, $AC = 80$ m, $AD = 90$ m and $CD = 150$ m. Given that B is due south of A and the bearing of C from A is 090°. Calculate

(a) the bearing of C from B, (b) the bearing of D from A.

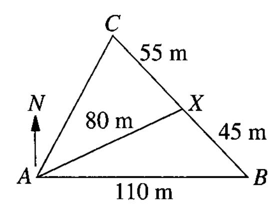
A vertical mast stands at point A and the angle of elevation of the top of this mast from C is 8.6°. Calculate the height of the mast and the angle of elevation of the top of the mast from D.



7. A, B and C are three points on level ground. B is 100 m due east of A. X is a point on BC such that $BX = 45$ m, $CX = 55$ m and $AX = 80$ m. Calculate

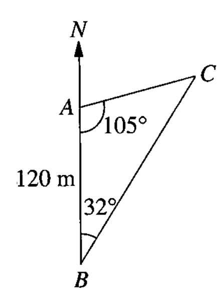
(a) $A\hat{B}X$, (b) the bearing of C from A.

A vertical mast of 20 m stands at point X. Calculate the angle of elevation of the top of the mast from A.



8. A, B, and C are three points on level ground such that B is due south of A, $A\hat{B}C = 32°$, $B\hat{A}C = 105°$ and $AB = 120$ m. Calculate the distance of BC.

A vertical mast CT of height 25 m stands at C. Calculate the angle of elevation of T from B.



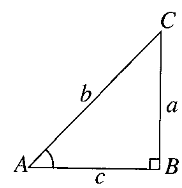
Summary

1. The trigonometrical ratios of an acute angle can be defined as:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{b}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{c}{b}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{c}$$



2. The trigonometrical ratios of an obtuse angle, A, can be expressed in terms of an acute angle as **(180° – A)**.

$$\sin A = \sin (180° - A)$$
$$\cos A = -\cos (180° - A)$$
$$\tan A = -\tan (180° - A)$$

3. Area of triangle $= \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B$.

4. In any triangle ABC, the Sine Rule states that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

5. In any triangle ABC, the Cosine Rule states that $a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$

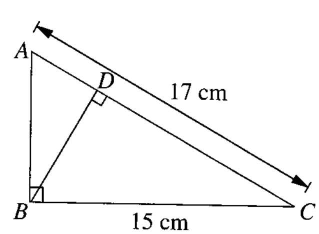
6. The bearing of a point A from another point O is an angle measured from the north, at O, in a clockwise direction and is always written as a three-figure number.

7. When solving three-dimensional problems, clearly-labelled diagrams are necessary.

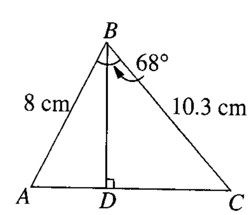
8. The basic technique used in solving a three-dimensional problem is to reduce it to a problem in a plane.

Review Questions 10

1. In $\triangle ABC$, $A\hat{B}C = 90°$, $BC = 15$ cm, $AC = 17$ cm and BD is perpendicular to AC. Calculate
   (a) AB;
   (b) the area of $\triangle ABC$;
   (c) BD.

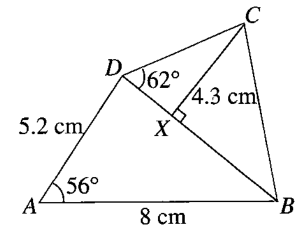


2. In $\triangle ABC$, $A\hat{B}C = 68°$, $AB = 8$ cm, $BC = 10.3$ cm and BD is perpendicular to AC. Calculate
   (a) AC;
   (b) area of $\triangle ABC$;
   (c) BD.



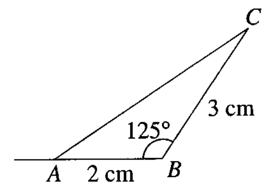
3. In the diagram, $ABCD$ is a quadrilateral and CX is perpendicular to BD. Given that $B\hat{A}D = 56°$, $B\hat{D}C = 62°$, $AB = 8$ cm, $AD = 5.2$ cm and $CX = 4.3$ cm, calculate

(a) BD;
(b) BX;
(c) $C\hat{B}X$;
(d) the area of $\triangle BCD$;
(e) the area of $\triangle ABD$.



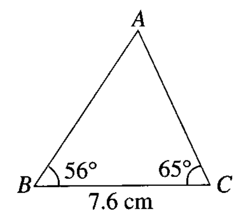
4. ABC is a triangle in which $AB = 2$ cm, $BC = 3$ cm and $A\hat{B}C = 125°$. Calculate

(a) AC;
(b) the area of $\triangle ABC$.



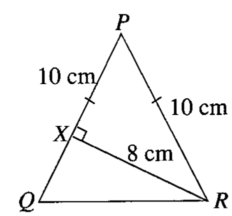
5. In $\triangle ABC$, $BC = 7.6$ cm, $A\hat{B}C = 56°$ and $A\hat{C}B = 65°$. Calculate

(a) AB;
(b) the area of $\triangle ABC$.



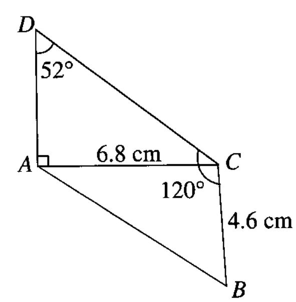
6. In $\triangle PQR$, $PQ = PR = 10$ cm, RX is perpendicular to PQ and $Q\hat{P}R$ is an acute angle. If $RX = 8$ cm, calculate

(a) the area of $\triangle PQR$;
(b) $Q\hat{P}R$;
(c) QR.



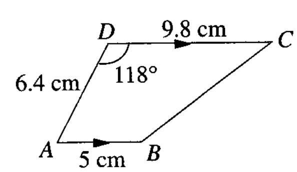
7. In the figure, $ABCD$ is a quadrilateral in which $A\hat{D}C = 52°$, $D\hat{A}C = 90°$, $B\hat{C}D = 120°$, $AC = 6.8$ cm and $BC = 4.6$ cm. Calculate

(a) CD;
(b) AB;
(c) the area of $ABCD$.



8. In the figure, $ABCD$ is a trapezium in which AB and DC are parallel. Given that $AB = 5$ cm, $AD = 6.4$ cm, $DC = 9.8$ cm and $A\hat{D}C = 118°$, calculate

(a) AC;
(b) BD;
(c) $A\hat{B}D$.



9. $ABCD$ is a rhombus of side 6.7 cm each and $A\hat{B}C = 102°$. Calculate the lengths of the diagonals AC and BD.

10. $ABCD$ is a rhombus of side 8 cm and the diagonal AC is of length 9.6 cm. Calculate $B\hat{A}D$.

11. $ABCD$ is a quadrilateral in which $AB = 4$ cm, $BC = 5$ cm, $CD = 8$ cm, $AD = 5.5$ cm and the diagonal $AC = 6.2$ cm. Calculate

(a) $B\hat{C}A$; (b) $A\hat{C}D$; (c) the area of $ABCD$.

12. In $\triangle ABC$, $B\hat{A}C = 78°$, $AB = AC$ and $BC = 10$ cm. If the bisector of the angle ACB meets AB at X, calculate

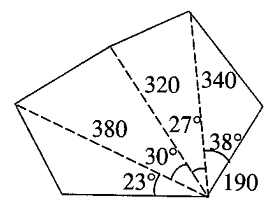
(a) $B\hat{X}C$; (b) BX.

13. The angles P, Q and R of $\triangle PQR$ are 48°, 53° and 79° respectively. The shortest side of $\triangle PQR$ is 14 cm. Calculate

(a) the length of the longest side of the triangle;
(b) the length of the shortest altitude of the triangle.

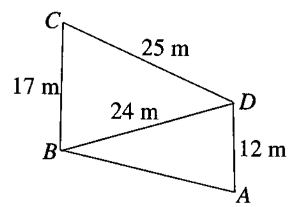
14. In $\triangle ABC$, $AB = 14$ cm, $BC = 9$ cm and $AC = 8$ cm. Calculate the value of $\cos A\hat{C}B$, giving your answer as a fraction in its lowest term.

15. At the end of a survey of a school field conducted by some boys, the result is as shown in the figure, where the lengths are in metres. Find the perimeter of the field, in metres, and its area, in m^2.

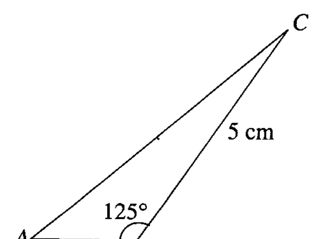


16. The diagram represents a plot of ground $ABCD$. Calculate $B\hat{D}C$. Given also that $\cos A\hat{D}B = \frac{1}{4}$, calculate

(a) AB;
(b) the area of $\triangle ABD$.



17. In the figure, $AB = 2$ cm, $BC = 5$ cm and $A\hat{B}C = 125°$. Calculate the length of AC and the area of $\triangle ABC$. Give your answer correct to two decimal places.

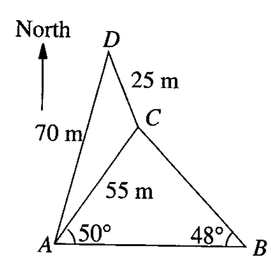


18. The sides of a parallelogram are of lengths 8 cm and 12 cm and the acute angles formed by the sides are 60° each. Find

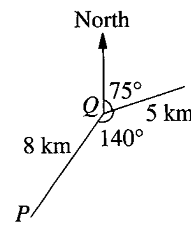
(a) the lengths of the diagonals; (b) the area of the parallelogram.

19. A, B, C and D are four points on level ground, with B due east of A. It is given that $AC = 55$ m, $CD = 25$ m, $AD = 70$ m, $C\hat{A}B = 50°$ and $A\hat{B}C = 48°$.

(a) Calculate (i) AB; (ii) $C\hat{A}D$; (iii) the bearing of D from A.
(b) A man walks due east from A until he reaches a point P, which is equidistant from A and C. Calculate the distance AP. (C)



20. A ship sails 8 km from P to Q. It then sails another 5 km from Q to R on a bearing of 075°.



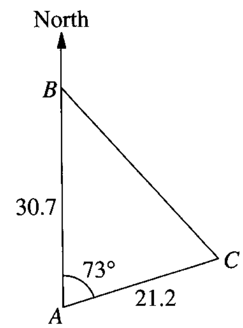
(a) Given that $P\hat{Q}R = 140°$, calculate

(i) the bearing of Q from P;
(ii) how far Q is east of P;
(iii) the distance PR.

(b) The ship finally sails from R to a position T, which is due north of Q. Given that $Q\hat{T}R = 40°$, calculate

(i) the distance of RT;
(ii) the shortest distance between the point Q and the ship as its sails from R to T. (C)

21. After an accident at sea, a rescue operation is carried out in the triangular region ABC as shown in the diagram.

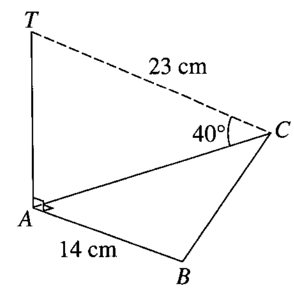


$AB = 30.7$ km, $AC = 21.2$ km, B is due north of A and the bearing of C from A is 073°.

(a) Calculate **(i)** the area of $\triangle ABC$; **(ii)** BC.

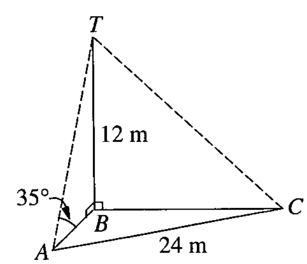
(b) A helicopter finds a life raft at a point X.
The bearing of X from A is 061°.
The bearing of X from B is 146°.
A rescue boat leaves A and travels directly to X at a speed of 40 km/h.
Calculate **(i)** AX;
(ii) the time, in minutes, that the boat takes to reach X. (C)

22. ABC is a triangle lying on a horizontal plane with $B\hat{A}C = 90°$ and $AB = 14$ cm. T is a point vertically above A and the angle of elevation of T from C is 40° and $TC = 23$ cm. Calculate



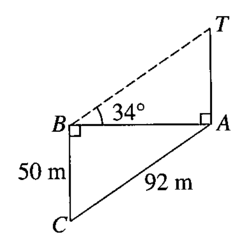
(a) AT,
(b) the angle of elevation of T from B,
(c) BC.

23. A, B and C are three points on level ground. BT is a vertical flag pole of height 12 m. Given that $T\hat{A}B = 35°$, $A\hat{B}C = 90°$ and $AC = 24$ m, calculate



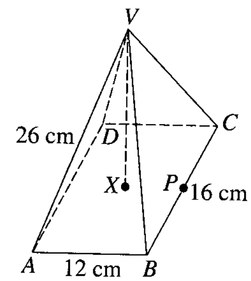
(a) AB,
(b) BC,
(c) the angle of elevation of T from C.

24. The diagram shows the points A, B and C lying on horizontal ground where $A\hat{B}C = 90°$, $BC = 50$ m and $AC = 92$ m. Given that A is the foot of a vertical pole AT and $T\hat{B}A = 34°$, calculate



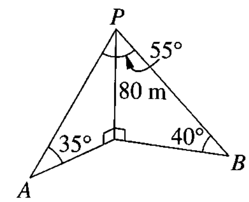
(a) AB,
(b) AT,
(c) the angle of depression of C from T.

25. The diagram shows a right pyramid on a horizontal rectangular base $ABCD$. Given that $AB = 12$ cm, $BC = 16$ cm and $VA = 26$ cm, calculate

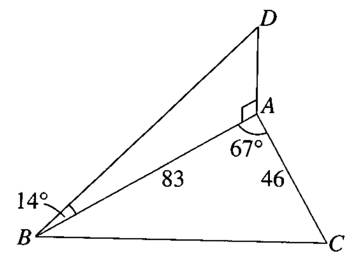


(a) the length of AX where X is the mid-point of AC,
(b) the vertical height, VX, of the pyramid,
(c) $A\hat{V}C$,
(d) VP where P is the mid-point of BC.

26. A man on top of a lighthouse, P, of height 80 m above sea level observes two boats A and B. If the angle of elevation of A is 35° and that of B is 40° and $APB = 55°$, calculate the distance between A and B giving your answer correct to the nearest 0.1 metre.



27. In the diagram, ABC represents a horizontal triangular field and AD represents a vertical tree in the corner of the field. A path runs along the edge BC of the field.

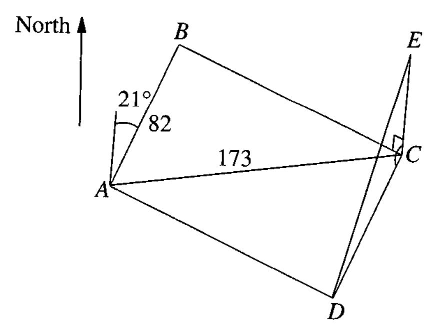


$AB = 83$ m, $AC = 46$ m and angle $BAC = 67°$.

(a) The angle of elevation of the top of the tree when viewed from B is 14°.
Calculate the height of the tree.
(b) Calculate the length of the path BC.
(c) Calculate the area of the field ABC.
(d) Calculate the shortest distance from A to the path BC.
(e) Calculate the greatest angle of elevation of the top of the tree when viewed from any point on the path.

(*C*)

28. The diagram shows A, B, C and D, the four corners of a horizontal rectangular field $ABCD$. The corner B is 82 metres from A on a bearing of 021° and C is 173 metres from A.



(a) Calculate

(i) the bearing of C from B,
(ii) $B\hat{A}C$,
(iii) the bearing of C from A.

(b) A hot air balloon was hovering at E, which is vertically above C. The angle of elevation of the bottom of the balloon from D was 35°. Calculate

(i) the height of the bottom of the balloon above C,
(ii) the angle of elevation of the bottom of the balloon from B.

(c) A bird is hovering at a height of 40 metres above the field. It spots its prey on the ground at an angle of depression of 63°. Calculate the distance that the bird must fly to catch its prey.

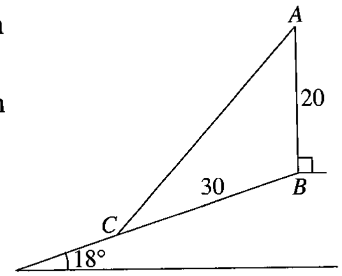
(*C*)

29. A radio mast AB, of height 20 m, stands at the top of a slope which is inclined at 18° to the horizontal.

(a) The mast is supported by a wire AC attached to a point C on the slope, which BC = 30 m. Calculate

(i) $A\hat{B}C$,

(ii) the length of the wire AC.

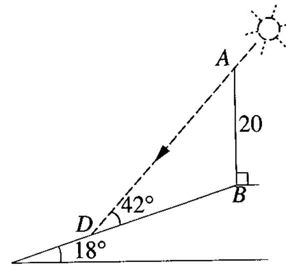


(b) When the sun is in a certain position, the shadow cast by the mast lies down the slope, shown in the diagram by the line BD. Given that $A\hat{D}B = 42°$, calculate

(i) the angle of elevation of the sun,

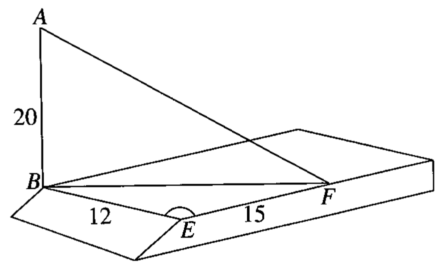
(ii) $D\hat{A}B$,

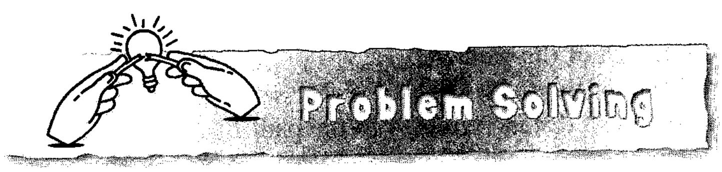
(iii) the length of the shadow BD.



(c) The mast is supported by another wire AF. The points B, E and F lie on horizontal ground. Given that $B\hat{E}F = 90°$, BE = 12 m and EF = 15 m, calculate the length of the wire AF.

(C)



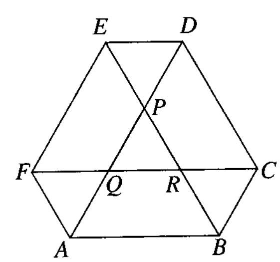


Example 20

The figure shows an equilateral $\triangle PQR$ with an area of 5 cm². The sides of the triangle are produced both ways to form a hexagon ABCDEF where

$$QA = RB = RC = PD = PE = QF = PQ.$$

Find the area of the hexagon ABCDEF.

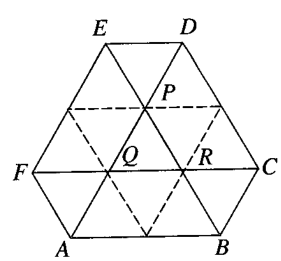


Solution

Method 1:

By constructing lines parallel to AB, CD and EF, we get 13 triangles congruent to $\triangle PQR$.

Hence, area of the hexagon = $13 \times 5 = 65$ cm².



Method 2:

$\triangle RBC$ is the image of $\triangle RPQ$, with R as the centre of rotational symmetry.

Similarly, $\triangle PED$ is the image of $\triangle PRQ$ with P as the centre of rotational symmetry and $\triangle QAF$ is the image of $\triangle QPR$ with Q as the centre of rotational symmetry.

$\therefore$ area of $\triangle PQR = \triangle RBC = \triangle QAF = \triangle PED = 5\text{ cm}^2$.

$\triangle PQR$ is similar to $\triangle PAB$

$\therefore$ area of $\frac{\triangle PQR}{\triangle PAB} = \left(\frac{PQ}{PA}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\therefore$ area of $\triangle PAB = 4$ (area of $\triangle PQR$) $= 20\text{ cm}^2$

$\therefore$ area of $QABR = 20 - 5 = 15\text{ cm}^2$.

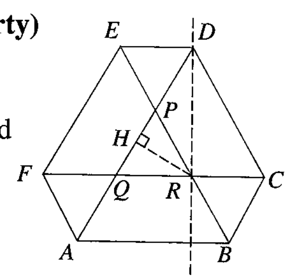
Similarly, $\triangle QPR$ is similar to $\triangle QDC$ and $\triangle RPQ$ is similar to $\triangle REF$.

$\therefore$ area of $QABR$ = area of $PRCD$
= area of $PQFE = 15\text{ cm}^2$.

$\therefore$ Area of $ABCDEF = 4(5) + 3(15) = 65\text{ cm}^2$.

Method 3:

$\triangle PQR$ is congruent to $\triangle PDE$ **(SAS Property)**
[$PD = PQ = PR = PE$ and $Q\hat{P}R = D\hat{P}E$.]



Similarly, $\triangle PQR$ is congruent to $\triangle AQF$ and $\triangle PQR$ is congruent to $\triangle BCR$.

$\therefore$ area of $\triangle PQR = \triangle PDE = \triangle AQF$
$= \triangle BCR = 5\text{ cm}^2$.

Area of $\triangle PDR$ = Area of $\triangle PQR$ as the base PQ
= PD and the height RH is common.

$\therefore$ Area of $\triangle PDR = 5\text{ cm}^2$.

Area of $\triangle RDC$ = Area of $\triangle RDQ$ as the base RC
= QR and the height DR is common.

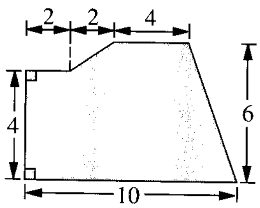
$\therefore$ Area of $DRC = 10\text{ cm}^2$ and area of $PRCD = 15\text{ cm}^2$.

Similarly, we can show that area of $QABR$ = area of $PQFE = 15\text{ cm}^2$.

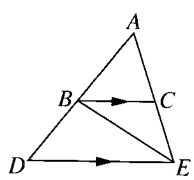
$\therefore$ Area of $ABCDEF = (4 \times 5) + (3 \times 15)\text{ cm}^2 = 65\text{ cm}^2$.

Problems

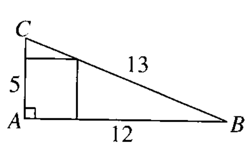
1. *Use as many different methods as possible to find the area of the shaded region in the diagram. Which of your methods do you think is the best?*



2. *In the figure below, BC is parallel to DE. If 2AB = 4BD and the area of $\triangle BDE$ = 24 cm², calculate the area of $\triangle ABC$.*



3. *The diagram below shows a right-angled $\triangle ABC$ with AB = 12 cm, AC = 5 cm and BC = 13 cm. A square is cut off from the side. Find the area of the square.*



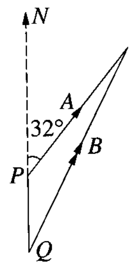
Exploration

1. The sides of a triangle are $24x$ cm, $28x$ cm and $19x$ cm. If the area of the triangle is 250 cm², calculate the value of x, giving your answer correct to 2 decimal places.

2. Given that x and y are positive, prove that the triangle, whose sides are of length $5x + 12y$, $12x + 5y$ and $13x + 13y$, contains an obtuse angle.

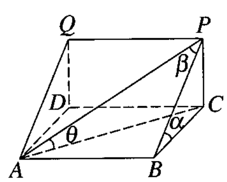
3. In a right-angled triangle, $\triangle ABC$, where $A\hat{B}C = 90°$, if BC is the mean of AB and AC, calculate the value of the smallest angle in $\triangle ABC$.

4. A boat A leaves a port P on a bearing of 032°, travelling at a speed of 9 km/h. Another patrol boat B, situated at another port Q, due south of P, sets sail to intercept the boat A. The patrol boat is capable of moving at 13.5 km/h. Find the bearing at which the patrol boat must sail in order to intercept A.

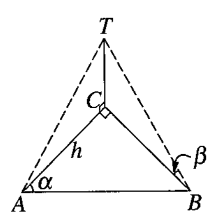


5. Three points A, B and C lie on horizontal ground. T is the top of a vertical tower standing on A. The bearing of B and C from A are 135° and 225° respectively, and the bearing of C from B is 250°. If the length of BC is 50 m and the angle of elevation of T from B is 35°, calculate the height of the tower and the angle of elevation of T from C.

6. In the figure, $ABCD$ is horizontal and $ABPQ$ is a rectangular plane. Given that $P\hat{C}A = P\hat{C}B = Q\hat{D}C = 90°$, $P\hat{B}C = a°$, $A\hat{P}B = b°$ and $P\hat{A}C = q°$, express $\sin q$ in terms of trigonometrical ratio of a and b only.

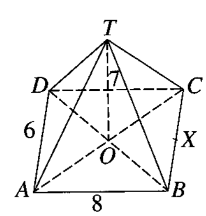


7. In the figure, $\triangle ABC$ is on horizontal ground and CT is vertical. Given that $A\hat{C}B = 90°$, $B\hat{A}C = \alpha°$, $T\hat{B}C = \beta°$ and $AC = h$ m, express the length of TB in terms of h and the trigonometrical ratio of α and β.



8. The diagram shows a rectangular pyramid with base $ABCD$ where $AB = 8$ cm and $AD = 6$ cm. The diagonals intersect at O. The vertex T is 7 cm vertically above O. The midpoint of BC is X. Calculate

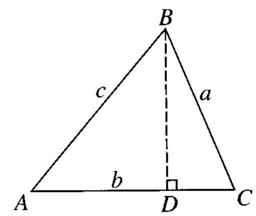
   (a) TX, (b) $T\hat{A}O$.



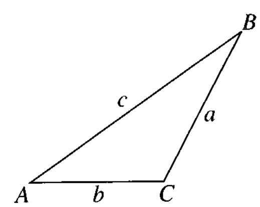
There are other ways of proving the sine rule besides using the method outlined earlier where we made use of the formula for the area of a triangle.

In $\triangle ABC$, BD is the perpendicular from B to AC.

By considering $\triangle ABD$ and $\triangle BCD$ to find the length of BD show that $\frac{a}{\sin A} = \frac{c}{\sin C}$.



(a) Can you show that $\frac{a}{\sin A} = \frac{b}{\sin B}$, by considering the perpendicular from C to AB?



(b) How can you show that $\frac{a}{\sin A} = \frac{c}{\sin C}$, for the case where one angle is obtuse?

CHAPTER 11

Geometrical Properties of Circles

In this chapter, you will learn how to use the angle properties of a circle to

- △ find the distance between two parallel chords in a circle;
- △ solve problems involving angles subtended at the centre and angles at the circumference;
- △ solve problems involving angles in a cyclic quadrilateral.

Preliminary Problem

A circle symbolises unity, harmony and continuity. It is one of the favourite shapes and is often used in the design of building, road signs, door entrances, stools and tables.

Symmetrical Properties of Circles

In Fig. 11.1, XY is called a **chord** of the circle. XAY is called a **minor arc** and XBY is called a **major arc**. The region bounded by the chord XY and the minor arc XAY is called a **minor segment** of the circle. The region bounded by the major arc XBY and the chord XY is called a **major segment** of the circle.

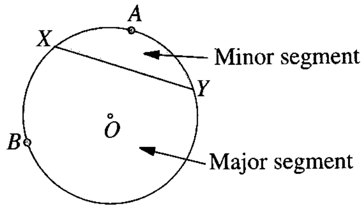


Fig. 11.1

For Your Information

*A chord is a line segment joining two distinct points on the circumference. It divides a circle into two segments. The larger part is called the **major segment** and the smaller part, the **minor segment**. If the chord happens to be a diameter, each segment is a semi-circle.*

We shall now study some properties of circles.

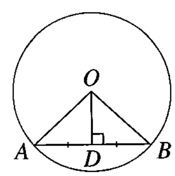
Property 1:

A straight line drawn from the centre of a circle to bisect a chord, which is not a diameter, is **perpendicular** to the chord. Conversely, the perpendicular to a chord, drawn from the centre of a circle, bisects the chord.

Given a circle, centre O and a chord, AB, with a mid-point D, we are required to show that $O\hat{D}B = 90°$.

Working

Join OA and OB. In $\triangle OAD$ and $\triangle OBD$,



$OA = OB$ *(radii of the circle)*
$AD = BD$ *(given)*
OD is common.

$\therefore$ $\triangle OAD$ is congruent to $\triangle OBD$ (**SSS Property**).

$\therefore$ $O\hat{D}A = O\hat{D}B$.

Since these are adjacent angles on a straight line, $O\hat{D}A = O\hat{D}B = 90°$.

The converse is left as an exercise for you.

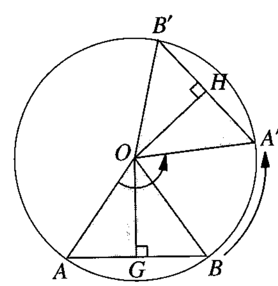
Property 2:

In equal circles or in the same circle, equal chords are **equidistant** from the centres or centre. Conversely, chords which are equidistant from the centres or centre are equal.

Working

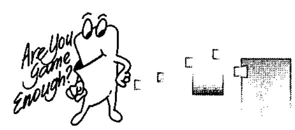
In the figure, $\triangle OAB$ is rotated through an angle AOA' to $\triangle OA'B'$ about O.

Since *rotation preserves shape and size*, we have $AB = A'B'$ and $OG = OH$.



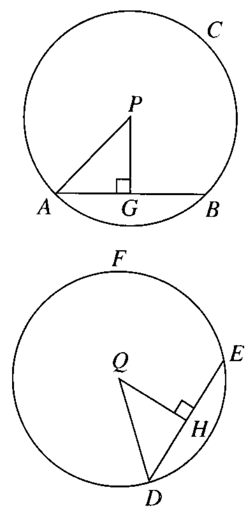
Example 7

*The lengths of two parallel chords of a circle, of radius 12 cm, are 14 cm and 8 cm respectively. Calculate the distance between the chords. (**Hint:** there are 2 possible answers.)*



The following shows two equal circles ABC and DEF, with centres P and Q, and two equal chords AB and DE.

Can you prove that PG = QH?



Solution

The two possible cases are shown in Fig. 11.2 and Fig. 11.3. (The 2 diagrams are not drawn to scale).

In $\triangle AON$, $(ON)^2 = 12^2 - 4^2$

$= 144 - 16$

$ON = \sqrt{128}$

$\therefore \quad ON \approx 11.31$ cm.

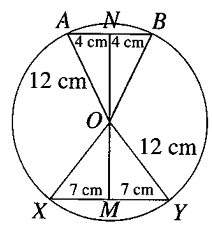


Fig. 11.2

In $\triangle YOM$, $(OM)^2 = 12^2 - 7^2$

$= 95$

$OM = \sqrt{95}$

$\therefore \quad OM \approx 9.747$ cm.

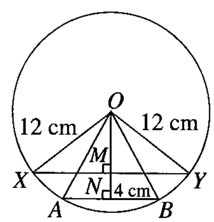


Fig. 11.3

In Fig. 11.2, $NM = ON + OM$

$\approx 11.31 + 9.747 \approx 21.1$ cm

In Fig. 11.3, $NM = ON - OM$

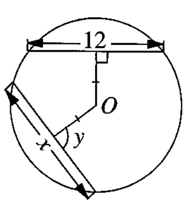
$\approx 11.31 - 9.747 \approx 1.56$ cm

$\therefore$ the distance between the chords can be either about 21.1 cm or 1.56 cm.

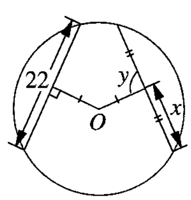
Exercise 11a

1. Given that O is the centre of each of the circles, find the values of x and y in each case.

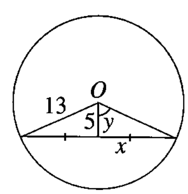
(a)



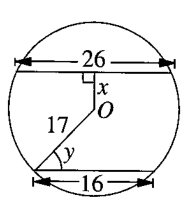
(b)



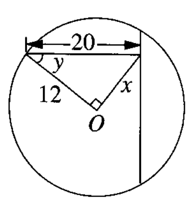
(c)



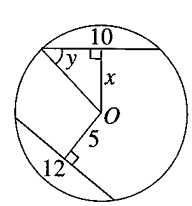
(d)



(e)



(f)



2. AB is a chord of a circle, centre O, with radius 17 cm. If AB = 16 cm, find the distance of O from AB.

3. A chord of length 24 cm is at a distance of 5 cm from the centre of a circle. Find the radius of the circle.

4. A chord of a circle of radius 8.5 cm is 5 cm from the centre. Find its length.

5. PQ is a chord 9.6 cm long. It lies 3 cm from the centre. What is the radius of this circle?

6. The radius of a circle is 17 cm. A chord XY lies 9 cm from its centre. Find the length of XY.

7. In a circle of radius 5 cm, there are two parallel chords of lengths 8 cm and 6 cm. Calculate the distance between the chords.

8. Two parallel chords in a circle of radius 7 cm measure 12 cm and 9 cm. Calculate the distance between them.

9. Two parallel chords PQ and MN are 3 cm apart on the same side of a circle where PQ = 7 cm and MN = 14 cm. Calculate the radius of the circle.

10. The perpendicular bisector of a chord XY cuts XY at N and the circle at P. If XY = 16 cm and NP = 2 cm, calculate the radius of the circle.

Angle Properties of Circles

In Fig. 11.4(a), O is the centre of the circle. We say that $A\hat{P}B$ is subtended by the minor arc AXB at the point P on the circumference. The minor arc AXB is also said to subtend $A\hat{O}B$ at the centre of the circle which is O. In Fig. 11.4(b), O is the centre of the circle. We say that $A\hat{P}B$ is subtended by the major arc AQB at the point P on the circumference. The major arc AQB is also said to subtend the reflex angle AOB at centre O of the circle.

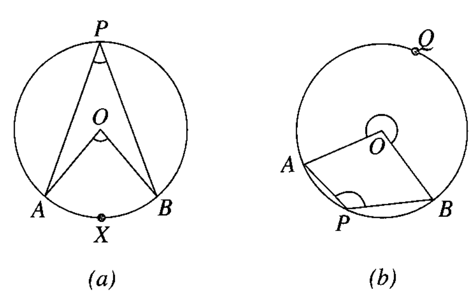


Fig. 11.4

In Fig. 11.5, P, Q and R are points on the major arc $APQRB$ of a circle. $A\hat{P}B$, $A\hat{Q}B$ and $A\hat{R}B$ are said to be angles subtended by the same arc AB or by the chord AB. These angles are called **angles in the same segment**.

For Your Information

A reflex angle is an angle between 180° and 360°.

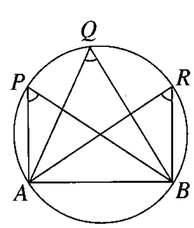


Fig. 11.5

Property 1:

An angle at the centre of a circle is **twice** any angle at the circumference subtended by the **same arc**.

We are given an arc AB of a circle with, centre O, and D being any point on the remaining part of the circumference. We are required to show that $A\hat{O}B = 2A\hat{D}B$.

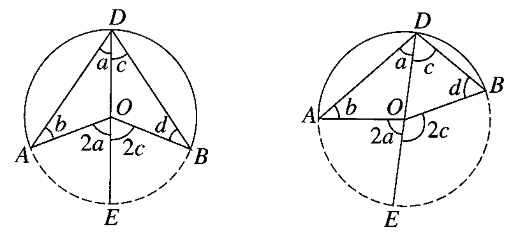


Fig. 11.6

Working

Join DO and produce it to E.

In Fig. 11.6(a), the angles are subtended by the minor arc AB.

In Fig. 11.6(b), the angles are subtended by the major arc AB.

Since $OA = OD$ *(radii of circle)*, $a = b$ *(base ∠s of isos. △)*.

But $A\hat{O}E$ is the exterior angle of $\triangle AOD$.

$\therefore \quad A\hat{O}E = 2a$

Similarly, $c = d$ *(base ∠s of isos. △)*.

$\therefore \quad B\hat{O}E = 2c$

Hence $A\hat{O}B = 2a + 2c = 2(a + c) = 2A\hat{D}B$.

Property 1 can be abbreviated as **∠ at centre = 2∠ at ⊙ce**.

Property 2:

Every angle at the circumference subtended by the diameter of a circle is a **right angle**.

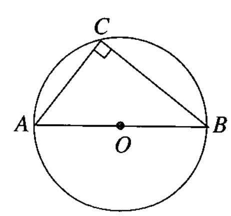
Given that AB is the diameter of a circle with centre O and a point C on the circumference, we are required to show that $A\hat{C}B = 90°$.

Working

$A\hat{O}B = 2A\hat{C}B$ *(∠ at centre = 2∠ at ⊙ce.)*

But $A\hat{O}B = 180°$ *(∠ on a straight line)*

$\therefore \quad A\hat{C}B = 90°$

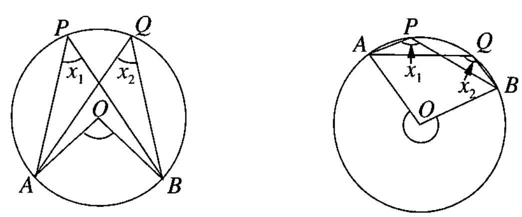


Property 2 can be abbreviated as **rt. ∠ in a semicircle**.

Property 3:

Angles in the same segment of a circle are **equal**.

Given 2 angles, $A\hat{P}B$ and $A\hat{Q}B$, which lie in the same segment of the circles, we are required to show that $A\hat{P}B = A\hat{Q}B$.



Working

Join OA and OB. Using the notations in the figures,

$A\hat{O}B = 2x_1 = 2x_2$ *(∠ at centre = 2∠ at ⊙ce)*

$\therefore \quad x_1 = x_2$

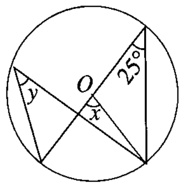
$\therefore \quad A\hat{P}B = A\hat{Q}B$

Property 3 can be abbreviated as **∠s in the same segment**.

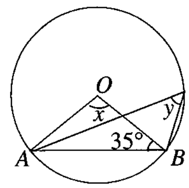
Example 2

Given that O is the centre of the circle, find the angles marked x and y in each case.

(a)



(b)



Solution

(a) $x = 2 \times 25°$ *(∠ at centre = 2∠ at ⊙ce)*

$= 50°$

$y = 25°$ *(∠s in the same segment)*

(b) $x = 180° - 35° - 35°$ *(∠ sum of △, OA = OB)*

$= 110°$

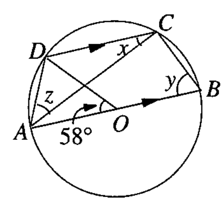
$y = \frac{110°}{2}$ *(∠ at centre = 2∠ at ⊙ce)*

$= 55°$

Example 3

In the figure, O is the centre of the circle, AOB is parallel to DC and $A\hat{O}D = 58°$. Find the angles marked x, y and z.

Solution



$x = \frac{58°}{2}$ *(∠ at centre = 2∠ at ⊙ce)*

$= 29°$

$A\hat{C}B = 90°$ *(rt. ∠ in a semicircle)*

$C\hat{A}B = 29°$ *(alt ∠s, AB // DC)*

$\therefore\ y = 180° - 90° - 29°$ *(∠ sum of △)*

$= 61°$

$A\hat{D}O = D\hat{A}O$ *(base ∠s of isos. △)*

$A\hat{D}O = \frac{1}{2}(180° - 58°) = 61°$

$\therefore\ z = 61° - 29° = 32°.$

Example 4

In the figure, O is the centre of the circle and ABN is a straight line. Find the obtuse angle $A\hat{O}C$.

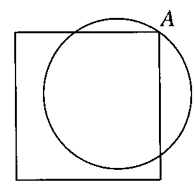
Solution

$b + 64° = 180°$ *(adj ∠s on a straight line)*

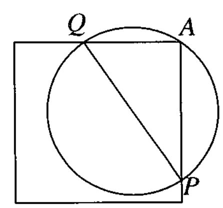
$\therefore\ b = 116°$

Do you know how the centre of a circle can be determined? Follow the instructions given below and discover the answer yourself.

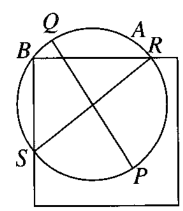
(a) Place a rectangular piece of paper under a circle such that one of its corners touches the circle, say at A.



(b) Join the two points, P and Q, as shown below.



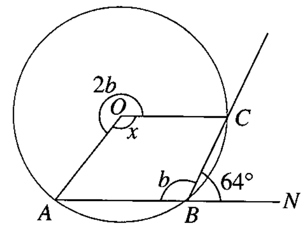
(c) Move the same sheet of paper such that another of its corners touches the circle, say at B. Join the two points, R and S, as shown below.



The result would show that the points of in-

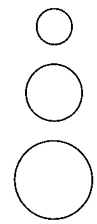
$\therefore \quad 2b = 2(116°)$ ($\angle$ *at centre* = $2\angle$ *at* $\odot^{ce}$)
$= 232°$
$x = 360° - 2b$ ($\angle$*s at a point*)
$= 360° - 232°$
$= 128°$

$\therefore$ obtuse $A\hat{O}C = 128°$.



tersection of PQ and RS gives the centre of the circle. Is the ***centre of a circle*** *also its* ***centre of rotation****?*

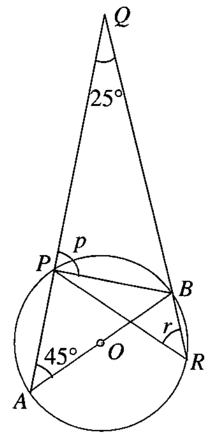
Trace the circles below and determine their respective centres using the method discussed earlier. Does it work? Use a pair of compasses to check your answers.



Example 5

In the following diagram, AB is a diameter and APQ and RBQ are straight lines. Find $B\hat{P}R$.

Solution



$A\hat{P}B = 90°$ (*rt.* $\angle$ *in a semicircle*)
$p = 180° - 90°$ (*adj* $\angle$*s on a straight line*)
$= 90°$
$r = 45°$ ($\angle$*s in the same segment*)

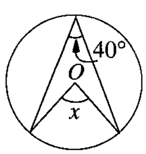
In $\triangle PQR$,

$B\hat{P}R + p + r + 25° = 180°$ ($\angle$ *sum of* $\triangle$)
$\therefore \quad B\hat{P}R = 180° - 90° - 25° - 45°$
$= 20°$.

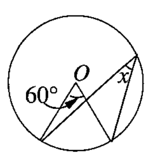
Exercise 11b

For questions 1, 2, and 3, find the value of x in each of the figures where O is the centre of each circle.

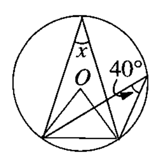
1. (a)



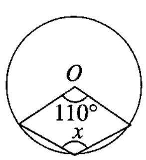
(b)



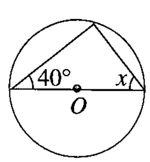
(c)



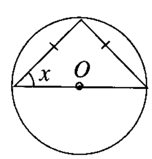
(d)



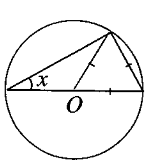
(e)



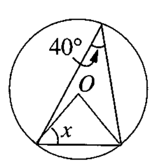
(f)



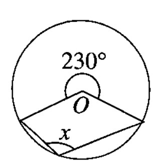
2. (a)

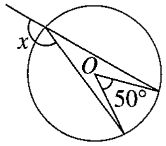


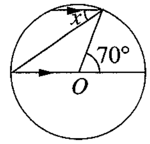
(b)

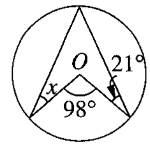


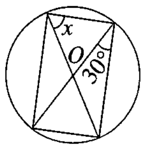
(c)

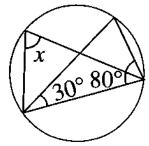


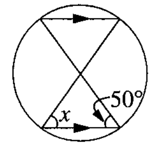
(d) 

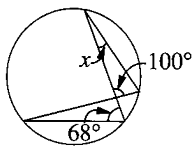
(e) 

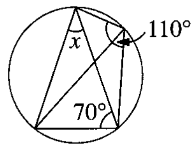
(f) 

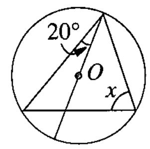
3. (a) 

(b) 

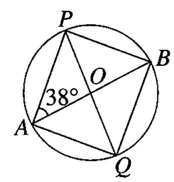
(c) 

(d) 

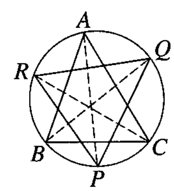
(e) 

(f) 

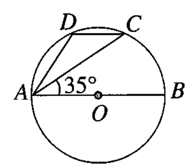
4. In the diagram, O is the centre of the circle and $P\hat{A}B = 38°$, find
(a) $P\hat{Q}B$; **(b)** $P\hat{Q}A$; **(c)** $Q\hat{P}B$.



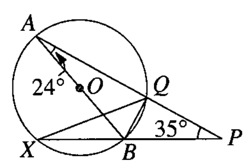
5. In the diagram, AP, BQ and CR are the angle bisectors of $\hat{A}$, $\hat{B}$ and $\hat{C}$ respectively. Given that $\hat{A} = 50°$, $\hat{B} = 70°$ and $\hat{C} = 60°$, find $\hat{P}$, $\hat{Q}$ and $\hat{R}$.



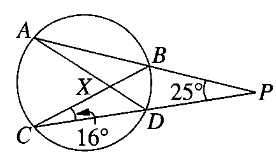
6. In the diagram, AB is a diameter of the circle. Given that $C\hat{A}B = 35°$, find $A\hat{D}C$.



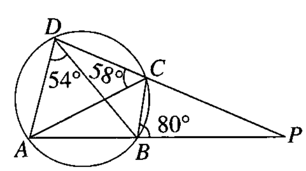
7. In the diagram, AB is a diameter of the circle. Given that $B\hat{A}P = 24°$ and $B\hat{P}A = 35°$, find $B\hat{Q}X$.



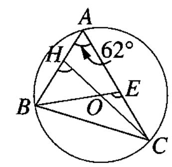
8. In the diagram, $A\hat{P}C = 25°$ and $B\hat{C}D = 16°$. Find $A\hat{X}B$.



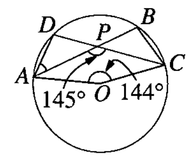
9. In the diagram, $A\hat{D}B = 54°$, $A\hat{C}D = 58°$ and $C\hat{B}P = 80°$. Find $A\hat{P}D$.



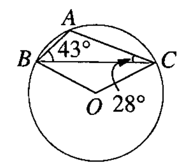
☆**10.** In the diagram, $\triangle ABC$ is inscribed in a circle, centre O. If $\hat{A} = 62°$, find the sum of $B\hat{H}C$ and $B\hat{E}C$ (i.e. $B\hat{H}C + B\hat{E}C$).



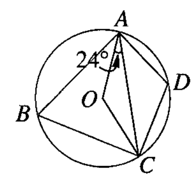
11. In the diagram, $A\hat{O}C = 144°$ and $A\hat{P}C = 145°$ where O is the centre of the circle. Find $B\hat{A}D$.



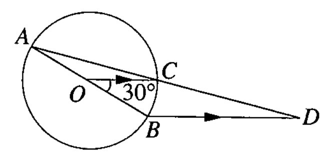
12. In the diagram, $A\hat{B}C = 43°$ and $A\hat{C}B = 28°$. Find $O\hat{B}A$ and $O\hat{C}A$.



13. In the diagram, O is the centre of the circle and $O\hat{A}C = 24°$. Find $A\hat{B}C$ and $A\hat{D}C$.



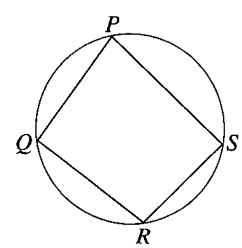
14. In the diagram, $OC \parallel BD$ and O is the centre of the circle. Given that $C\hat{O}B = 30°$, find $A\hat{B}D$ and $A\hat{D}B$.



Cyclic Quadrilaterals

A quadrilateral with its four vertices lying on the circumference of a circle is called a **cyclic quadrilateral**.

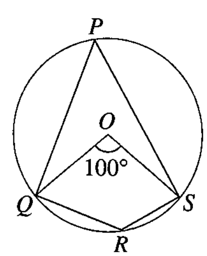
In the diagram, $PQRS$ is a cyclic quadrilateral. The points P, Q, R and S are said to be **concyclic**.



Example 6

In the figure, O is the centre of the circle and $Q\hat{O}S = 100°$. Find

(a) the reflex angle $Q\hat{O}S$; *(b) $Q\hat{P}S$;*
(c) $Q\hat{R}S$; *(d) $Q\hat{P}S + Q\hat{R}S$.*



Solution

(a) Reflex angle $QOS = 360° - 100°$
$= 260°$

(b) $\hat{QPS} = \frac{100°}{2}$ *(∠ at centre = 2∠ at ⊙ce)*

$= 50°$

(c) $\hat{QRS} = \frac{260°}{2}$ *(∠ at centre = 2∠ at ⊙ce)*

$= 130°$

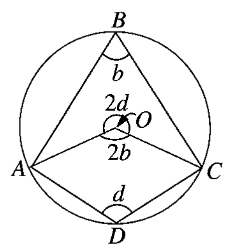
(d) $\hat{QPS} + \hat{QRS} = 50° + 130°$

$= 180°$

The results of Example 6 suggest that:

Property 1:

In a cyclic quadrilateral, the opposite angles are supplementary, i.e., they add up to two right angles.



Given that a quadrilateral $ABCD$ is inscribed in a circle, with centre O, we are required to show that $\hat{ABC} + \hat{ADC} = 180°$.

Do the proof yourself, following what is being done in Example 6.

Back in Time

*Circles can be found in innumerable places in nature. The early Greek mathematicians were interested in circles. Properties of circles were specially mentioned in the third volume of a series called **Elements**. This series was the masterpiece of Euclid, a Greek mathematician, over 2 000 years ago. He collected all the known geometric facts of his day and then arranged the subject matter in a completely logical way.*

Property 2:

If one side of a cyclic quadrilateral is produced, the exterior angle so formed is **equal** to the interior opposite angle.

Given that a quadrilateral $ABCD$ is inscribed in a circle, with AD produced to E, we are required to show that $\hat{ABC} = \hat{CDE}$.

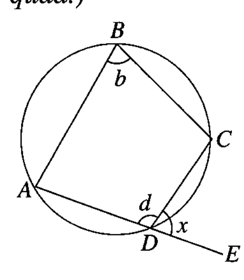
Working

In the figure, $b + d = 180°$ *(opp. ∠s of a cyclic quad.)*

But $x + d = 180°$ *(adj. ∠s on a straight line)*

$\therefore \quad b + d = x + d \Rightarrow b = x$

$\therefore \quad \hat{ABC} = \hat{CDE}$



NB: The converse of these properties are also true, i.e.,

(1) if the opposite angles of a quadrilateral are supplementary, then the quadrilateral is a cyclic quadrilateral;

(2) if the exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is a cyclic quadrilateral.

For Your Information

*(1) The property which states that, in a cyclic quadrilateral, the opposite angles add up to two right angles, can be **abbreviated** as:*

opp. ∠s of cyclic quad.

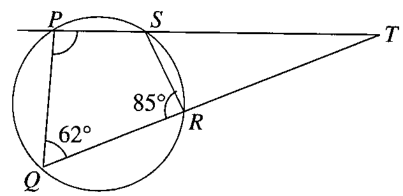
*(2) The property which states that, if one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle, can be **abbreviated** as:*

ext. ∠ of cycylic quad.

Example 7

In the figure, PQRS is a cyclic quadrilateral, PST and QRT are straight lines and $P\hat{Q}R = 62°$ and $Q\hat{R}S = 85°$. Find

(a) $Q\hat{P}S$; *(b)* $R\hat{S}T$; *(c)* $S\hat{T}R$.



Solution

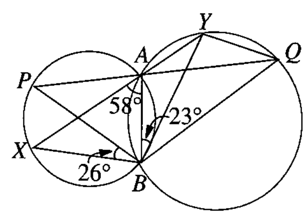
(a) $Q\hat{P}S = 180° - 85°$ *(opp. ∠s of a cyclic quad.)*
$= 95°$

(b) $R\hat{S}T = 62°$ *(ext. ∠ of a cyclic quad.)*

(c) $S\hat{T}R = 85° - 62°$ *(ext. ∠ = sum of int. opp. ∠s)*
$= 23°$

Example 8

In the figure, PAQ and XAY are straight lines. $B\hat{A}X = 58°$, $P\hat{B}X = 26°$ and $A\hat{B}Y = 23°$. Find $A\hat{Q}B$ and $A\hat{Y}Q$.



Solution

$B\hat{Q}Y = B\hat{A}X = 58°$ *(ext. ∠ of a cyclic quad.)*

$A\hat{Q}Y = A\hat{B}Y = 23°$ *(∠s in the same segment)*

$A\hat{Q}B = B\hat{Q}Y - A\hat{Q}Y$
$= 58° - 23° = 35°.$

$Y\hat{B}Q = Y\hat{A}Q$ *(∠s in the same segment)*

But $Y\hat{A}Q = X\hat{A}P$ *(vert. opp. ∠s)*

and $X\hat{A}P = P\hat{B}X$ *(∠s in the same segment)*

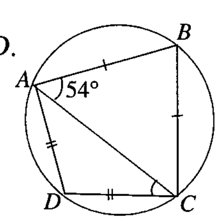
$\therefore$ $Y\hat{B}Q = 26°$

$A\hat{B}Q = A\hat{B}Y + Y\hat{B}Q$
$= 23° + 26° = 49°$

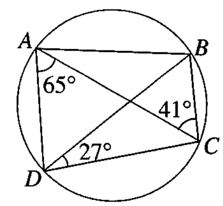
$\therefore$ $A\hat{Y}Q = 180° - A\hat{B}Q$ *(opp. ∠s of a cyclic quad.)*
$= 180° - 49° = 131°.$

Exercise 11c

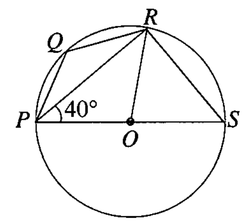
1. In the diagram, $AB = BC$, $CD = DA$ and $B\hat{A}C = 54°$. Find the value of $A\hat{C}D$.



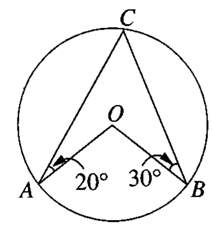
2. In the diagram, $D\hat{A}C = 65°$, $A\hat{C}B = 41°$ and $B\hat{D}C = 27°$. Find $A\hat{B}D$.



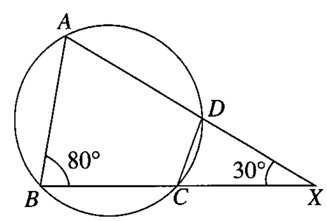
3. In the diagram, O is the centre of the circle and $R\hat{P}S = 40°$. Calculate $P\hat{Q}R$ and $O\hat{R}S$.



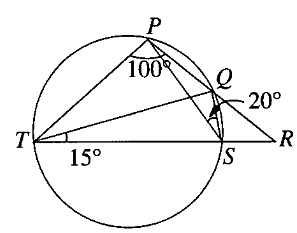
4. In the figure, O is the centre of the circle ABC. Given that $C\hat{A}O = 20°$ and $C\hat{B}O = 30°$, find $A\hat{C}B$.



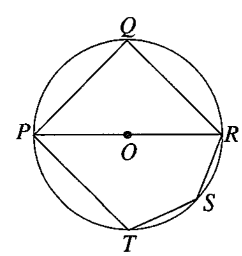
5. In the diagram, $A\hat{B}C = 80°$ and $A\hat{X}B = 30°$. Find $B\hat{A}D$ and $X\hat{C}D$.



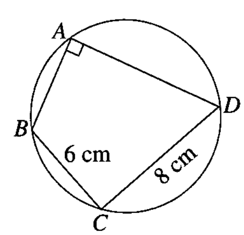
6. In the figure, $PQST$ is a cyclic quadrilateral, in which $T\hat{P}Q = 100°$, $Q\hat{T}S = 15°$ and $P\hat{S}Q = 20°$. Find $P\hat{Q}T$ and $R\hat{Q}S$.



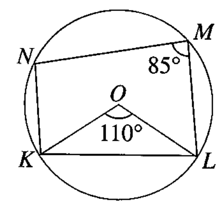
7. In the figure, O is the centre of the circle. Find the sum of $P\hat{Q}R$, $P\hat{R}S$ and $P\hat{T}S$.



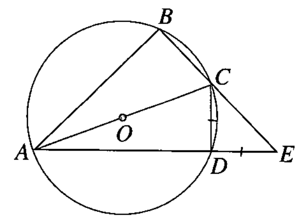
8. In the diagram, $B\hat{A}D = 90°$, $BC = 6$ cm and $CD = 8$ cm. Find the radius of the circle.



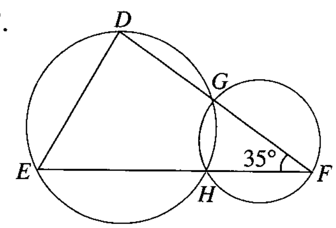
9. In the diagram, O is the centre of the circle. Given that $K\hat{O}L = 110°$ and $L\hat{M}N = 85°$, find $O\hat{K}N$.



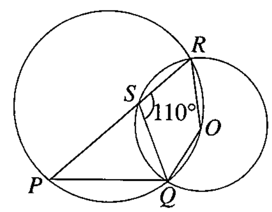
10. In the figure, O is the centre of the circle. If $CD = DE$, find $B\hat{A}D$.



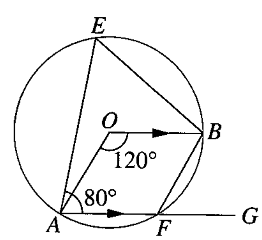
11. In the figure, GF is a diameter of the circle GHF and $G\hat{F}H = 35°$. Find $E\hat{D}G$ and $D\hat{E}F$.



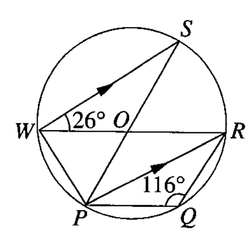
12. In the figure, O is the centre of the circle SQR, $R\hat{S}Q = 110°$ and PSR is a straight line. Calculate $Q\hat{P}S$.



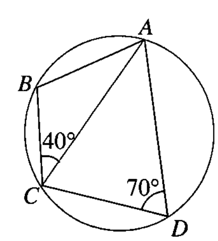
13. In the figure, O is the centre of the circle, with AFG // OB, $A\hat{O}B = 120°$ and $E\hat{A}G = 80°$. Find BFG and EBO.



14. In the figure, O is the centre of the circle, with WS // PR, $S\hat{W}R = 26°$ and $P\hat{Q}R = 116°$. Find $P\hat{S}W$, $P\hat{W}R$ and $S\hat{P}W$.



15. In the diagram, $A\hat{D}C = 70°$ and $A\hat{C}B = 40°$. Find $B\hat{A}C$.



Problems on Angle Properties of Circles

The following examples involve the angle properties of circles.

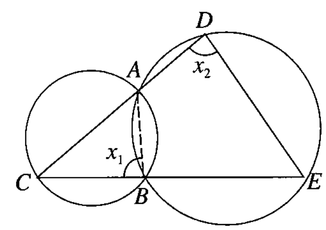
Example 9

In the figure, CAD and CBE are straight lines. If CA is the diameter of the circle ABC, explain why $A\hat{D}E$ is a right angle.

Solution

Join AB.

$x_1 = 90°$ (*rt. ∠ in a semicircle*)
$x_1 = x_2$ (*ext. ∠ of a cyclic quad.*)
$\therefore\ x_2 = 90°$
$\therefore\ A\hat{D}E = 90°$

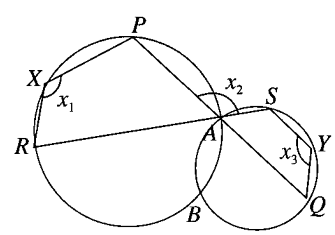


Example 10

In the figure, PAQ and RAS are straight lines. Show that $P\hat{X}R = Q\hat{Y}S$.

Solution

$x_1 = x_2$ (*ext. ∠ of a cyclic quad.*)
and $x_2 = x_3$ (*ext. ∠ of a cyclic quad.*)
$\therefore\ x_1 = x_3$
$\therefore\ P\hat{X}R = Q\hat{Y}S$



Example 11

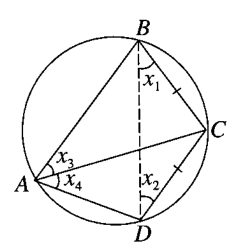
ABCD is a cyclic quadrilateral and BC = CD. Show that AC bisects $B\hat{A}D$.

Solution

Join BD.

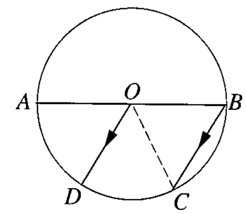
$x_1 = x_2$ (*base ∠s of isos. ∠*)
$x_1 = x_4$ (*∠s in the same segment*)
$x_2 = x_3$ (*∠s in the same segment*)
$\therefore\ x_3 = x_4$

Hence, AC bisects $B\hat{A}D$.

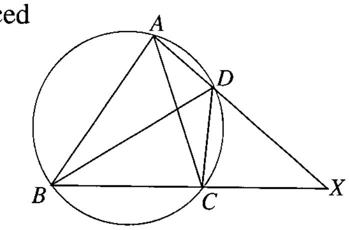


Exercise 11d

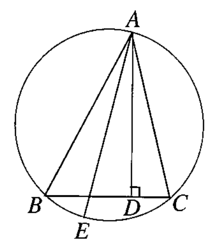
1. In the figure, $OD \parallel BC$ and AOB is the diameter of the circle. Show that $A\hat{O}D = C\hat{O}D$.



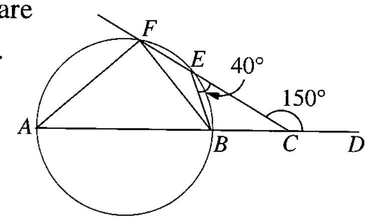
2. In the figure, $ABCD$ is a cyclic quadrilateral. BC and AD are produced to meet at X and CD bisects $B\hat{D}X$. Show that $AC = CB$.



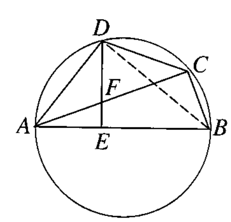
3. In the figure, AD is the altitude of $\triangle ABC$ and AE is the diameter of the circle. Show that $C\hat{A}D = B\hat{A}E$.



4. In the figure, AB is the diameter of the circle. $ABCD$ and CEF are straight lines. $B\hat{E}C = 40°$ and $F\hat{C}D = 150°$. Show that $BE = EF$.



5. In the figure, AB is the diameter of the circle. If $A\hat{D}E = D\hat{C}A$, prove that
   (a) $D\hat{E}B = 90°$;
   (b) $BCFE$ is a cyclic quadrilateral.



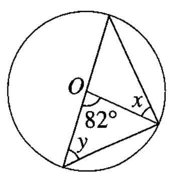
6. Draw a quadrilateral, $PQRS$, with diagonals PR and QS. If $P\hat{Q}R = 70°$, $P\hat{R}Q = 35°$ and $Q\hat{S}R = 75°$, prove that P, Q, R and S are concyclic, and find $P\hat{S}Q$.

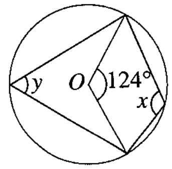
Summary

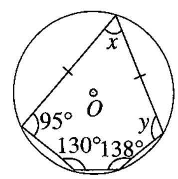
1. A straight line drawn from the centre of a circle to bisect a chord, which is not a diameter, is perpendicular to the chord.
2. In equal circles (or in the same circle) equal chords are equidistant from the centres (or centre).
3. An angle at the centre of a circle is twice that of any angle at the circumference subtended by the same arc.
4. Every angle subtended by the diameter of a circle at the circumference is a right angle.
5. Angles in the same segment of a circle are equal.
6. In a cyclic quadrilateral, the opposite angles are supplementary.
7. If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle.

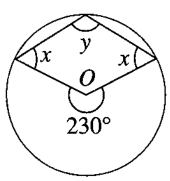
Review Questions 11

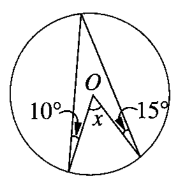
1. Given that O is the centre of the circle, find the values of x and/or y in each case.

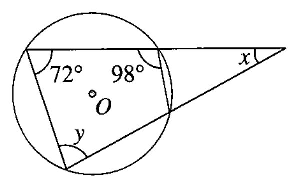
(a)


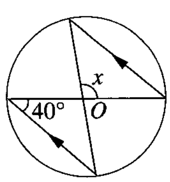
(b)


(c)


(d)


(e)


(f)


(g)


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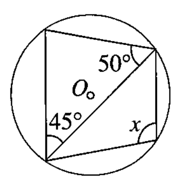
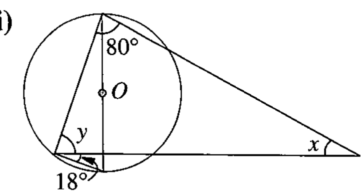
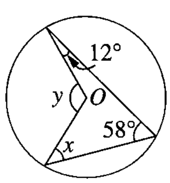
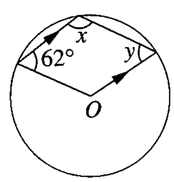
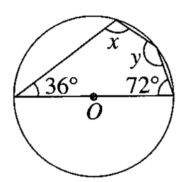


(i)


2. Given that O is the centre of the circle, find the values of x and/or y and/or z in the following figures.

(a)


(b)


(c)


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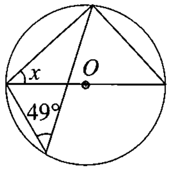

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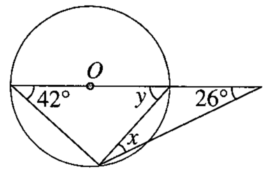

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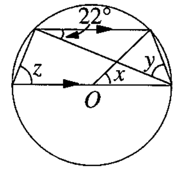
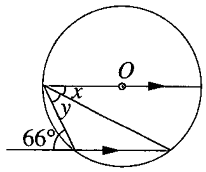


(g)



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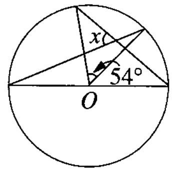
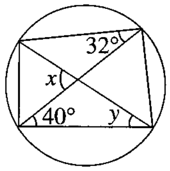
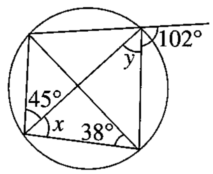


3. Find the angles marked x and/or y in each of the following figures.

(a)



(b)



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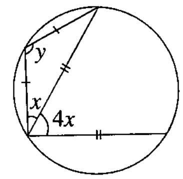
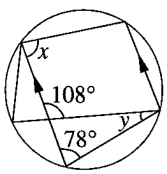


(d)



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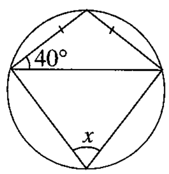

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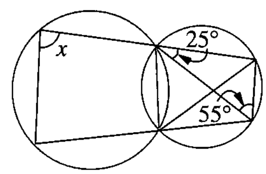

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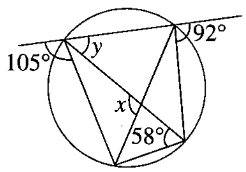

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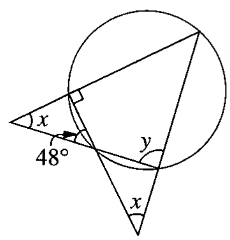
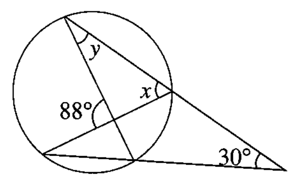
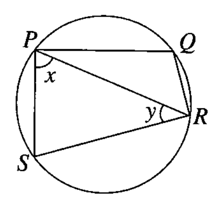


(i)



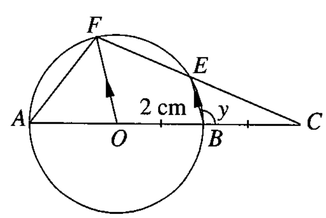
4. In the figure, $PQRS$ is a circle. Calculate $P\hat{Q}R$ in terms of x and y.



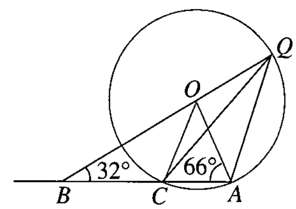
☆5. In the figure, O is the centre of the circle $ABEF$, FEC and $AOBC$ are two straight lines, BE // OF, $OB = BC = 2$ cm and $C\hat{B}E = y$. Find

(a) BE;

(b) $F\hat{A}O$ in terms of y.

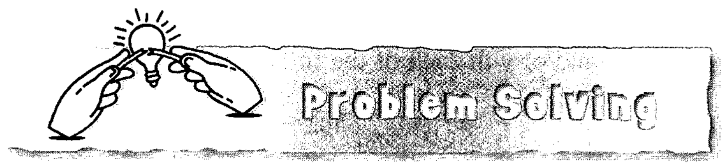


6. In the figure, O is the centre of the circle, BOQ and BCA are straight lines, $O\hat{A}C = 66°$ and $O\hat{B}C = 32°$. Calculate $C\hat{Q}A$ and $Q\hat{C}A$.



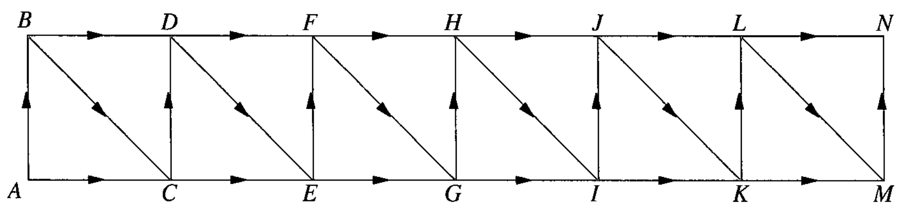
7. AOD is the diameter of a circle, centre O. The cyclic quadrilateral $ABCD$ is such that $C\hat{O}B$ is 42° and OD is parallel to BC. Calculate

(a) $C\hat{O}D$; (b) $B\hat{C}D$.



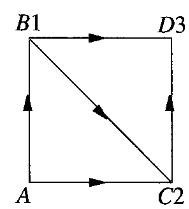
Example 12

The diagram shows a map of the streets in a housing estate. Given that all the streets are one-way, as indicated by the arrows, find the number of different possible routes to take from A to N.



Solution

We begin with an easier problem and draw a diagram showing the street joining A, B, C and D.



There is only one way to go from A to B. We indicate this by writing 1 beside B as shown. Similarly, we write 2 beside C since there are 2 ways to go from A to C ($A \to B \to C$ and $A \to C$) and 3 is written beside D because there are 3 ways to go from A to D ($A \to B \to C \to D$, $A \to B \to D$ and $A \to C \to D$).

Next, consider the diagram showing the streets connecting A, B, C, D, E and F.

There are five ways from A to E and eight ways from A to F.

From the above discussion, we can see the pattern required to obtain the total number of different possible routes from A to N. Note that number of ways at any one point is the sum of the previous two.

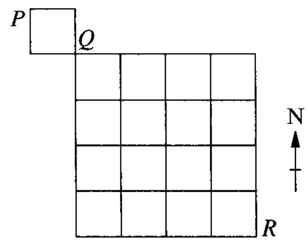
$$B1, C2, D3, E5, F8, G13, H21, \ldots, L144, M233, N377$$

Thus, there are altogether 377 different possible routes from A to N.

In solving the above problem, we used the strategies of *simplifying the problem, drawing diagrams, using symbols* and *looking for a pattern.*

Example 13

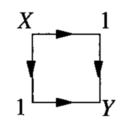
Look at the diagram. How many ways can a person move from P to R if he is only allowed to move south or east?



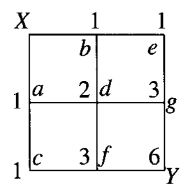
Solution

Method: Look for a pattern

First, consider a 1×1 square.
From the given figure, we can see that there is only 1 way to go from X to each adjacent corner. (The number at each corner denotes the number of ways to get to that corner from X.) Thus, there are $1 + 1 = 2$ ways to go from X to the opposite corner, Y.



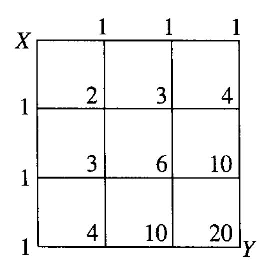
Next, consider a 2×2 square. From the figure below, we can see the following:



To get from X to:	No. of ways
a	1 ($X \to a$)
b	1 ($X \to b$)
c	1 ($X \to a \to c$)
d	2 ($X \to a \to d$; $X \to b \to d$)
e	1 ($X \to b \to e$)
f	3 ($X \to b \to d \to f$; $X \to a \to c \to f$; $X \to a \to d \to f$)
g	3 ($X \to b \to e \to g$; $X \to a \to d \to g$; $X \to b \to d \to g$)
Y	6 (try to figure out the 6 ways for yourself.)

Did you notice that the numbers on the diagram are related to the ones in Pascal's triangle?

For a 3×3 square, as shown in the figure, there are 20 ways to go from X to Y.

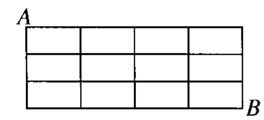


Can you show, following the pattern, that there are 70 possible paths to take for a 4×4 square?

A person can move from P to Q in 2 ways, and for each of these two ways he can move from Q to R in 70 ways. Thus, there are altogether $2 \times 70 = 140$ paths for him to take to get from P to R.

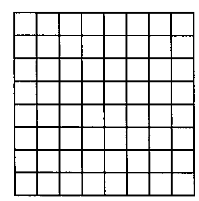
Problems

1. *The diagram shows a map of the streets in a city. If we want to go from A to B by the shortest distance, how many possible routes are there?*



2. *There are 10 students at a class gathering. If each student shakes hands with each of the other students once and only once, find the total number of handshakes by the end of the gathering.*

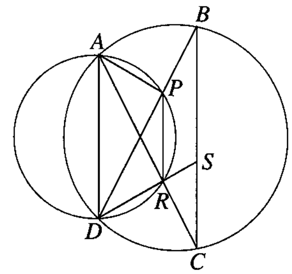
3. *How many squares (of various sizes) are there in this chessboard?*



Exploration

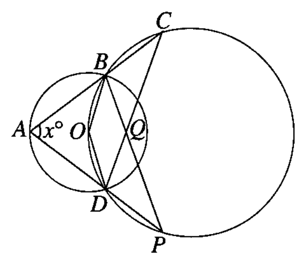
1. A square is inscribed in a circle of diameter x cm. Find the area of the square in terms of x.

2. In the diagram, $ABCD$ and $APRD$ are two circles intersecting at A and D. ARC and BPD are straight lines. DR produced meets BC at S. Prove that PR is parallel to BC.

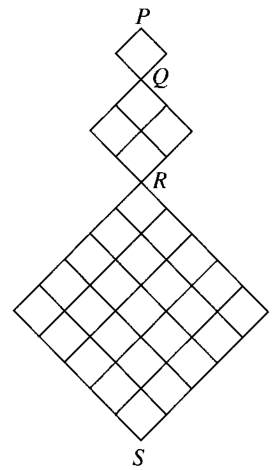


3. In the figure, O is the centre of the small circle ABD. ABC and ADP are straight lines, and BP and CD intersect at point Q. Given that $B\hat{A}D = x°$, find the value of each of the following angles in terms of x:

   (a) $A\hat{C}D$ **(b)** $A\hat{D}C$ **(c)** $P\hat{Q}C$



4. Find the number of possible paths from P to R and, hence, the number of possible paths from P to S, counting only paths along the lines in the given figure and going in the downward direction.

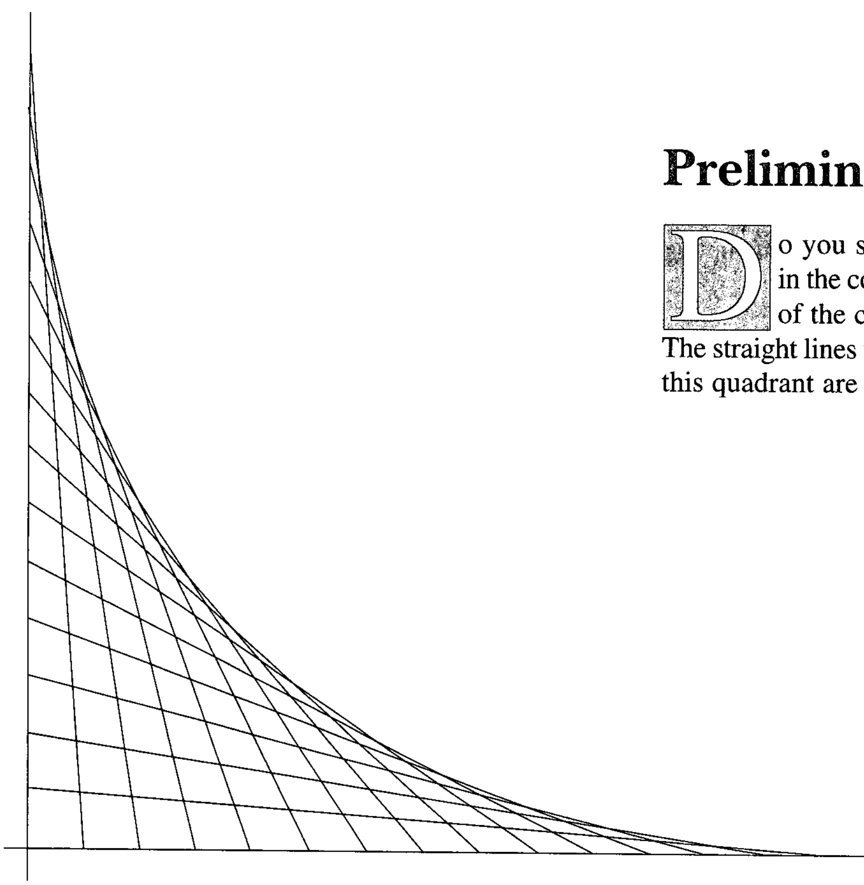


CHAPTER 12

Tangents and the Alternate Segment Theorem

In this chapter, you will learn how to

- △ use the properties of tangents of a circle to find angles in the circle;
- △ make use of the alternate segment theorem to find angles.



Preliminary Problem

Do you see any curved lines used in the construction of the quadrant of the circle?
The straight lines that are drawn to construct this quadrant are called tangents.

Tangents

A straight line cutting a circle at two distinct points is called a **secant**. In Fig. 12.1(a), A_1B_1 is a secant.

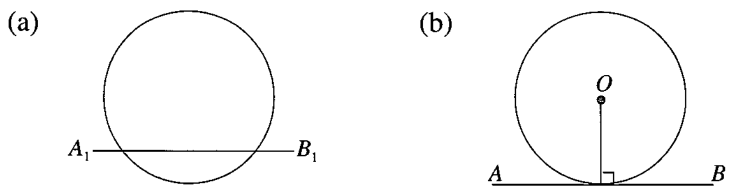


Fig. 12.1

If a straight line and a circle have only one point of contact, then that line is called a **tangent**. In Fig. 12.1(b), AB is a tangent and X is the point of contact.

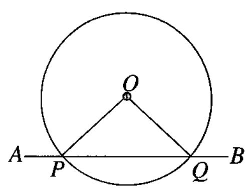


Fig. 12.2

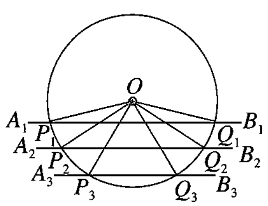


Fig. 12.3

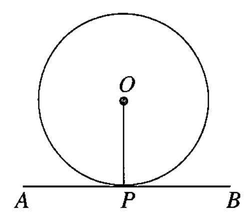


Fig. 12.4

Back in Time

For over a thousand years, Greek mathematicians used to consider the tangent of a circle as a line touching the circle at only one point. To them, a circle was a stagnant figure and no more than a geometrical shape. However, Isaac Newton, an Englishman, considered the tangent at a point to be the limiting position of a line passing through that point, as well as another point, which was running closer and closer to the former point. Newton's circle was a circular path which allowed motion and thus, a dynamic one. With his idea of a running point on the circle, he established the theory of calculus.

Fig. 12.2 shows a secant AB cutting the circle at points P and Q. Now $\triangle POQ$ is an isosceles triangle and $O\hat{P}Q = O\hat{Q}P$. Hence $O\hat{P}A = O\hat{Q}B$.

Fig. 12.3 shows a series of secants, A_1B_1, A_2B_2 and A_3B_3, cutting the circle at various positions, P_1Q_1, P_2Q_2 and P_3Q_3, respectively.

As AB moves away from the centre O, P and Q become closer to each other. Eventually, the two points will coincide in the single point P, as in Fig. 12.4. Since the angles OPA and OQB are equal, it follows that $O\hat{P}A = O\hat{P}B$ in Fig. 12.4. Since $O\hat{P}A$ and $O\hat{P}B$ are supplementary angles, $O\hat{P}A = O\hat{P}B = 90°$.

In general, we have the following property:

A tangent to a circle is perpendicular to the radius drawn to the point of contact.

This property can be abbreviated as **tan $\perp$ rad**.

Tangents from an External Point

Fig. 12.5 shows that P is a point outside the circle, with centre O. PA and PB are two tangents drawn from P to touch the circle at A and B respectively. We can find that

(a) $AP = BP$;
(b) $A\hat{P}O = B\hat{P}O$;
(c) $A\hat{O}P = B\hat{O}P$.

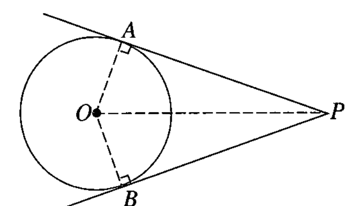


Fig. 12.5

Working

In $\triangle$s AOP and BOP,

$O\hat{A}P = O\hat{B}P = 90°$ *(tan ⊥ rad.)*

$OA = OB$ *(radii of the same circle)*

OP is common.

$\therefore$ $\triangle AOP$ and $\triangle BOP$ are congruent **(RHS Property)**

$\therefore$ $AP = BP$

$A\hat{P}O = B\hat{P}O$ and $A\hat{O}P = B\hat{O}P$

We can, therefore, conclude that:

(i) tangents drawn to a circle from an external point are equal;
(ii) the tangents subtend equal angles at the centre;
(iii) the line joining the external point to the centre of the circle bisects the angle between the tangents.

We can also prove the properties by using reflection. Can you find out how this is done with reference to Fig. 12.5?

Since every angle in a semicircle is a right angle, we can make use of this property to construct tangents to a circle from an external point P.

Given a point P outside a circle, with centre O, we are required to construct tangents from P to the circle.

Join OP and bisect it at point M.
With radius OM and centre M, draw a circle, passing through O and P, to cut the given circle at A and B. Join AP and BP.

Since OP is a diameter of the circle, centre M,

$O\hat{A}P = O\hat{B}P = 90°$ *(rt. ∠ in a semicircle).*

But OA and OB are radii of the given circle. Therefore, AP and BP are **tangents** to the given circle.

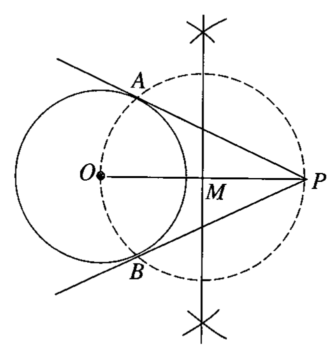
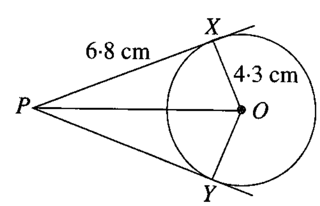


Fig. 12.6

Example

In the figure, PX and PY are tangents to the circle, centre O. Given that PX = 6.8 cm and OX = 4.3 cm, calculate

(a) $O\hat{P}X$; *(b)* OP;
(c) the area of the quadrilateral PXOY.



Solution

(a) $O\hat{X}P = 90°$ *(tan ⊥ rad.)*

In $\triangle OPX$, $\tan O\hat{P}X = \dfrac{OX}{PX} = \dfrac{4.3}{6.8}$

$\therefore$ $O\hat{P}X \approx 32.3°$

(b) In $\triangle OPX$, $OP^2 = 6.8^2 + 4.3^2 = 64.73$

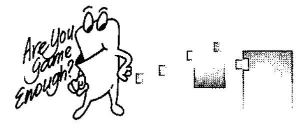
$OP \approx 8.05$ cm

Alternatively, in $\triangle OPX$, $\sin 32.3° = \frac{4.3}{OP}$

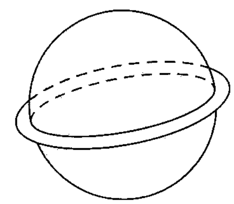
$$OP = \frac{4.3}{\sin 32.3°} \approx 8.05 \text{ cm}$$

(c) Area of $\triangle OPX = \frac{1}{2}(6.8)(4.3) = 14.62$ cm^2

$\therefore$ area of $PXOY = 2(14.62) = 29.24$ cm^2.



Imagine there is a wire long enough to go round the equator of the Earth, which we will assume to be a perfect sphere of radius 6 400 km. This wire is then stretched tightly around the surface of the earth. If we were to place the wire 1 m off the ground all the way round, how much longer must we increase the length of the wire?

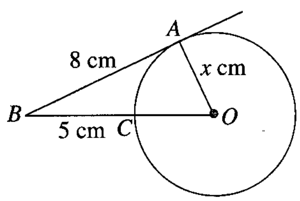


Example 2

In the figure, AB is a tangent to the circle, with centre O. Given that AB = 8 cm, BC = 5 cm and OA = x cm, find

(a) the value of x; *(b) $A\hat{O}B$;*

(c) the area bounded by AB, BC and the arc AC.



Solution

(a) $O\hat{A}B = 90°$ *(tan ⊥ rad.)*

$OB = (x + 5)$ cm

Now, $(x + 5)^2 = x^2 + 8^2$

$x^2 + 10x + 25 = x^2 + 64$

$10x = 64 - 25 = 39$

$\therefore x = 3.9$

(b) In $\triangle AOB$, $\tan A\hat{O}B = \frac{8}{3.9}$

$\therefore A\hat{O}B = 64.01° = 64.0°$ (to 1 dec. place)

(c) Area of $\triangle AOB = \frac{1}{2}(8)(3.9)$ cm^2 = 15.6 cm^2

Area of minor sector $AOC = \frac{64.01}{360} \times \pi(3.9)^2$ cm^2 = 8.496

= 8.50 cm^2 (to 3 sig. figures)

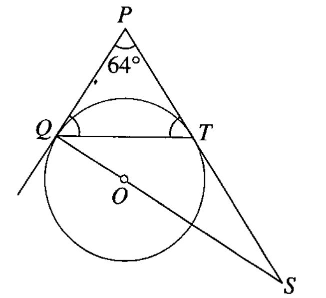
$\therefore$ area bounded by AB, BC and the arc AC = (15.6 − 8.496) cm^2

= 7.104

= 7.10 cm^2 (to 3 sig. figures)

Example 3

In the figure, O is the centre of the circle. PQ and PT are tangents to the circle at Q and T respectively. PT produced meets QO produced at S. Given that $Q\hat{P}T = 64°$, calculate $S\hat{Q}T$.



Solution

$PQ = PT$ *(tangents from an external point)*

Hence $P\hat{Q}T = P\hat{T}Q$ *(base ∠s of isos △PQR)*

$P\hat{Q}T = \frac{180° - 64°}{2}$ *(∠ sum of a △)*

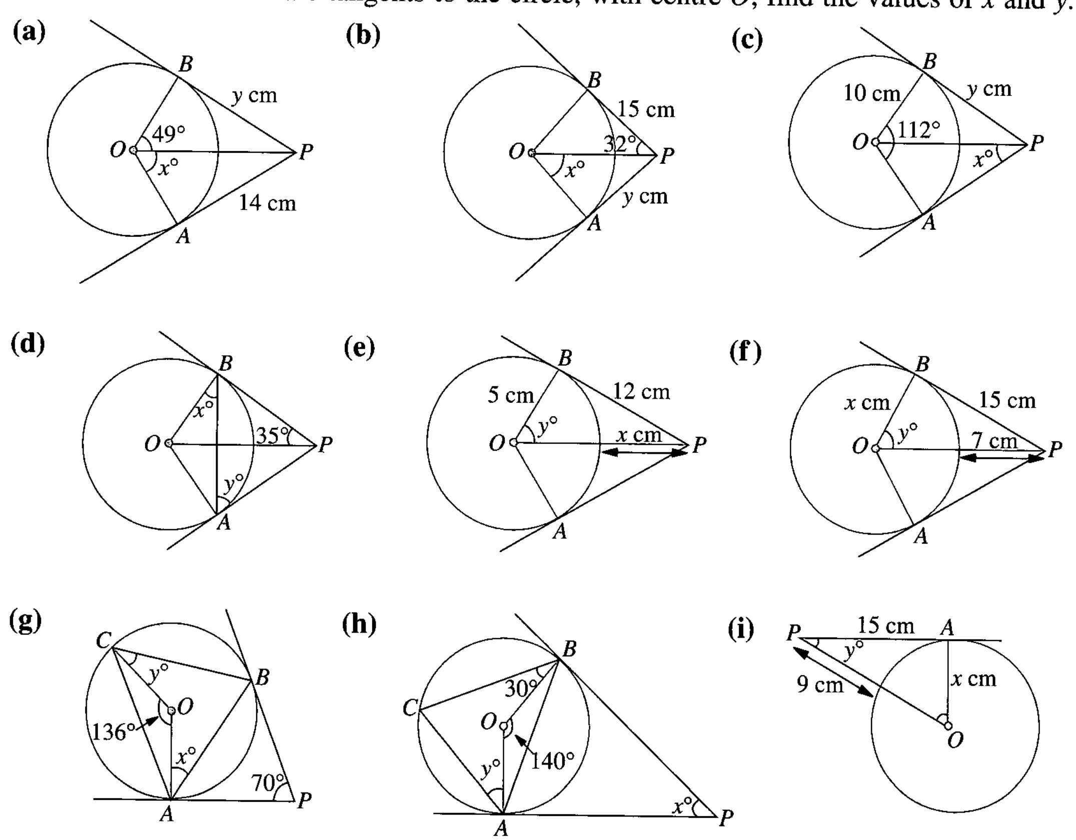
$= 58°$

$O\hat{Q}P = 90°$ *(tan ⊥ rad.)*

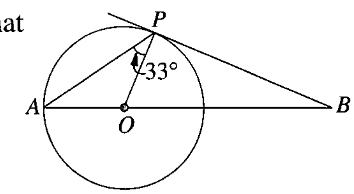
$S\hat{Q}T = 90° - 58° = 32°$

Exercise 12a

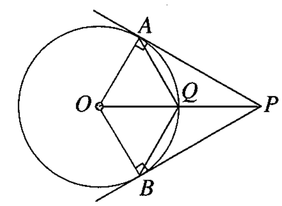
1. Given that PA and PB are tangents to the circle, with centre O, find the values of x and y.



2. In the diagram, BP is a tangent to the circle, centre O. Given that $A\hat{P}O = 33°$, find $P\hat{B}A$.



3. In the figure, O is the centre of the circle and PA and PB are tangents. Find



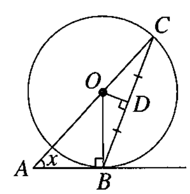
   (a) $A\hat{O}P$ when $A\hat{B}P = 54°$;
   (b) $A\hat{Q}O$ when $A\hat{P}O = 42°$;
   (c) $Q\hat{A}P$ when $A\hat{Q}P = 126°$;
   (d) $A\hat{P}B$ when $O\hat{A}Q = 73°$.

4. PQ is a chord and O is the centre of a circle. If $P\hat{O}Q = 84°$, find the obtuse angle between PQ and the tangent at P.

5. O is the centre of a circle. The tangents from a point T touch the circle at A and B. If $A\hat{O}T = 51°$, find $B\hat{A}T$.

6. X, Y and Z are three points on a circle and $Y\hat{X}Z = 43°$. The tangents at Y and Z meet at W. Find $Y\hat{W}Z$.

7. In the diagram, AB is a tangent to the circle, centre O. D is the mid-point of the chord BC. Given that $B\hat{A}C = x$, find $C\hat{O}D$ in terms of x.

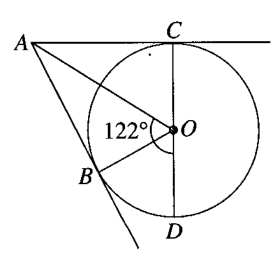


8. L, M and N are three points on a circle. The tangents at L and M intersect at P. If $L\hat{P}M = 58°$, calculate $L\hat{N}M$.

9. PA is the diameter of a circle and PT is a tangent. S is a point on the circle, such that $S\hat{P}T = 46°$. Find $P\hat{A}S$. Hence, calculate $P\hat{R}S$, where R is any other point on the minor arc, PS, of the circle.

10. The tangent from a point P touches a circle at N. Given that the radius of the circle is 5.6 cm and that P is 10.6 cm away from the centre, find the length of the tangent PN.

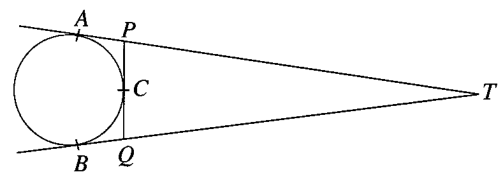
11. In the diagram, AB and AC are tangents to the circle at B and C respectively. O is the centre of the circle and $A\hat{O}D = 122°$. Find $B\hat{A}C$.



12. A point T is 9.1 cm away from the centre of a circle. The tangent from T is 8.4 cm long. Find the radius of the circle.

13. Two circles have the same centre, their radii being 12 cm and 25.5 cm. A tangent to the inner circle cuts the outer circle at points H and K. Find the length of HK.

14. In the figure, the straight lines TPA, TQB and PCQ are the tangents to the circle. Show that $TP + PC = TQ + QC$.



The Alternate Segment Theorem (Optional)

In Fig. 12.7, MXN is the tangent to the circle at X. The major segment XPY is known as the **alternate segment** of $Y\hat{X}N$, since it is on the other side of the chord XY from $Y\hat{X}N$.

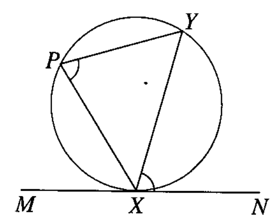


Fig. 12.7

In Fig. 12.8, MXN is the tangent to the circle at X. The minor segment XQR is known as the alternate segment of $R\hat{X}N$.

Observing the angles in the two figures, what can you find?

In fact, in Fig. 12.7, $Y\hat{X}N = X\hat{P}Y$ and $P\hat{X}M = P\hat{Y}X$. Also, in Fig. 12.8, $R\hat{X}N = R\hat{Q}X$ and $Q\hat{X}M = Q\hat{R}X$.

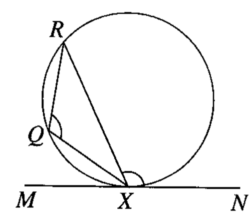


Fig. 12.8

The following shows the steps taken to establish the result:

Given, in Fig. 12.9, that ATB is the tangent to the circle at T and O is the centre, we are required to show that $P\hat{T}B = P\hat{Q}T$.

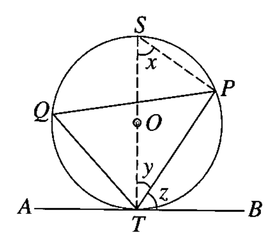


Fig. 12.9

Working

Join TO and produce to get the diameter TOS.

$S\hat{P}T = 90°$ *(rt. ∠ in a semicircle)*

$\therefore\ \hat{x} + \hat{y} = 90°$ *(∠ sum of a △)*

But $\hat{y} + \hat{z} = 90°$ *(tan ⊥ rad.)*

$\therefore\ \hat{x} = \hat{z}$

$\therefore\ P\hat{T}B = P\hat{S}T$

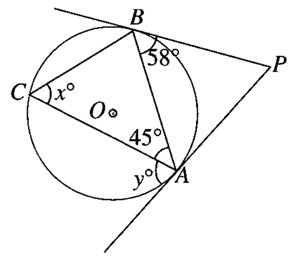
But $P\hat{S}T = P\hat{Q}T$ *(∠s in the same segment)*

$\therefore\ P\hat{T}B = P\hat{Q}T$

The **alternate segment theorem** states that an angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

Example 4

Given that PA and PB are tangents to a circle, with centre O, and that $A\hat{B}P = 58°$ and $B\hat{A}C = 45°$, find the values of x and y.



Solution

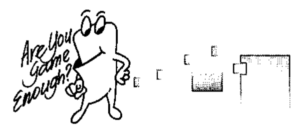
$x° = 58°$ (∠ in alt. segment)

$P\hat{A}B = 58°$ (base ∠ of isos △PAB)

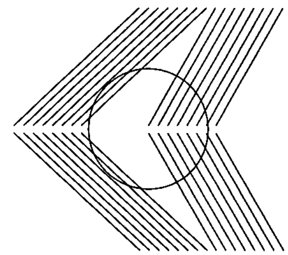
$\therefore\ y° = 180° - 58° - 45°$ (adj. ∠s on a str. line)

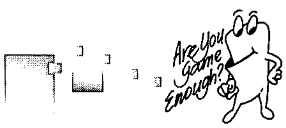
$= 77°$

Hence, $x = 58$ and $y = 77$.



Do you see a perfect circle in the figure below?

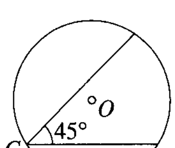




The diagram below shows the plan of a circular hall of a jewellery exhibition. C is a hidden video camera which scans an angle of 45°. How many more such video cameras must be installed on the walls of the hall so that they will cover the entire hall? Indicate the position where each video camera must be mounted.

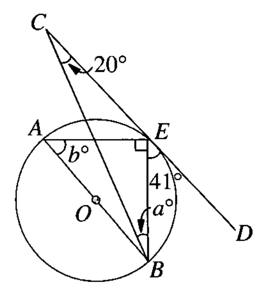
How many video cameras do you need if each one can scan an angle of

(a) 35° (b) 60°
(c) 90° (d) 100°?



Example 5

In the figure, AB is a diameter of the circle ABE and CD is the tangent at E. If $E\hat{C}B = 20°$ and $B\hat{E}D = 41°$, find the value of $A\hat{B}C$.



Solution

$b = 41$ (∠ in alt. segment)

$41° = 20° + a°$ (ext. ∠ of △BCE)

$\therefore\ a = 21$

$A\hat{E}B = 90°$ (rt. ∠ in a semicircle)

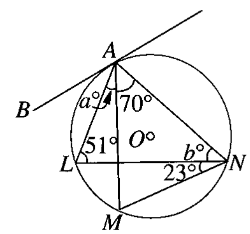
$b° + a° + A\hat{B}C = 90°$

$\therefore\ A\hat{B}C = 90° - 21° - 41°$

$= 28°$

Example 6

In the figure, AB is the tangent to the circle at A. $A\hat{L}N$ = 51°, $N\hat{A}M$ = 70° and $L\hat{N}M$ = 23°. Calculate $B\hat{A}L$.



Solution

$a = 23$ (*∠s in the same segment*)

$b° + 51° + a° + 70° = 180°$ (*∠ sum of a △*)

$\therefore \quad b° = 180° - 51° - 23° - 70°$

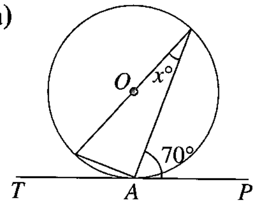
$\therefore \quad b = 36$

$B\hat{A}L = b° = 36°$ (*∠ in alt. segment*)

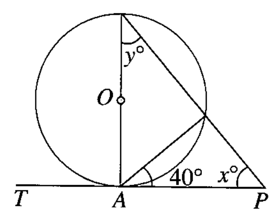
Exercise 12b

1. Given that PAT is a tangent to the circle, with centre O, find the values of x and/or y and/or z.

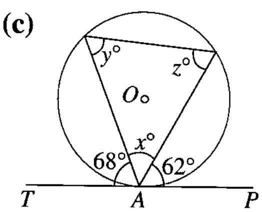
(a)



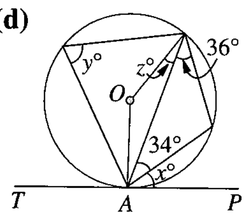
(b)



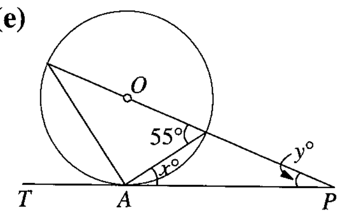
(c)



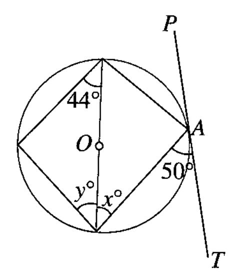
(d)



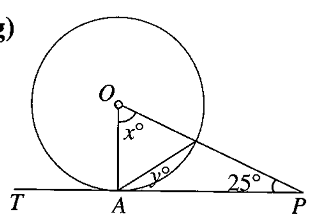
(e)



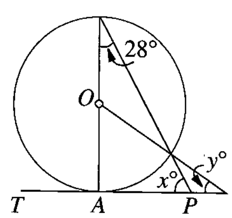
(f)



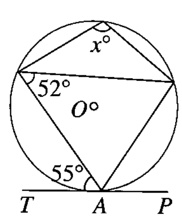
(g)



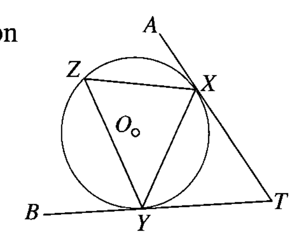
(h)



(i)



2. In the figure, TXA and TYB are tangents to the circle and Z is a point on the major arc XY.

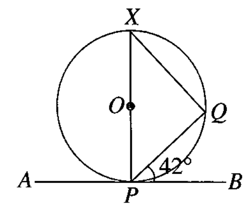


(a) Given that $X\hat{Y}Z = 48°$ and $Y\hat{X}Z = 70°$, calculate $X\hat{T}Y$.

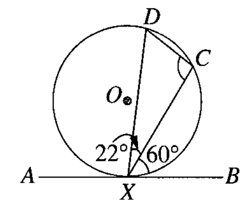
(b) Given that $X\hat{Z}Y = 60°$ and $Z\hat{Y}B = 70°$, calculate $Z\hat{X}A$.

(c) Given that $Z\hat{X}A = 75°$ and $Z\hat{Y}B = 65°$, calculate $X\hat{T}Y$.

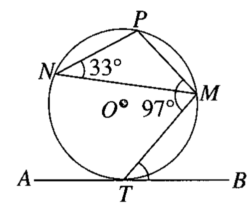
3. In the diagram, AB is the tangent to the circle at P and PX is the diameter. Given that $B\hat{P}Q = 42°$, find $P\hat{Q}X$, $P\hat{X}Q$ and $X\hat{P}Q$.



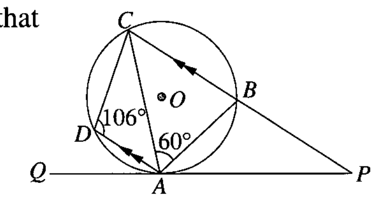
4. In the diagram, O is the centre of the circle. AB is the tangent to the circle at X, $C\hat{X}B = 60°$ and $C\hat{X}D = 22°$. What is the size of $X\hat{C}D$?



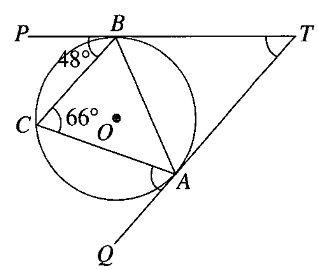
5. In the diagram, ATB is the tangent to the circle at point T. Given that $P\hat{N}M = 33°$ and $T\hat{M}P = 97°$, find $M\hat{T}B$.



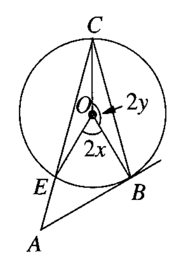
6. In the diagram, PQ is the tangent to the circle at A. Given that AD // BC, $A\hat{D}C = 106°$ and $B\hat{A}C = 60°$, find $Q\hat{A}D$ and $A\hat{P}C$.



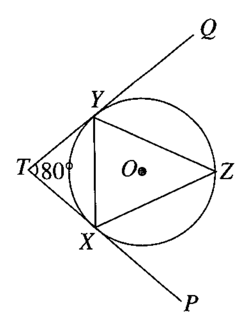
7. In the diagram, TP and TQ are two tangents drawn from a point T outside the circle to touch it at B and A respectively. If $A\hat{C}B = 66°$ and $C\hat{B}P = 48°$, find $P\hat{T}Q$ and $C\hat{A}Q$.



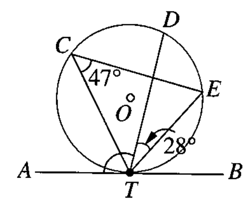
8. In the diagram, AB is a tangent to the circle, centre O. Given that $E\hat{O}B = 2x$ and $B\hat{O}C = 2y$, find $B\hat{A}C$ in terms of x and y.



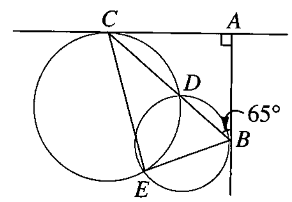
9. In the diagram, TP and TQ are two tangents touching the circle, with centre O, at X and Y respectively. If $P\hat{T}Q = 80°$, find $X\hat{Z}Y$.



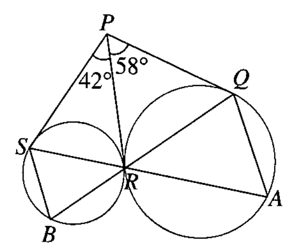
10. In the diagram, AB is a tangent to the circle, meeting it at T, $T\hat{C}E = 47°$ and $D\hat{T}E = 28°$. Find $A\hat{T}D$.



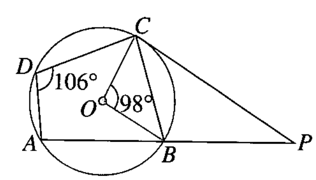
11. In the diagram, AB and AC are tangents, $B\hat{A}C = 90°$ and $A\hat{B}C = 65°$. Find $B\hat{E}C$.



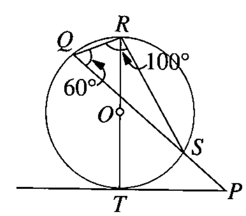
12. In the figure, PQ, PR and PS are tangents, $R\hat{P}S = 42°$ and $Q\hat{P}R = 58°$. Find the sum of $Q\hat{A}R$ and $R\hat{B}S$.



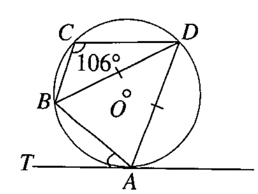
13. In the diagram, O is the centre of the circle and PC is the tangent to the circle at C. Given that $A\hat{D}C = 106°$ and $B\hat{O}C = 98°$, find $A\hat{P}C$.



14. In the diagram, RT is the diameter of the circle, centre O. PT is the tangent to the circle at T. Given that $R\hat{Q}S = 60°$ and $Q\hat{R}S = 100°$, find $S\hat{P}T$.



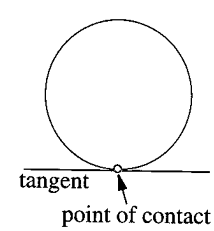
15. In the diagram, $ABCD$ is a cyclic quadrilateral and TA is the tangent to the circle at A. If $BD = DA$ and $B\hat{C}D = 106°$, find $B\hat{A}T$.



Summary

1. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

2. Tangents drawn to a circle from an external point are equal; they subtend equal angles at the centre, and the line joining the external point to the centre of the circle bisects the angle between the tangents.

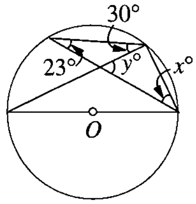


3. An angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

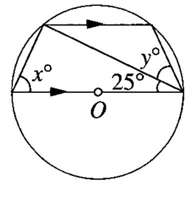
Review Questions 12

1. Given that O is the centre of the circle, find the values of x and y in each case.

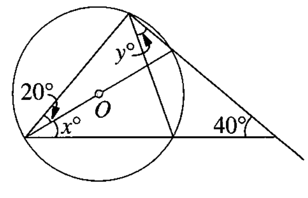
(a)



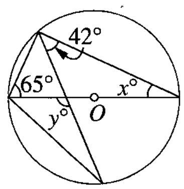
(b)



(c)



(d)



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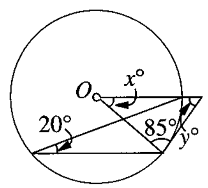

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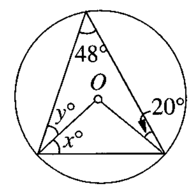

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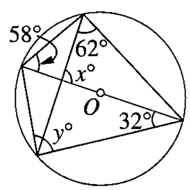
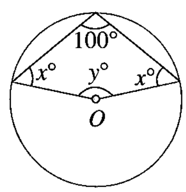
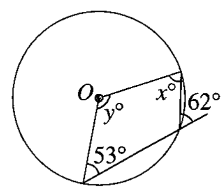


(h)

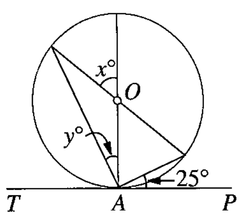


(i)

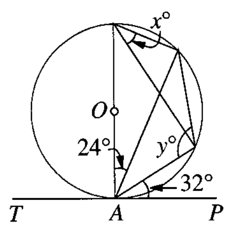


2. Given that PAT is a tangent at A to each of the following circle, with centre O, find the values of x and y in each case.

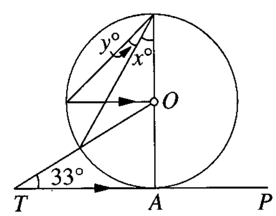
(a)



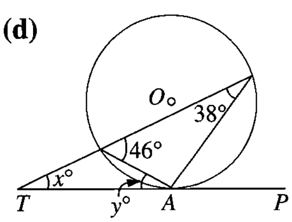
(b)



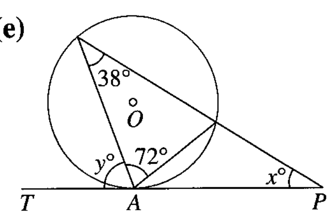
(c)



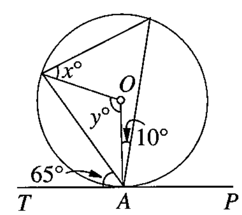
(d)



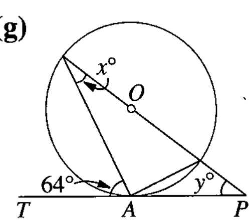
(e)



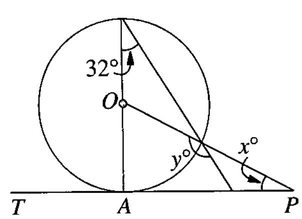
(f)



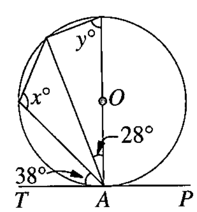
(g)



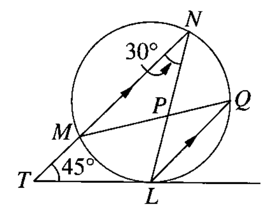
(h)



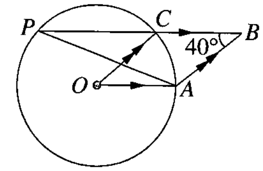
(i)



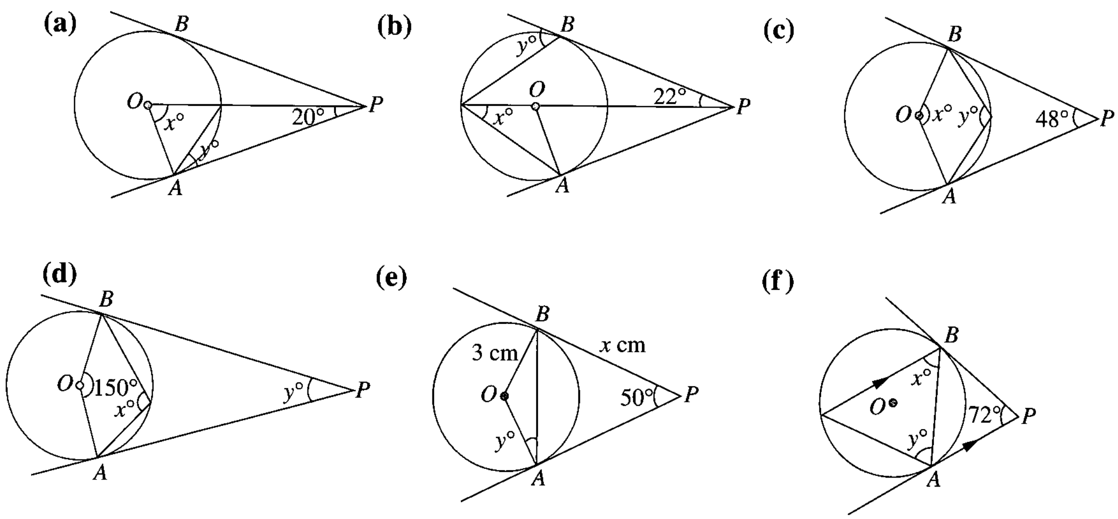
3. In the figure, TL is a tangent to the circle at L. TMN is a straight line. $L\hat{N}M = 30°$ and $L\hat{T}M = 45°$. Calculate $L\hat{M}T$ and $L\hat{P}M$.



4. In the figure, O is the centre of the circle. $OABC$ is a parallelogram and BCP is a straight line. Find $O\hat{A}P$.



5. Given that PA and PB are tangents to the circle, with centre O, find the values of x and y in each case.

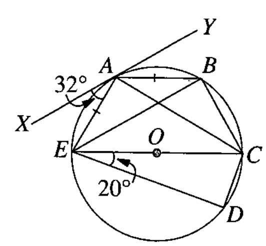


6. Given a cyclic quadrilateral whose diagonals AC and BD intersect at E, such that $D\hat{A}C = a°$, $C\hat{A}B = 2a°$ and $A\hat{E}B = 60°$, prove that $D\hat{C}A = 2A\hat{C}B$. If $A\hat{B}D = 2B\hat{D}C$, calculate the value of a.

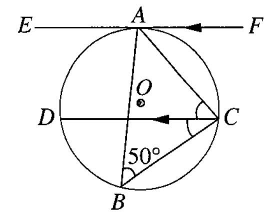
7. The tangents from T touch a circle at X and Y. A chord YZ, parallel to TX, is drawn. If $Y\hat{X}T = 60°$, calculate $Y\hat{X}Z$.

8. The angles of a triangle are 50°, 60° and 70°. A circle touches the sides at P, Q and R. Calculate the angles of $\triangle PQR$.

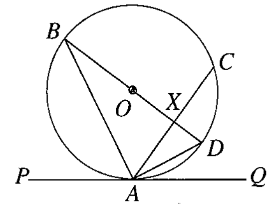
9. In the figure, EC is the diameter of the circle, and $AE = AB$. The line XAY is the tangent to the circle at A. Given that $X\hat{A}E = 32°$ and that $C\hat{E}D = 20°$, calculate $A\hat{E}D$, $E\hat{C}D$, $B\hat{A}C$ and $B\hat{E}D$.



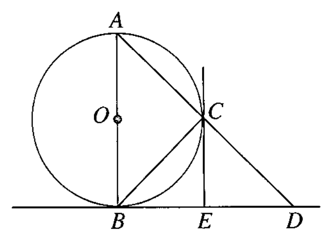
10. In the figure, EAF is the tangent to the circle at A. CD bisects $A\hat{C}B$ and CD // FE. If $A\hat{B}C = 50°$, find $D\hat{C}B$.



11. PQ is a tangent to the circle $ABCD$. If $Q\hat{A}B = P\hat{A}C$ and $C\hat{A}D = Q\hat{A}D = 30°$, show that
   (a) $AD = CD$;
   (b) BDA is a semicircle;
   (c) $B\hat{X}C = 90°$.

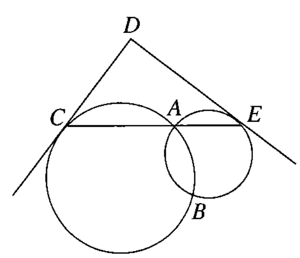


12. In the figure, BD and CE are tangents to the circle, in which AB is a diameter and ACD is a straight line. Show that $A\hat{B}C = E\hat{C}D$ and $BE = ED$.

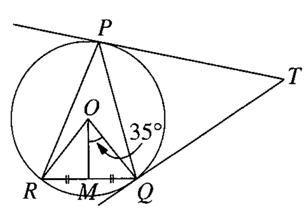


Problem Solving

1. Two circles intersect at A and B. CD and DE are tangents to the circles at C and E, and CAE is a straight line. Prove that $CDEB$ is a cyclic quadrilateral.



2. A circle, with centre O, passes through P, Q and R. The tangents to the circle at P and Q meet at T. The midpoint of RQ is M. $Q\hat{O}M = 35°$.
   (a) Calculate
      (i) $O\hat{R}M$; (ii) $R\hat{P}Q$.
   (b) Given, also, that $P\hat{Q}O = 15°$, calculate $P\hat{T}Q$, giving a reason for each step of this calculation. (*C*)

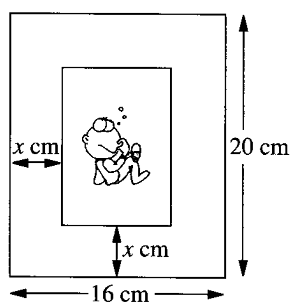


Revision Exercise III No. 1

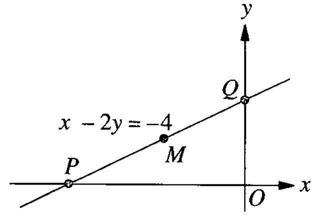
1. **(a)** Find the value of $4\frac{1}{2} - \frac{2}{5} \div \frac{7}{10}$, giving your answer as a fraction.
   (b) Find the value of 0.321×0.654, giving your answer correct to 4 decimal places.

2. Make n the subject of the formula $m = \frac{2n - 3}{n + 4}$. Calculate
   (a) the value of m when $n = \frac{9}{2}$;
   (b) the value of n when $m = -\frac{1}{4}$.

3. In the diagram, the width of the border of the picture is x cm. The picture has an area of 160 cm^2. Form an equation in x and show that it simplifies to $x^2 - 18x + 40 = 0$. Solve this equation, giving your answers correct to two significant figures. Hence, write down the width of the border.

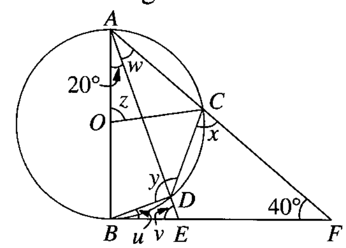


4. The line $x - 2y = -4$ cuts the x-axis and y-axis at P and Q respectively. M is the mid-point of PQ. Find the
   (a) coordinates of the points P and Q;
   (b) coordinates of M, and, hence, write down the equation of the line through M which is parallel to the y-axis;
   (c) equation of the line which is the reflection of the line PQ in the x-axis.

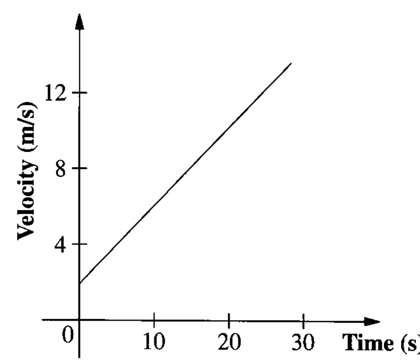


5. Given that $3 \leqslant x \leqslant 6$ and that $-9 \leqslant y \leqslant -1$, find the
   (a) least possible value of $x + y$;
   (b) greatest possible value of $x^2 - y^2$.

6. In the diagram, AB is the diameter of the circle, centre O. BEF is tangent to the circle, $B\hat{A}E = 20°$ and $A\hat{F}B = 40°$. Find u, v, w, x, y and z in the figure.

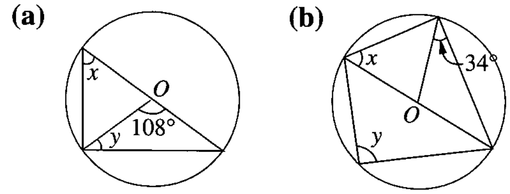


7. **(a)** The velocity-time graph shows a motorist travelling with constant acceleration.
   (i) What is the initial velocity?
   (ii) What is the acceleration?
   (iii) What is the distance moved in the first 30 seconds?



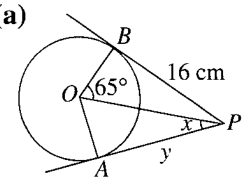
   (b) Copy and complete each of the following number sequence:
   (i) 5, 6, 8, 11, 15, ______, ______
   (ii) 6, 7, 9, 13, 21, ______, ______

8. Given that O is the centre of the circle, find the values of x and y in each case.

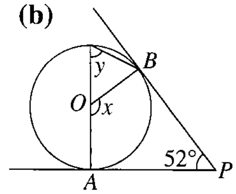


9. Given that O is the centre of the circle, and that PA and PB are tangents to the circle, find the values of x and y in each case.

(a)



(b)



10. If y varies directly as x^2 and $y = 75$ when $x = 15$, express y in terms of x and find the value of y when $x = 8$.

Revision Exercise III No. 2

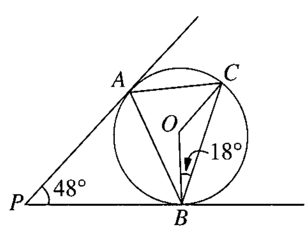
1. (a) Express 0.038 596
   (i) as a decimal, correct to 3 decimal places;
   (ii) as a decimal, correct to 4 significant figures;
   (iii) exactly, in the form $A \times 10^n$, where $1 \leqslant A < 10$ and n is the integer.

   (b) Find the exact value of $\frac{0.04 \times 0.49}{56}$, giving your answer as a decimal.

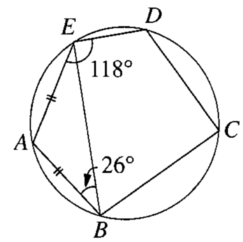
   (c) A man buys an article for \$520 and sells it later for \$572. Calculate his percentage gain on the transaction.

2. The weight of a body (W) varies directly as its height (h). Given that $W = 96$ when $h = 3$, find the value of W when $h = 4$. Express W in terms of h.

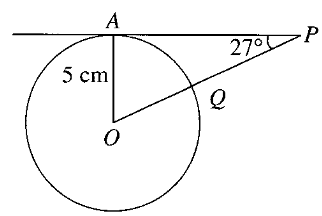
3. PA and PB are the tangents to the circle whose centre is O. Given that $A\hat{P}B = 48°$ and $O\hat{B}C = 18°$, calculate (a) $B\hat{A}C$; (b) $A\hat{B}C$.



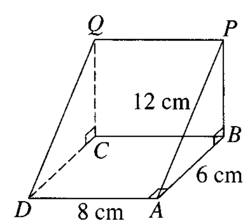
4. (a) A, B, C, D and E are the points on a circle. Given that $AB = AE$, $A\hat{B}E = 26°$ and $AED = 118°$, calculate
   (i) $B\hat{A}E$;
   (ii) $B\hat{C}D$.



   (b) In the diagram, O is the centre of the circle of radius 5 cm. If $O\hat{P}A = 27°$, calculate
   (i) AP;
   (ii) PQ.



5. The diagram shows a right triangular prism with $A\hat{B}P = 90°$ and $ABCD$ lying on a horizontal table. If $AB = 6$ cm, $AD = 8$ cm and $AP = 12$ cm, calculate (a) $P\hat{A}B$, (b) PB, (c) $P\hat{D}B$.



6. (a) Solve the equation $4x^2 - 3x - 7 = 0$, giving your answer correct to 2 decimal places.

   (b) Make k the subject of the formula
   $$a = \frac{2k - 5}{5 - 6k}.$$

7. Evaluate (a) $9^{\frac{1}{2}} + (0.25)^{\frac{1}{2}}$;

   (b) $9^{2.5} \div 27^{\frac{1}{3}}$.

8. Given that $y = \frac{5}{x} + 2x - 3$, copy and complete the following table.

x	0.5	1	2	4	5	6	7
y	8						11.7

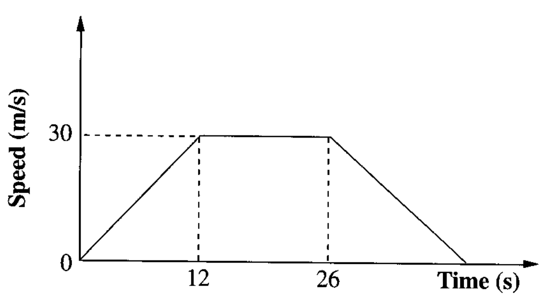
Taking 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = \frac{5}{x} + 2x - 3$ from $x = 0.5$ to $x = 7$.

(a) By drawing a tangent, find the gradient of the graph at the point where $x = 3$.
(b) Using your graph, estimate the solutions to the equation

(i) $\frac{5}{x} + 2x - 8 = 0$;

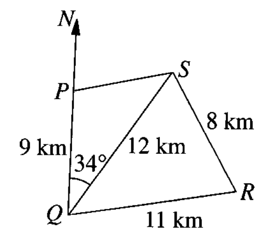
(ii) $\frac{5}{x} + x - 6 = 0$.

9. The diagram shows the speed-time graph of a moving object.
(a) Calculate the acceleration of the moving object during the first 12 seconds.
(b) Calculate the distance the moving object travels from rest before it begins to decelerate.
(c) Given that the moving object decelerates at 2 m/s², calculate the total time taken for the journey.
(d) Calculate the average speed of the moving object for the whole journey.



10. P, Q, R and S are four points on level ground. P is due north of Q and the bearing of S from Q is 034°. Given that $PQ = 9$ km, $QS = 12$ km, $QR = 11$ km and $RS = 8$ km, calculate

(a) PS;
(b) the bearing of R from Q;
(c) the bearing of S from R.

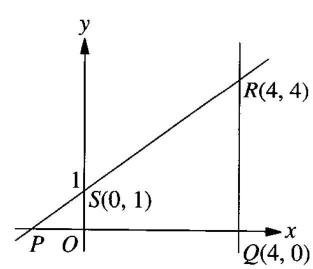


Revision Exercise III No. 3

1. (a) Write the following fractions in order of size, starting with the smallest: $\frac{3}{5}, \frac{7}{10}, \frac{5}{8}$.
(b) Evaluate (i) $18 - (6 - 2)$;
(ii) $(18 - 6) - 2$.

2. Solve the equations:
(a) $5x + 3 = 28$ (b) $6(x - 5) = 24$
(c) $2x^2 + 5x - 12 = 0$

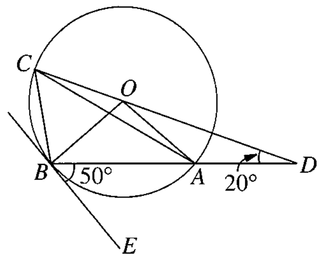
3. In the diagram, Q is (4, 0), R is (4, 4) and S is (0, 1). PSR is a straight line.
(a) Find the gradient of PR.
(b) Write down the equation of PR and the coordinates of P.
(c) Find the area of the quadrilateral $QRSO$.
(d) Calculate the length of PR.



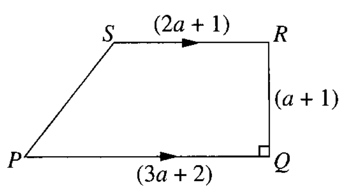
4. (a) Given that y is inversely proportional to x, and that $y = 4$ when $x = \frac{1}{2}$, find the value of y when $x = 2\frac{1}{2}$.
(b) Given that $z^2 \propto y$ and that $z = 9$ when $y = 27$, find y in terms of z. Also find z when $y = 48$, and y when $z = 6$.

5. In the diagram, EB is the tangent at B to the circle, centre O. DAB and DOC are straight lines.
Given that $A\hat{B}E = 50°$ and $C\hat{D}A = 20°$,

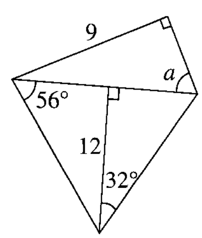
(a) state the value of $A\hat{C}B$;
(b) find the value of $A\hat{O}B$;
(c) show that $AD = AO$.



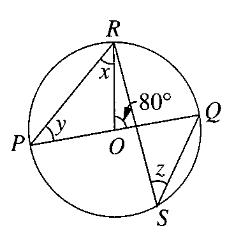
6. In the diagram, $P\hat{Q}R = 90°$ and SR is parallel to PQ. The side $PQ = (3a + 2)$ cm, $QR = (a + 1)$ cm and $RS = (2a + 1)$ cm.
Find, in terms of a, an expression for the area of the trapezium $PQRS$. Given that the area of the trapezium is 9 cm^2, form an equation in a and show that it simplifies to $5a^2 + 8a - 15 = 0$. Solve this equation and hence find the length of PQ, giving your answer in cm and correct to two decimal places.



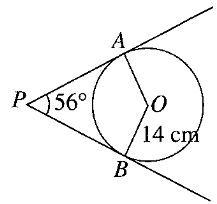
7. (a) Find the value of a in the diagram.



(b) O is the centre of the circle. Find the values of x, y and z in the diagram.

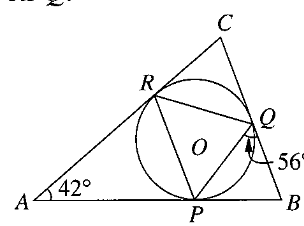


8. In the figure, PA and PB are tangents to the circle, centre O. Given that the radius of the circle is 14 cm and $A\hat{P}B = 56°$, calculate the area of the shaded region.



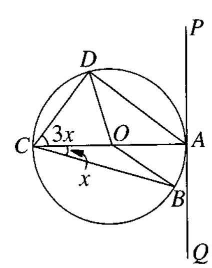
9. (a) The tangents to a circle, centre O, at P, Q and R intersect at A, B and C as shown in the diagram. Given that $B\hat{A}C = 42°$ and $P\hat{Q}B = 56°$, find

(i) $A\hat{C}B$;
(ii) $P\hat{Q}R$;
(iii) $R\hat{P}Q$.



(b) In the diagram, O is the centre of the circle, PAQ is a tangent and AC is a diameter. If $A\hat{C}B = x$ and $A\hat{C}D = 3x$, express the following angles in terms of x.

(i) $C\hat{O}D$
(ii) $P\hat{A}D$
(iii) $O\hat{A}B$



10. Given that y varies directly as $(2x - 3)$, and that the difference in the values of y when $x = 2$ and $x = 4$, is 20, find the value of y when $x = 7$.

CHAPTER 13

Frequency Distribution

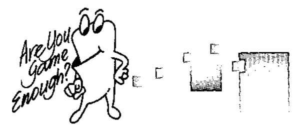
In this chapter, you will learn how to

- △ construct a grouped frequency table;
- △ construct a histogram representing a grouped frequency table;
- △ construct a frequency polygon representing a grouped frequency table.

Preliminary Problem

If we are to classify this group of 13-year-old secondary pupils according to their heights, then we will get a frequency distribution that is representative of all the pupils in the age group with a minority that are extremely tall or short, while the majority are of average height. Will the frequency distribution of their weights also follow this pattern?

Revision



Which letters of the alphabet are used most frequently in written material?

Choose an article from a newspaper. Making use of tally marks, record the frequency for each letter by going through the first 600 successive letters in the article. Tabulate the results in a frequency distribution and construct a histogram to illustrate your findings.

Example 7

The data for the number of days the pupils in a class of 45 were absent from school in a year are as follows:

4 3 0 5 2 2 0 1 6 2 7 6 3 3 0
4 4 2 0 1 4 0 2 1 5 1 6 3 2 5
2 2 4 0 6 3 2 0 1 7 5 0 1 2 3

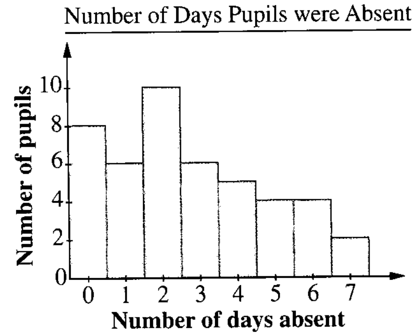
Construct a frequency table and draw a histogram representing it.

Solution

Number of days absent	*Tally*	*Number of students*
0	𝍸 |||	8
1	𝍸 |	6
2	𝍸 𝍸	10
3	𝍸 |	6
4	𝍸	5
5	||||	4
6	||||	4
7	||	2
Total number of pupils		45

The frequency table above shows the distribution of the number of days the pupils were absent from school. It is called the **frequency distribution** of the number of days pupils were absent. The figure on the right shows the histogram representing the frequency distribution.

Number of Days Pupils were Absent



Exercise 13a

1. A company claims that it produces 50 matches per box. Fifty boxes were selected from a large number of boxes manufactured by the company and their contents counted, giving the following results:

53	50	48	52	52	50	49	51	50	52
49	49	50	51	48	51	50	52	52	49
49	51	51	50	50	51	48	50	52	49
50	50	51	52	49	53	50	52	49	52
51	49	50	53	52	50	49	50	48	50

(a) Construct a frequency distribution table for the results.
(b) Draw a histogram to represent the data.
(c) What percentage of the boxes contain exactly 50 matches?

2. The number of siblings that each employee in a group of 40 has is shown below:

2	2	3	5	3	2	1	4	2	1	1	0	3	2	1	1	2	3	1	4
3	2	3	1	0	1	2	1	2	3	1	1	2	2	1	3	5	1	1	5

(a) Construct a frquency distribution table for the results.
(b) Represent the results with a histogram.

3. The table below shows the number of fire incidents per day in a city over a period of 60 days.

Number of fire incidents	0	1	2	3	4	5	6	7
Number of days	16	12	11	10	6	2	1	2

(a) Represent the data using a histogram.
(b) Express the number of days when there were no fire incidents as a percentage of the total number of days.

4. The table below shows the number of hours 60 workers work per week.

Number of hours	37	38	39	40	41	42	43	44	45
Frequency	3	4	9	18	10	6	3	5	2

(a) Draw a histogram to represent the data.
(b) What percentage of workers work at least 40 hours per week?

5. The table below shows the amount of vitamin C, in milligrams, found in 102 oranges.

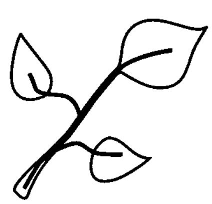
Amount of vitamin C (mg)	26	27	28	29	30	31	32	33	34
Number of oranges	10	7	16	20	24	12	8	2	3

Draw a histogram to represent the above frequency distribution.

Grouped Frequency Distribution

The following set of raw data shows the lengths, in millimetres, measured to the nearest mm, of 40 leaves taken from plants of a certain species.

40	54	25	50	58	45	47	49	30	28
52	31	52	41	47	44	46	39	41	59
49	38	43	48	43	43	40	51	40	56
31	53	44	37	35	37	33	38	46	36



Back in Time

The study of statistics began when an Englishman, John Graunt (1620–1674), collected and studied the death records in various cities of Britain and was fascinated by the patterns he found in the whole population even though people died randomly.

Data represented in this form can be confusing and, as a result, we cannot possibly grasp the important patterns in the data. To better understand the data and to get as much information as we can out of it, we need to arrange the data in an *orderly* fashion. The number of different measurements that appear in the data makes it impractical to organise the data in the same way as we did in Example 1.

One way is to group this kind of data into a number of **class intervals**. If possible, we want all the intervals to have *equal* size.

Grouping a set of data into class intervals requires the following steps:

(1) Glance through the data to pick out the largest and the smallest observations. An observation can be the result of a measurement, the mark scored by a pupil in a test or the sales of a certain product in a particular year. In our example above,

the *largest* observation = 59
the *smallest* observation = 25

(2) Find the difference between the largest and the smallest observations. In our example,

the *difference* = 59 – 25 = 34.

(3) Decide on the number of class intervals. This depends on the amount of data available. The difference between the largest and smallest observations will help us make the choice. In our example, a 10-mm interval size will give us 3 class intervals as $\frac{34}{10} = 3.4 \approx 3$.

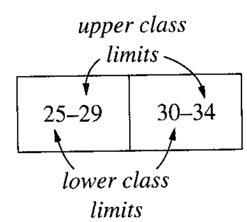
A 5-mm interval size will give us 7 class intervals as $\frac{34}{5} \approx 7$.

A 5-mm interval size is more appropriate for our data. As a working guide, we should choose 5 to 20 class intervals depending on the amount of data available.

(4) Choose the first class interval such that the smallest observation is included. In our example, we will take 25–29 as the first class interval. The other class intervals are 30–34, 35–39, 40–44, 45–49, 50–54 and 55–59. The last class interval 55–59 must include the largest observation.

(a) Class Limits

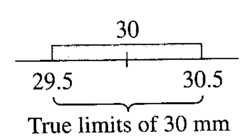
25 and 29 are called the **lower class limit** and the **upper class limit**, respectively, of the class interval 25–29.



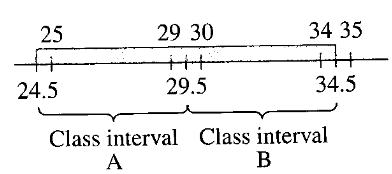
30 and 34 are the lower and upper class limits, respectively, of the interval 30–34.

Can we take 25–30, 30–35, 35–40, 40–45, 45–50, 50–55 and 55–60 as our class intervals?

For our set of data, the answer is no because each length was measured to the nearest mm i.e., each figure in the data has been rounded off from the original measurement. Hence, a length recorded as 30 mm will have the actual length varying from 29.5 mm to 30.5 mm.



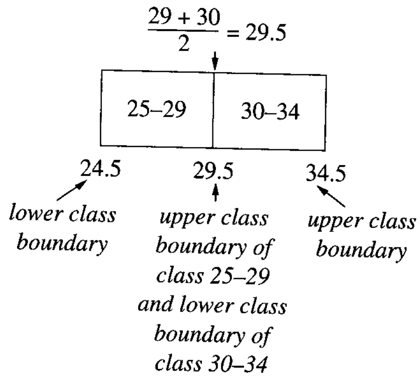
A case of ambiguity arises when we try to classify the figure 30, i.e., should we include it in the class interval 25–30 or 30–35?



Thus, 30 would be included in class interval B.

(b) Class Boundaries

To avoid any ambiguity, a class interval like 25–29 will be taken as 24.5–29.5. Similarly, the class interval 30–34 will be written as 29.5–34.5 and so on. 24.5 and 29.5 are called the **lower class boundary** and the **upper class boundary**, respectively, of the class interval 25–29.



In general, the mid-point between the upper class limit of the first interval and the lower class limit of the second interval is the ***upper class boundary*** *of the first class and the* ***lower class boundary*** *of the second class.*

Do you notice that the upper class boundary of the class interval 25–29 and the lower class boundary of the class interval 30–34 are the same? It is obtained by dividing the sum of 29 and 30 by 2 i.e.,

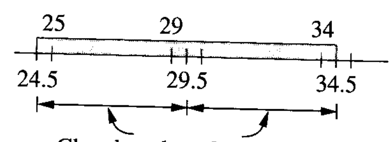
$$\frac{29 + 30}{2} = 29.5.$$

For measurements recorded to the nearest whole number, the interval 25–29 will include an original measurement of x, where $24.5 \leqslant x < 29.5$ and the interval 30–34 will include any original measurement x where $29.5 \leqslant x < 34.5$.

(c) Class Width

The **class length**, or **class width**, of a class interval is the difference between its upper and lower class boundaries.

In our example, the width of each class interval is 5 mm.

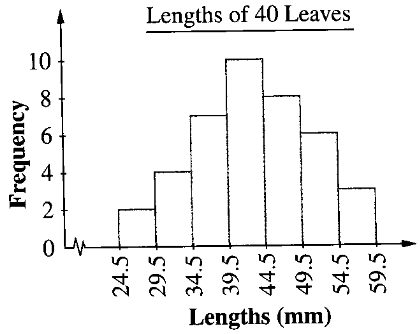


(5) Read through the list of observations, making a stroke in the tally column against the class interval to which each belongs. The table below shows the frequency distribution of the lengths of the 40 leaves.

Lengths (mm)	*Class boundaries*	*Tally*	*Frequency*
25–29	24.5–29.5	//	2
30–34	29.5–34.5	////	4
35–39	34.5–39.5	~~////~~ //	7
40–44	39.5–44.5	~~////~~ ~~////~~	10
45–49	44.5–49.5	~~////~~ ///	8
50–54	49.5–54.5	~~////~~ /	6
55–59	54.5–59.5	///	3
Total Frequency = 40			

Histograms

The diagram below shows the histogram representing the frequency distribution of the lengths of 40 leaves.



For Your Information

Today, statistical methods have been widely applied in psychology, sociology, education, economics, medicine, agriculture, industry, business as well as in many other fields.

Whenever we construct a histogram representing a set of figures which have been rounded off from the original measurements, we have to make the class intervals continuous by using class boundaries. This is to avoid having gaps in between the bars.

Example 2

The fluoride levels, measured in parts per million (PPM), of drinking water treated in a certain water treatment plant were monitored for 30 days. The results are given below:

0.76	*0.75*	*0.84*	*0.82*	*0.98*	*0.88*	*0.71*	*0.87*	*0.79*	*0.91*
0.87	*0.91*	*0.83*	*0.84*	*0.88*	*0.99*	*0.84*	*0.83*	*0.83*	*0.90*
0.93	*0.85*	*0.78*	*0.77*	*0.81*	*0.92*	*1.04*	*0.92*	*0.79*	*0.87*

The measurements are given correct to 2 decimal places.

Construct a frequency table and draw a histogram representing it.

Solution

The largest measurement = 1.04
The smallest measurement = 0.71
The range = 1.04 – 0.71 = 0.33

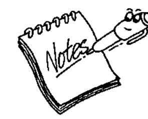
An interval size of 0.05 will give us 7 class intervals as $\frac{0.33}{0.05} = 6.6 \approx 7$.

The first class interval 0.71–0.75 will include the smallest measurement. The other classes are 0.76–0.80, 0.81–0.85, 0.86–0.90, 0.91–0.95, 0.96–1.00 and 1.01–1.05.

NB: A fluoride level reported as 0.76 PPM correct to 2 decimal places will have an actual fluoride level between 0.755 PPM and 0.765 PPM.

The table below shows the frequency distribution of the fluoride levels:

Fluoride levels (PPM)	*Class boundaries*	*Tally*	*Frequency*
0.71–0.75	0.705–0.755	//	2
0.76–0.80	0.755–0.805	~~////~~	5
0.81–0.85	0.805–0.855	~~////~~ ////	9
0.86–0.90	0.855–0.905	~~////~~ /	6
0.91–0.95	0.905–0.955	~~////~~	5
0.96–1.00	0.955–1.005	//	2
1.01–1.05	1.005–1.055	/	1
			Total = 30



Divide the data into different classes of the same class widths.

0.71 is called the lower class limit of the class interval 0.71–0.75 in Example 2. 0.75 is called the upper class limit of that same interval.

Fig. 13.1 below shows the histogram.

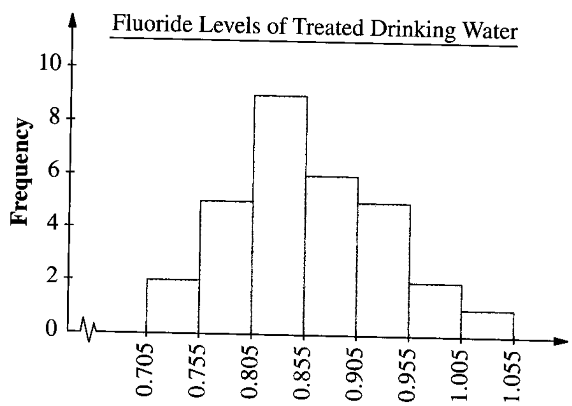


Fig. 13.1

If the fluoride levels given were original measurements, we may take the class intervals to be 0.70–0.75, 0.75–0.80, 0.80–0.85, 0.85–0.90, 0.90–0.95, 0.95–1.00 and 1.00–1.05. In this case, the lower and upper class limits of a class interval will coincide with the lower and upper class boundaries. Hence, for the class interval 0.70–0.75, the lower and upper class limits or boundaries are 0.70 and 0.75 respectively.

Again, ambiguity arises when we try to classify a certain measurement like 0.75 PPM. Should we include it in the class interval 0.70–0.75 or 0.75–0.80?

Thus, there is a need to define a class interval clearly. We may specify the class interval 0.70–0.75 to include any measurement x PPM, where $0.70 \leqslant x < 0.75$ or $0.70 < x \leqslant 0.75$. If we adopt the former, then 0.75 should be placed in the class interval 0.75–0.80 and the resulting frequency table is as shown below:

Fluoride levels (PPM)	*Fluoride levels* (x PPM)	*Frequency*
0.70–0.75	$0.70 \leqslant x < 0.75$	1
0.75–0.80	$0.75 \leqslant x < 0.80$	6
0.80–0.85	$0.80 \leqslant x < 0.85$	8
0.85–0.90	$0.85 \leqslant x < 0.90$	6
0.90–0.95	$0.90 \leqslant x < 0.95$	6
0.95–1.00	$0.95 \leqslant x < 1.00$	2
1.00–1.05	$1.00 \leqslant x < 1.05$	1

Fig. 13.2 gives the corresponding histogram as shown below:

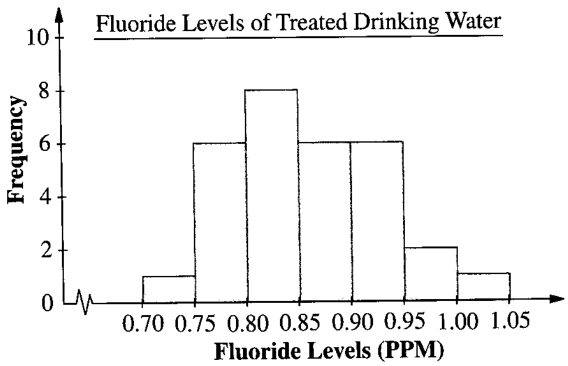


Fig. 13.2

In a histogram, the ***frequency*** *is represented by* ***(height of bar)*** × ***(number of standard intervals)****.*

Histograms with Unequal Class Intervals

Ismail and Vasu carried out a survey together to find out the number of children in the families residing in their neighbourhood. They displayed the findings somewhat differently in the form of frequency distribution tables shown below:

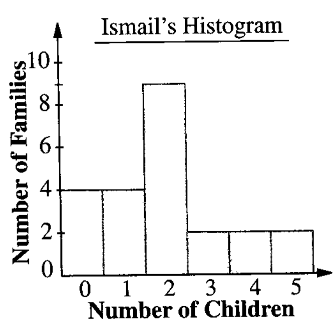
Ismail's frequency table →

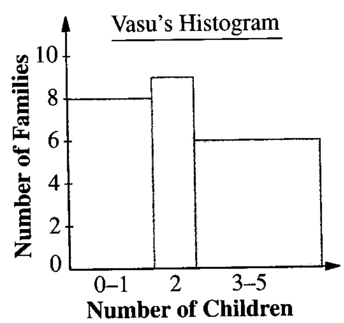
Number of children	0	1	2	3	4	5
Number of families	4	4	9	2	2	2

Vasu's frequency table →

Number of children	0–1	2	3–5
Number of families	8	9	6

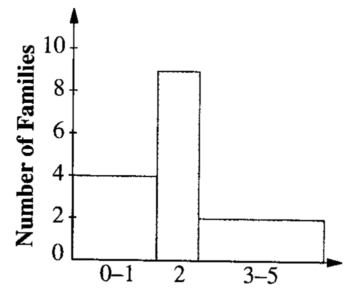
They constructed histograms which are shown below:





NB: Ismail's and Vasu's frequency tables represent the same data and thus both Ismail's and Vasu's histograms should give us the same impression of the distribution of the number of children. Our eyes compare areas and not heights in a histogram. Thus, for a histogram to give a correct impression of a frequency distribution, the *area* and not height of each rectangle need to be **proportional** to the *frequency*. In our example, Vasu's histogram gives the wrong impression of the distribution of the number of children. Can you explain why? Vasu's correct histogram is as shown below. Can you explain how the wrong histogram is adjusted to give the correct impression of the frequency distribution?

The Correct Version of Vasu's Histogram



The next example will illustrate the proper way of constructing histogram of frequency distribution with unequal class intervals.

Example 3

The table below shows the heights of 44 students.

Height (cm)	$130 < x \leqslant 145$	$145 < x \leqslant 150$	$150 < x \leqslant 155$	$155 < x \leqslant 160$	$160 < x \leqslant 170$	$170 < x \leqslant 180$
Frequency	9	7	8	6	10	4

Construct a histogram representing this distribution.

Solution

NB: The class intervals are not equal. In constructing the histogram, we must ensure that the areas of the rectangles are *proportional* to the class frequencies as the frequency in a histogram is represented by the area of each rectangle. In examining the sizes of the classes, we find that the interval size of 5 is the smallest. Three class intervals are of this size: $145 < x \leqslant 150$, $150 < x \leqslant 155$ and $155 < x \leqslant 160$. The class intervals $160 < x \leqslant 170$ and $170 < x \leqslant 180$ are each of size 10 and therefore they contain 2 class intervals of size 5 grouped together. The size of the class interval $130 < x \leqslant 145$ is 15, i.e. there are 3 class intervals of size of 5 grouped together.

The table below shows the calculation of the heights of the rectangles.

Height (*x* cm)	*Class width* (cm)		*Frequency*	*Rectangle's height in histogram*
$130 < x \leqslant 145$	15	$3 \times$ standard	9	$9 \div 3 = 3$
$145 < x \leqslant 150$	5	$1 \times$ standard	7	$7 \div 1 = 7$
$150 < x \leqslant 155$	5	$1 \times$ standard	8	$8 \div 1 = 8$
$155 < x \leqslant 160$	5	$1 \times$ standard	6	$6 \div 1 = 6$
$160 < x \leqslant 170$	10	$2 \times$ standard	10	$10 \div 2 = 5$
$170 < x \leqslant 180$	10	$2 \times$ standard	4	$4 \div 2 = 2$

The corresponding histogram is shown in Fig. 13.3 below.

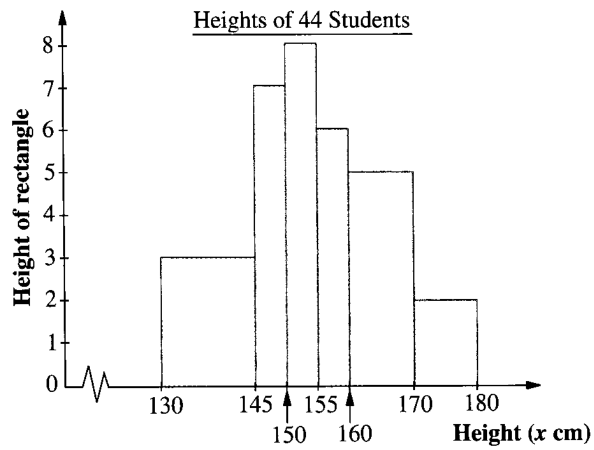
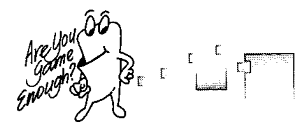
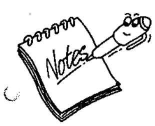


Fig. 13.3



A farmer has $5\frac{1}{2}$ haystacks in one corner of his farm and $4\frac{1}{2}$ haystacks in another corner. If he puts them all together, how many haystacks will he have?



Since the areas of the rectangles in a histogram must be proportional to the class frequencies, height of rectangle × class width = class frequency

OR

height of rectangle $= \frac{\text{class frequency}}{\text{class width}}$.

Thus, the height of a rectangle may be found by dividing the class frequency by the class width. The ratio, $\frac{\text{class frequency}}{\text{class width}}$ *is called the* ***frequency density****.*

Another way of presenting the above data is using the idea of *frequency density*. The table below shows the calculation of the frequency density in the distribution of the heights of 44 students.

Height (x cm)	*Frequency*	*Class width*	*Frequency density*
$130 < x \leqslant 145$	9	15	$9 \div 15 = 0.6$
$145 < x \leqslant 150$	7	5	$7 \div 5 = 1.4$
$150 < x \leqslant 155$	8	5	$8 \div 5 = 1.6$
$155 < x \leqslant 160$	6	5	$6 \div 5 = 1.2$
$160 < x \leqslant 170$	10	10	$10 \div 10 = 1$
$170 < x \leqslant 180$	4	10	$4 \div 10 = 0.4$

Fig. 13.4 shows the corresponding histogram.

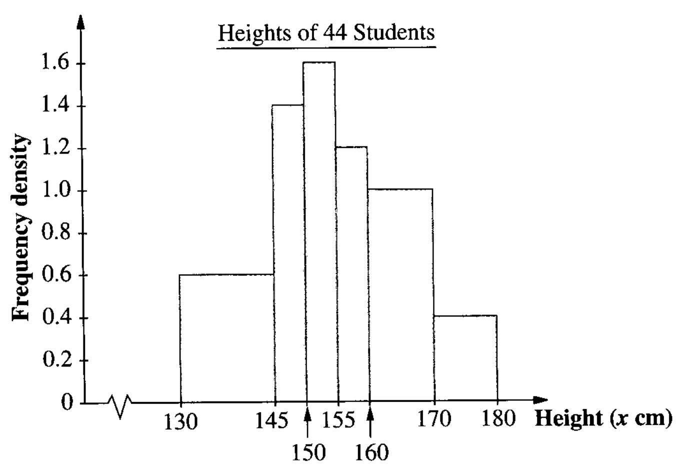


Fig. 13.4

Compare the two histograms in Fig. 13.3 and Fig. 13.4. What do you notice?

Exercise 13b

1. The following table shows the distribution of marks of some students who took part in a science quiz.

Marks	*Tally*	*Lower class boundary*	*Upper class boundary*	*Frequency*
56–60	𝍸 //			
61–65	𝍸 //			
66–70	𝍸			
71–75	𝍸 𝍸			
76–80	𝍸			
81-85	𝍸			
86–90	//			
91–95	///			
96–100	///			

(a) Copy and complete the table.
(b) To which classes do the marks 90.9, 66.2 and 81.5 belong?
(c) Draw a histogram to represent this distribution.

2. The lengths, in mm, of 48 rubber tree leaves are given below.

137	152	127	134	147	141	157	132	153	166	147	136
146	142	162	169	149	135	166	148	157	141	146	147
163	133	148	150	136	127	162	152	143	138	142	153
145	154	144	126	139	126	158	147	136	144	159	161

Copy and complete the following table:

Lengths (x mm)	*Tally*	*Frequency*
$125 < x \leqslant 130$		
$130 < x \leqslant 135$		
$135 < x \leqslant 140$		
$140 < x \leqslant 145$		
$145 < x \leqslant 150$		
$150 < x \leqslant 155$		
$155 < x \leqslant 160$		
$160 < x \leqslant 165$		
$165 < x \leqslant 170$		

(a) Determine the class width of the second class.
(b) Draw a histogram to illustrate the frequency distribution.

3. The daily wages of 50 workers, in dollars, are given below. Construct a frequency table with class intervals 10–14, 15–19, 20–24, and so on. Draw a histogram to represent the data.

12 21 13 17 29 33 26 47 10 17 36 31 32 27 25 16 36 29 22 24 21 25 45 18 37
42 35 28 20 44 34 43 22 36 34 20 15 26 17 21 25 30 27 32 26 28 30 38 19 26

4. The waiting times, x minutes, for 60 patients at a certain clinic are as follows:

25 12 53 8 26 5 19 73 67 18 87 42 6 21 14 19 12 15 13 36
36 16 72 36 13 37 11 51 39 32 30 47 6 22 68 25 98 23 45 22
7 9 26 35 27 48 58 56 29 20 32 62 80 41 58 17 54 15 14 74

(a) Construct a frequency table using class intervals $0 < x \leqslant 10$, $10 < x \leqslant 20$, $20 < x \leqslant 30$ and so on.
(b) Draw a histogram for the frequency distribution.

5. In a survey on the different prices of an article sold in the shops of a certain city, the following results were obtained.

Cost (x cents)	*Frequency*
$90 \leqslant x < 95$	4
$95 \leqslant x < 100$	11
$100 \leqslant x < 105$	15
$105 \leqslant x < 110$	24
$110 \leqslant x < 115$	18
$115 < x \leqslant 120$	9
$120 < x \leqslant 125$	3

(a) How many shops were included in the survey?
(b) Construct a histogram to represent the data.

6. The circumferences, in cm, of the trunks of 100 beech trees from a certain forest are measured. Draw a histogram representing this information.

Circumference (x cm)	*Number of trees*
$40 < x \leqslant 50$	4
$50 < x \leqslant 60$	8
$60 < x \leqslant 70$	20
$70 < x \leqslant 80$	28
$80 < x \leqslant 90$	20
$90 < x \leqslant 100$	10
$100 < x \leqslant 110$	6
$110 < x \leqslant 120$	4

7. (a) Copy and complete the following frequency distribution table.

Class interval	*Class width*		*Frequency*	*Height of rectangle*
10–14	5	1 standard	5	5 ÷ 1 = 5
15–24			8	
25–29			6	
30–34			11	
35–39			13	
40–54			3	
55–64	10	2 standard	4	4 ÷ 2 = 2

(b) Construct a histogram to represent the distribution.

8. (a) Copy and complete the following table.

Class interval	*Class width*	*Frequency*	*Frequency density* $= \frac{\text{Frequency}}{\text{Class width}}$
$0 < x \leqslant 20$	20	4	4 ÷ 20 = 0.2
$20 < x \leqslant 30$		12	
$30 < x \leqslant 40$	10	14	14 ÷ 10 = 1.4
$40 < x \leqslant 50$		11	
$50 < x \leqslant 70$		8	
$70 < x \leqslant 100$		6	

(b) Draw a histogram to represent the distribution.

9. (a) Draw a histogram representing the distribution in Question 7 using frequency densities.

(b) Draw a histogram representing the distribution in Question 8 using an appropriate standard class width.

10. The following table shows the weekly earnings of 68 employees of Meteor Pte Ltd.

Weekly earnings ($)	*Number of workers*
$180 \leqslant x < 185$	4
$185 \leqslant x < 190$	6
$190 \leqslant x < 210$	8
$210 \leqslant x < 230$	16
$230 \leqslant x < 250$	20
$250 \leqslant x < 280$	6
$280 \leqslant x < 320$	8

Draw a histogram representing this information.

11. On a particular day, the length of stay of each car at a car park, measured to the nearest minute, was recorded.

(a) Find the total number of cars parked in the car park that day.
(b) Draw a histogram representing the information.

Length of stay (min)	*Number of cars*
5–24	60
25–59	50
60–79	86
80–104	150
105–114	60
115–129	105
130–149	60
150–189	48

Frequency Polygons

The value mid-way between the class boundaries of a class is called the **class mark**, or the **mid-value**, of the class.

25–29	30–34

24.5 27 29.5 32 34.5
↑ mid-value ↑ mid-value

$$27 = \frac{25 + 29}{2} = \frac{24.5 + 29.5}{2}$$

$$32 = \frac{30 + 34}{2} = \frac{29.5 + 34.5}{2}$$

NB: The mid-value of a class is given by

$$\frac{\text{lower class limit + upper class limit}}{2} \text{ or } \frac{\text{lower class boundary + upper class boundary}}{2}$$

The table (Page 287) is reproduced as shown below. It shows, in addition, the class mark for each class interval in the frequency distribution of the fluoride levels discussed in Example 2.

Fluoride levels (PPM)	*Class boundaries*	*Class mark*	*Tally*	*Frequency*
0.71–0.75	0.705–0.755	0.73	//	2
0.76–0.80	0.755–0.805	0.78	~~////~~	5
0.81–0.85	0.805–0.855	0.83	~~////~~ ////	9
0.86–0.90	0.855–0.905	0.88	~~////~~ /	6
0.91–0.95	0.905–0.955	0.93	~~////~~	5
0.96–1.00	0.955–1.005	0.98	//	2
1.01–1.05	1.005–1.055	1.03	/	1
				Total = 30

For Your Information

The total area of the rectangle in the histogram is equal to the ***area*** *under the frequency polygon.*

The corresponding histogram in Fig. 13.1 (Page 287) is also reproduced in Fig. 13.5 at the right.

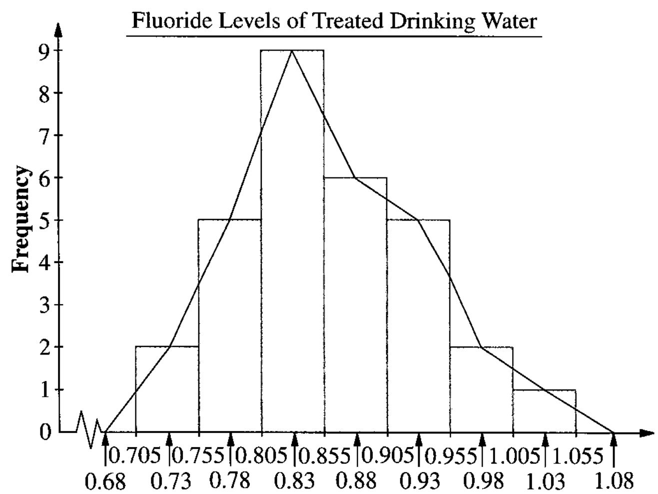


Fig. 13.5

NB: *A frequency polygon is drawn by joining all the mid-points at the top of each rectangle.* The mid-points at both ends are joined to the horizontal axis to accommodate the end points of the polygon. This will make the graph neater with the end points falling off to zero on the horizontal axis.

Example 4

The weights, in kg, of 50 boys were recorded as shown in the table below:

Weight (x kg)	$40 < x \leqslant 45$	$45 < x \leqslant 50$	$50 < x \leqslant 55$	$55 < x \leqslant 60$	$60 < x \leqslant 65$	$65 < x \leqslant 70$	$70 < x \leqslant 75$
Number of boys	4	5	10	14	8	6	3

Draw a frequency polygon of the distribution.

Solution

We can draw the frequency polygon of a distribution without first drawing the histogram. We plot each class frequency against the *mid-value* of the class to obtain points which we join by straight lines to form the frequency polygon.

The table below shows the mid-value of each class.

Weight (x kg)	*Mid-value*	*Frequency*
$40 < x \leqslant 45$	42.5	4
$45 < x \leqslant 50$	47.5	5
$50 < x \leqslant 55$	52.5	10
$55 < x \leqslant 60$	57.5	14
$60 < x \leqslant 65$	62.5	8
$65 < x \leqslant 70$	67.5	6
$70 < x \leqslant 75$	72.5	3

For Your Information

When the classes in a distribution are of equal class width, the frequency polygon can be constructed without first drawing the histogram.

The points to be plotted are (37.5, 0), (42.5, 4), (47.5, 5), (52.5, 10), (57.5, 14), (62.5, 8), (67.5, 6), (72.5, 3) and (77.5, 0). The first and last points are added to give a complete polygon.

The frequency polygon is as shown in Fig. 13.6.

A frequency polygon is often useful when we wish to *observe trends*. What do you notice from the frequency polygon in Fig. 13.6 about the relationship between the weights and the number of boys? Does the number of boys increase as the weight increases? If yes, is this trend maintained throughout? If not, after what weight does the number of boys start to decline?

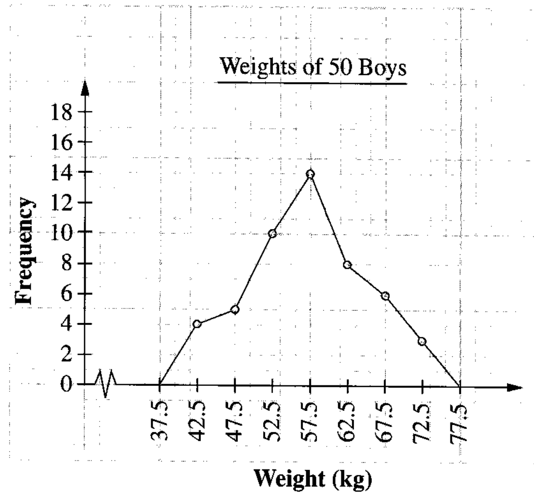


Fig. 13.6

A frequency polygon is also useful when we wish to compare two distributions by displaying two overlapping frequency polygons on the same set of axes. For example, we may want to compare the weight distributions among boys and girls of a particular age group in this way.

Exercise 13c

1. Copy and complete the following table which gives the frequency distribution of the lengths of 40 fishes of a certain species, measured to the nearest mm.

Draw the frequency polygon of the distribution.

Length (mm)	*Mid-value*	*Frequency*
25–29	27	2
30–34		4
35–39		7
40–44		10
45–49		8
50–54		6
55–59		3

2. The following table gives the frequency distribution of the marks obtained by 40 students in an English test.

(a) Write down the mid-value of the class interval $40 < x \leqslant 50$.

(b) Draw the frequency polygon of the distribution.

Mark (x)	*Number of students*
$20 < x \leqslant 30$	2
$30 < x \leqslant 40$	3
$40 < x \leqslant 50$	8
$50 < x \leqslant 60$	9
$60 < x \leqslant 70$	11
$70 < x \leqslant 80$	5
$80 < x \leqslant 90$	2

3. 120 teachers have been in the teaching service for 10 years or longer. The following table gives the frequency distribution of the length of service of these 120 teachers.

Length of service (x years)	$10 \leqslant x < 15$	$15 \leqslant x < 20$	$20 \leqslant x < 25$	$25 \leqslant x < 30$	$30 \leqslant x < 40$	$40 \leqslant x < 50$
Number of teachers	32	40	25	12	7	4

(a) Draw a histogram for the frequency distribution.
(b) Using the same diagram, draw the frequency polygon of the distribution.

4. The weights, in kg, of 80 members of a sports club were measured and recorded as shown in the table.

(a) Draw a histogram for the frequency distribution.
(b) Using a separate diagram, draw a frequency polygon to represent the data.

Weight (x kg)	*Number of members*
$40 < x \leqslant 50$	7
$50 < x \leqslant 60$	10
$60 < x \leqslant 70$	14
$70 < x \leqslant 80$	27
$80 < x \leqslant 90$	12
$90 < x \leqslant 100$	6
$100 < x \leqslant 110$	4

5. The table gives the frequency distribution of the masses of 200 steel bars, to the nearest kg.

(a) Draw a histogram to represent the information.
(b) Using a separate diagram, draw a frequency polygon to display the data.

Mass (kg)	*Number of steel bars*
10–29	32
30–39	38
40–49	64
50–59	35
60–69	22
70–99	9

Summary

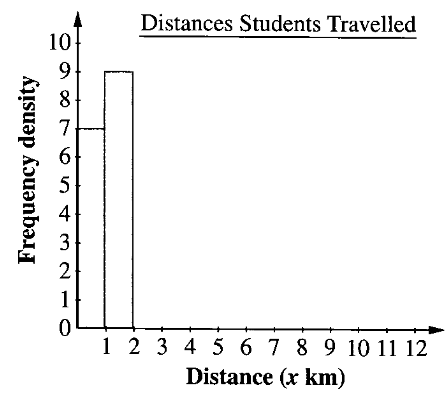
1. A set of unsummarised data, or raw data, can be organized and then arranged in an orderly way in the form of a **frequency table**.
2. A frequency table can be represented graphically by a **histogram**.
3. A histogram is a vertical bar graph with no space in between its bars.
4. The area of each bar is proportional to the frequency it represents.
5. If the mid-points of the tops of the consecutive bars in a histogram are joined by line segments and the mid-points at both ends are joined to the horizontal axis, a **frequency polygon** is obtained.

Review Questions 13

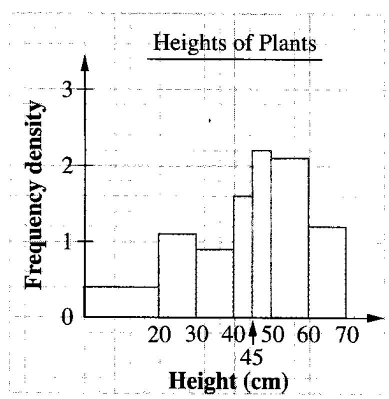
1. Each of the 44 students in a class was asked how far he or she travelled from home and the results are tabulated below:

Distance (x km)	*Number of students*
$0 < x \leqslant 1$	7
$1 < x \leqslant 2$	9
$2 < x \leqslant 3$	10
$3 < x \leqslant 5$	8
$5 < x \leqslant 8$	6
$8 < x \leqslant 12$	4

Copy and complete the histogram which represents this information.



2. The heights of plants grown under experimental conditions were measured and the histogram illustrates the results.

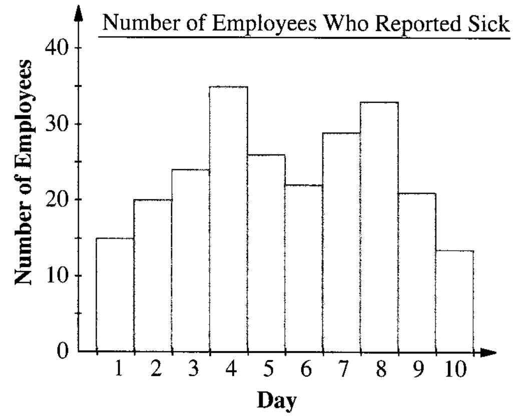


(a) Copy and complete the following table.

Height (x cm)	*Number of plants*
$0 < x \leqslant 20$	
$20 < x \leqslant 30$	11
$30 < x \leqslant 40$	9
$40 < x \leqslant 45$	8
$45 < x \leqslant 50$	
$50 < x \leqslant 60$	
$60 < x \leqslant 70$	

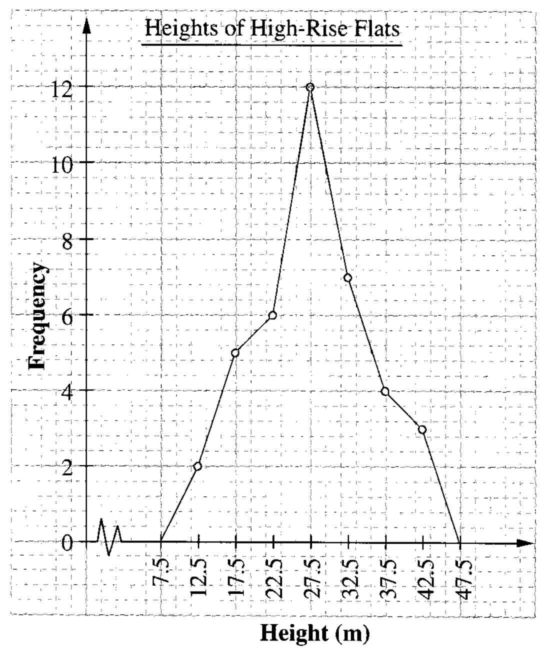
(b) Find the number of plants in the distribution.
(c) Using a separate diagram, draw the frequency polygon for the distribution.

3. During an outbreak of flu in a city, the number of employees who reported sick over ten successive days in a company was illustrated by the histogram.



(a) On which day did the greatest number of employees report sick? How many employees reported sick on that day?
(b) On which day did the least number of employees report sick? How many employees reported sick on that day?
(c) On which days were the number of sick employees more than 30?

4. The heights of high-rise flats (in metres) in a certain housing estate are represented by the frequency polygon below:



(a) Copy and complete the following table:

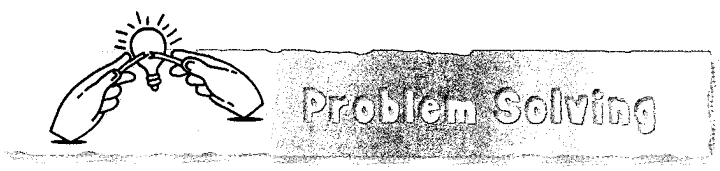
Height (x m)	*Number of flats*
$10 < x \leqslant 15$	2
$15 < x \leqslant 20$	5
	6
$25 < x \leqslant 30$	

(b) Find the total number of flats in the distribution.
(c) Using the graph, construct a frequency table for the distribution.
(d) Draw the corresponding histogram, using a separate diagram.

5. The time, in minutes, taken by an auditing firm to audit each of the 60 accounts is given below:

51	35	52	36	36	57	57	35	49	42
21	31	41	25	44	41	33	42	39	48
52	41	41	45	22	28	38	46	42	35
43	53	35	39	44	31	47	47	27	25
31	54	42	47	32	43	46	35	42	31
58	41	31	44	39	42	42	31	33	38

(a) Construct a grouped frequency table for the information using a class width of 5, the first class having a lower limit of 20.
(b) Draw a histogram to represent the information.
(c) Using a separate diagram, draw the frequency polygon representing the data.



Example 5

Given that the median of five different integers 4, 9, 13, x and (2x – 3) is 9, find the value of x.

Solution

Method: **Use logical deduction**

Since 9 is the median, x and $(2x - 3)$ must be on either side of 9. Let us now find out which side they should be on, i.e., which one of them is greater than 9 and which is less.

$$x < 9 < 2x - 3 \Rightarrow 2x - 3 > 9 \quad \text{and} \quad x < 9$$
$$2x > 12 \quad \text{and} \quad x < 9$$
$$x > 6 \quad \text{and} \quad x < 9$$
$$\therefore \quad 6 < x < 9$$

The possible values of x are 7 and 8.

$$2x - 3 < 9 < x \Rightarrow 2x - 3 < 9 \quad \text{and} \quad x > 9$$
$$2x < 12 \quad \text{and} \quad x > 9$$
$$x < 6 \quad \text{and} \quad x > 9$$

There is no solution in this case.

Problems

1. *Write down a set of 5 different positive integers whose median is 3 and whose mean is 6.*
2. *Write down a set of 5 consecutive odd integers whose median is x. What is the mean of these 5 odd integers? Find the difference between the mean of the squares of these 5 integers and the median of the squares.*
3. *The distribution of marks scored by students in a class is as follows:*

Marks	2	3	4	5	6	7	8
Number of Students	5	7	6	4	x	3	6

(a) If the mode is 6, write down an inequality in x.
(b) If the median is 5, write down the possible values of x.
(c) If the mean is 4.95, find the value of x.

Hence, $(2x - 3)$ must be greater than 9 and x must be less than 9, where x is 7 or 8.

When $x = 7$, $2x - 3 = 11$

When $x = 8$, $2x - 3 = 13$, which is inadmissible because all the five integers must be different.

From the above, when $x = 7$, the five integers are 4, 7, 9, 11 and 13, which gives the median 9.

This confirms that the value of x is 7.

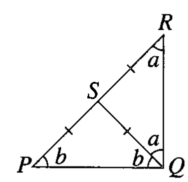
Example 6

Given that a point lying on one side of a triangle is equidistant from the three vertices of the triangle, prove that the triangle is a right-angled triangle.

Solution

Method 1: Use algebra

In the diagram, the point S lies on the side PR of $\triangle PQR$ and is equidistant from P, Q and R.



We label the two equal angles, a, in the isosceles triangle QRS and the two equal angles, b, in the isosceles triangle PQS, as shown.

We have $a + b + b + a = 180°$ (∠ sum of △)

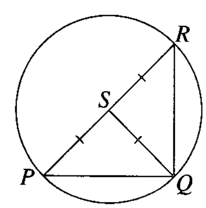
$2a + 2b = 180°$

$\therefore\ a + b = 90°$

$\therefore\ P\hat{Q}R = 90°$ and, thus, $\triangle PQR$ is right-angled.

Method 2: Use geometry

Since S is equidistant from P, Q and R, taking S as the centre and PR as the diameter we can inscribe $\triangle PQR$ in a circle as shown in the diagram.



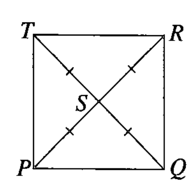
$\therefore\ P\hat{Q}R = 90°$ (rt. ∠ in semi-circle)

Thus, $P\hat{Q}R$ is right-angled.

Method 3: Use logical reasoning

By extending QS to a point T, such that $QS = ST$, and joining PT and RT, we obtain the quadrilateral $PQRT$.

Do the diagonals of quadrilateral $PQRT$ bisect each other?



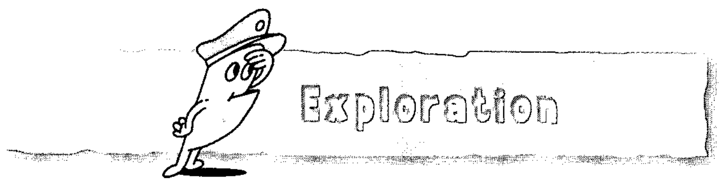
What kind of quadrilateral is $PQRT$?

Are the diagonals PR and QT equal in length?

What more can we say about quadrilateral $PQRT$?

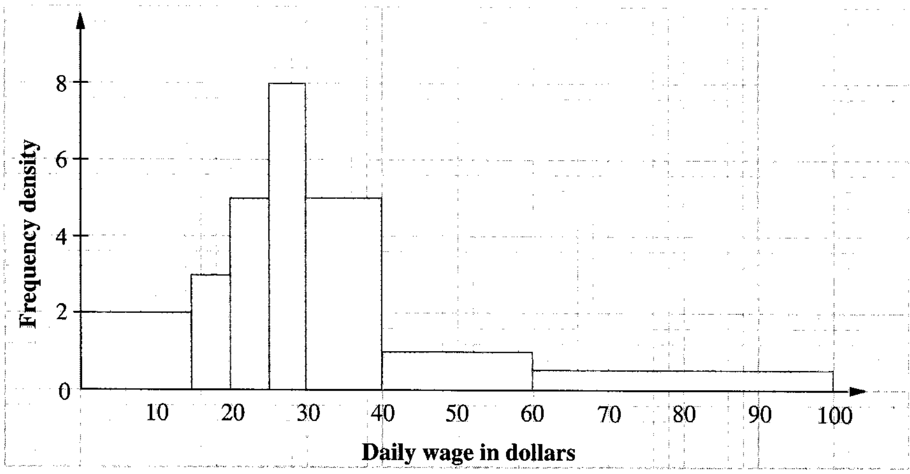
Can we conclude that $P\hat{Q}R = 90°$?

Thus, we can also solve the problem through *logical reasoning*.



Exploration

1. This histogram illustrates the daily wage of a group of people.



(a) Find the number of people in the group.

(b) Work out an estimate of the mean daily wage of the group.

2. (a) Ask 15 of your classmates how many hours in a week he or she spends

   (i) watching television; (ii) on homework;

   and record your findings.

   (b) Organise each set of data obtained by constructing a grouped frequency table.

   (c) Display each set of data by means of a histogram.

   (d) Using your histograms, answer the following questions:

   (i) What is the class of time for watching television with the highest frequency? What is this frequency?

   (ii) How many of them watch between 5 to 10 hours of television a week?

   (iii) How many of them watch more than 10 hours of television a week?

   (iv) What is the class of time for homework with the lowest frequency? What is this frequency?

   (v) What percentage of your class spends less than 10 hours a week on homework?

   (e) Show the frequency polygons representing the two sets of data on the same axes.

   (i) What are the differences between the time patterns of the two activities?

   (ii) What recommendations, if any, do you have for your classmates?

3. The table shows the mark distribution of 360 candidates in an examination in 1998. The candidates were awarded a grade, either A, B, C or F.

Grade	F	C	B	A
Mark	1–30	31–50	51–75	76–100
Number of candidates	21	95	225	19

(a) Draw a pie-chart of radius 6 cm to represent the data.
(b) Draw a histogram to illustrate the mark distribution of these 360 candidates.

A corresponding pie-chart is drawn to illustrate the grades obtained by the candidates taking the examination in 1999.

Given that the radius of this pie-chart is 8 cm and that the angle of the sector representing the grade B remains the same, calculate the number of candidates obtaining the grade B in 1999.

CHAPTER 14

Measures of Central Tendency

In this chapter, you will learn how to

- △ calculate the mean of a grouped frequency distribution;
- △ calculate the mean of a grouped frequency distribution using an "assumed mean" method.

Preliminary Problem

Do you know that the average Singaporean eats 65 kg of rice per year, that the average Singaporean man wears size 7 shoes and that the average Singaporean family has two children? However, the method used to arrive at each of the above figures may differ.

Revision

In lower secondary, we learnt how to *collect*, *classify* and *tabulate* data. We also learnt how to find the averages or the measures of central tendency of a given set of data. The three most common measures of central tendency are the mean, mode and median.

The **mean** of a set of observations is the sum of the observations divided by the number of observations in the set.

The **mode** of a set of observations is the observation which occurs most frequently in the set.

The **median** of a set of observations which is arranged in order of magnitude is

(a) the middle observation if the number of observations is odd.
(b) the mean of the two middle observations if the number of observations is even.

We shall now revise the above concepts with two examples before proceeding to find measures of central tendency of grouped data and the use of an *assumed mean*.

For Your Information

*The **mean** is the most commonly used measure of central tendency.*

*The **mode** is the data with the highest frequency. Usually, the mode is used when we need a quick and approximate measure of the centre. The mode does not exist when every item in a set of data has the same frequency.*

*The **median** is a better measure of the centre compared to the mean in cases where there are extreme values but not in cases where the data are widely separated.*

Example 1

The total number of points scored by a basketball team in eight matches are 42, 52, 48, 44, 54, 55, 42 and 63.

(a) State the modal score.
(b) Find the median score.
(c) Calculate the mean score.
(d) Find the number of points the team needs to score in its next match in order for its mean score in the nine matches to be exactly 51.

Notes

*The **modal score** must be one of the data entries.*

*To get the **median score**, we make use of the middle two data entreis.*

*To obtain the **mean score**, we must make use of all the data entries.*

Solution

(a) By observation, the modal score = 42.

(b) Arranging the eight scores in ascending order, we have
42, 42, 44, 48, 52, 54, 55, 63.

$\therefore$ the median score $= \frac{48 + 52}{2} = 50.$

(c) The total score in eight matches $= 42 + 42 + 44 + 48 + 52 + 54 + 55 + 63$
$= 400$

$\therefore$ the mean score $= \frac{1}{8} \times 400 = 50.$

(d) Given the mean score in nine matches is 51, the total score in nine matches = $51 \times 9 = 459$.
$\therefore$ the number of points the team needs to score in the ninth match = $459 - 400 = 59$.

Example 2

A die is tossed 49 times and the table below gives the results:

Number shown on the die (x)	*1*	*2*	*3*	*4*	*5*	*6*
Frequency (f)	*14*	*10*	*8*	*4*	*5*	*8*

(a) For these results, write down
(i) the mode; (ii) the median.
(b) Calculate the mean of the 49 tosses.
(c) The die is then tossed one more time. Find the number shown on the die if the mean of the 50 tosses is to be the same as that of the 49 tosses obtained in (b).

Check this out: *Choose an article of at least 500 words from a newspaper. Record the word lengths (in terms of number of letters in a word) of 500 successive words in the article. Tabulate the results in a frequency distribution. Find the mean, mode and median for the word lengths used.*

Solution

(a) (i) By observation, the mode = 1.
(ii) In the ascending sequence of numbers shown on the die, the median is the 25th number of the sequence.
$\therefore$ the median = 3.

(b) $\sum fx = 14 \times 1 + 10 \times 2 + 8 \times 3 + 4 \times 4 + 5 \times 5 + 8 \times 6 = 147$
$\sum f = 49$

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{147}{49} = 3$$

$\therefore$ the mean of the 49 tosses = 3.

(c) If the mean of the 50 tosses is 3, then for 50 tosses,
$\sum fx = 50 \times 3 = 150$.
$\therefore$ the number shown on the die after the 50th toss = $150 - 147 = 3$.

Notes: *The 14th score is 1, the 24th score is 2 and the 32nd score is 3.*

If the result of the 50th toss is less than 3, it will reduce the mean of the 50 tosses. Similarly, if the result is more than 3, it will ***increase*** *the mean.*

Exercise 14a

1. In a test marked out of a maximum possible mark of 10, a class of 40 students obtained marks as shown in the table below:

Mark	0	1	2	3	4	5	6	7	8	9	10
Frequency	1	2	5	8	4	3	7	5	2	1	2

(a) Find the median mark.
(b) Write down the modal mark.
(c) Calculate the mean mark, giving your answer correct to 1 decimal place.

2. For the distribution 8, 11, 15, 13, 8, 6, 10, 8, 23, 13, find
(a) the mode; **(b)** the mean; **(c)** the median.

3. The table gives the daily pocket money received by 45 pupils in a primary one class.

Amount of pocket money (cents)	40	50	60	75	90	100	120
Number of pupils receiving this amount	4	7	10	6	7	9	2

(a) Write down the mode of this distribution.
(b) Find the median.
(c) Calculate the mean.
(d) The mean daily pocket money of 35 pupils in another class is m cents. Given that the mean daily pocket money of the 80 pupils from these two classes is 72.25 cents, calculate the value of m.

4. The mean of a set of nine numbers is 4 and the mean of another set of 20 numbers is m. Given that the mean of the combined set of 29 numbers is 14, calculate m.

5. The marks scored by 12 children in a mathematics test are as follows:

19, 14, 13, 17, 16, 8, 16, 14, 19, 16, 11, 20

For this set of marks, find

(a) the mode; **(b)** the median; **(c)** the mean.

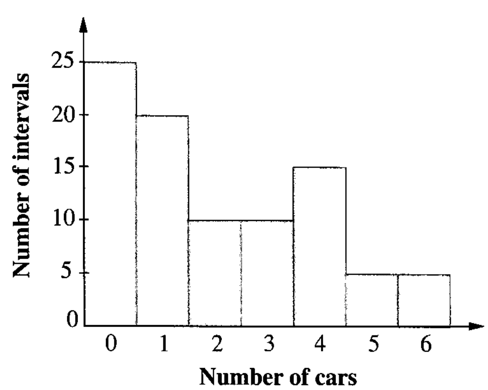
6. The table gives the number of video cassette tapes borrowed by 120 members from a video library.

Number of tapes	1	2	3	4	5
Number of members	18	63	m	19	n

(a) Show that $m + n = 20$.
(b) Given that the mean number of tapes borrowed per member is 2.4, show that $3m + 5n = 68$.
(c) Solve the equation in **(a)** and **(b)** simultaneously to find the values of m and n.
(d) State **(i)** the mode;
(ii) the median number of tapes borrowed.

7. The diagram illustrates the results of a survey conducted to find the number of cars passing by a road junction during 90 equal intervals, each of half a minute. For example, one car passed by the junction during each of 20 intervals.

(a) State the mode.
(b) Find the median number of cars.
(c) Calculate the mean number of cars per interval of time.



8. A group of students were asked how many pencils they had with them. The results are shown in the table:

Number of pencils	0	1	2	3
Number of students	5	28	39	x

(a) (i) If the mode is 2, write down an inequality satisfying x.
(ii) Using the largest possible value of x, find the mean and median.
(b) If the median is 2, write down the largest and smallest possible values of x.

Finding the Mean of Grouped Data

If we are given a set of data grouped into classes, chances are we will not know the individual numbers assigned to each class interval, unless we have the original set of data to refer to. A class interval 44.5–49.5 having a frequency of 7, where 44.5 and 49.5 are the class boundaries, will only indicate that there are seven numbers with actual values *greater than* or *equal* to 44.5, but less than 49.5, assigned to the interval. Hence, to calculate the mean of a grouped data, we instead represent all the values in a given class interval by the mid-value of the interval.

For Your Information

For grouped data, the median can be found by using the histogram, or by using the cumulative frequency polygon.

NB: In general, the mean of a set of n numbers grouped into class intervals is $\bar{x} = \frac{\Sigma fx}{\Sigma f}$, where x represents the mid-value of the class interval and f, the frequency of the interval.

Example 3

The table below shows the results of a survey on the prices of an article sold in different shops.

Price (cents)	*90–94*	*95–99*	*100–104*	*105–109*	*110–114*	*115–119*	*120–124*
Frequency	*4*	*11*	*15*	*24*	*18*	*9*	*3*

Estimate the mean of the distribution.

Solution

Class interval	*Mid-value (x)*	*Frequency (f)*	*fx*
90–94	92	4	368
95–99	97	11	1 067
100–104	102	15	1 530
105–109	107	24	2 568
110–114	112	18	2 016
115–119	117	9	1 053
120–124	122	3	366
		$\Sigma f = 84$	$\Sigma fx = 8\,968$

The mean cost of the article $\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{8\,968}{84} = 106.8$ cents.

Modal Class

The mode of a distribution cannot be determined exactly from the grouped frequency distribution. An estimate of the mode can be obtained from the grouped frequency distribution. However, this is beyond the scope of this book.

From the grouped frequency distribution we can identify the class which corresponds to the highest frequency. It is called the **modal class**. For the distribution in Example 3, the modal class is the class 105 – 109 which has the highest frequency of 24.

Exercise 14b

1. The following tables give the frequency distribution of the lifespan (in arbitrary units) of 200 insects.

Lifespan (l units)	*Number of insects*
$0 < l \leqslant 2$	15
$2 < l \leqslant 4$	47
$4 < l \leqslant 6$	50
$6 < l \leqslant 8$	35
$8 < l \leqslant 10$	24
$10 < l \leqslant 12$	14
$12 < l \leqslant 14$	8
$14 < l \leqslant 16$	7

(a) Copy and complete the table below

Lifespan (l units)	*Mid-value* (x)	*Frequency* (f)	fx
0 – 2	1	15	15
2 – 4		47	
4 – 6		50	
6 – 8	7	35	245
8 – 10		24	
10 – 12		14	
12 – 14		8	
14 – 16	15	7	105
		$\Sigma f = 200$	$\Sigma fx =$

(b) Hence, calculate the mean lifespan of the insects.

(c) Write down the modal class.

If you are living in an HDB flat or a private apartment, find out the number of children under the age of 16 in each household in your block of flat. If you are living in a landed property, find out the number of children in each household along your street. Tabulate your findings using the following as a guide:

Number of children	***Number of households***
0	
1	
2	
3	
4	
5	
more than 5	

Present your findings in the form of a bar graph. Compare it with your friends' results and then find the average number of children in each household for

(a) a block of 2-room flats
(b) a block of 3-room flats
(c) a block of 4-room flats
(d) a block of 5-room flats
(e) a block of private apartments
(f) a street of landed properties

What conclusions can you draw from your survey?

2. The lengths of 35 earthworms found in a garden are given in the following distribution table:

Length (l cm)	$0 < l \leqslant 4$	$4 < l \leqslant 6$	$6 < l \leqslant 10$	$10 < l \leqslant 12$	$12 < l \leqslant 16$
Number of earthworms	1	3	5	8	18

(a) Identify the modal class.
(b) Copy and complete the following table:

Length (cm)	*Mid-value (x)*	*Frequency (x)*	*fx*
0–4		1	
4–6		3	
6–10		5	
10–12	11	8	88
12–16	14	18	252
		$\sum f = 35$	

(c) Calculate the mean length of the 35 earthworms.

3. The distribution table below gives the ages of a group of 700 people:

Age (in completed years)	20–29	30–39	40–49	50–59	60–69	70–79	80–89
Number of people	31	100	180	225	108	46	10

Identify the modal class and calculate the mean of the distribution.

4. The grouped frequency distribution of the diameters (measured to the nearest 0.01 cm) of wooden sticks produced by a certain machine is given by the following table:

Diameter (cm)	*Number of sticks*
3.25–3.27	16
3.28–3.30	30
3.31–3.33	57
3.34–3.36	61
3.37–3.39	24
3.40–3.42	12

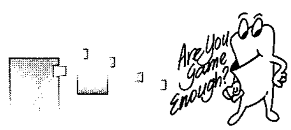
Identify the modal class and calculate the mean of the distribution.

5. The distribution table below shows the speed of 100 cars:

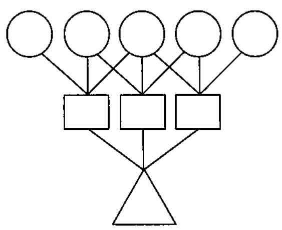
Speed (v km/h)	$0 < v \leqslant 30$	$30 < v \leqslant 40$	$40 < v \leqslant 50$	$50 < v \leqslant 60$	$60 < v \leqslant 70$	$70 < v \leqslant 80$	$80 < v \leqslant 90$
Number of cars	8	8	25	35	14	6	4

Identify the modal class and calculate the mean of the distribution.

Computation of the Mean



Fill in the five circles in the diagram with the following five numbers: 1.2, 3.7, 6.5, 2.9 and 4.6.



Next, fill in each square with the mean of the three numbers in the circles joined to the square. Then, fill in the triangle with the mean of the numbers in the three squares joined to the triangle. Find out what are the numbers to be put in the circles so as to get the smallest number possible in the triangle.

Consider the two sets of numbers below:

3, 5, 8, 12, 22
103, 105, 108, 112, 122

How do the numbers in the second set compare in size with those in the first set? Let us calculate the *mean* of each set of numbers.

The mean of 3, 5, 8, 12 and 22 is given by $\bar{d} = \frac{3+5+8+12+22}{5} = \frac{50}{5} = 10$.

The mean of 103, 105, 108, 112 and 122 is given by

$$\bar{x} = \frac{103+105+108+112+122}{5} = \frac{550}{5} = 110.$$

Now, how does the mean of the numbers in the second set compare with the mean of the numbers in the first set?

We find that each number in the second set is larger than the corresponding number in the first set by 100, i.e., a number x in the second set can be written as $x = 100 + d$ where d denotes a number in the first set. For example, $103 = 100 + 3$, $105 = 100 + 5$ and so on. We also note that the mean of the second set is **larger** than the mean of the first set of numbers by 100. i.e. $\bar{x} = 100 + \bar{d}$.

The above observation suggests a simplified method for finding the mean of a set of numbers. The method is often called the assumed mean method.

NB: In general, given a set of n numbers $x_1, x_2, x_3, \ldots, x_n$, we choose a number a and assume it to be the mean. We then find the deviation d of each number x in the set from the assumed mean a, so that

$$d_1 = x_1 - a,\ d_2 = x_2 - a,\ \ldots,\ d_n = x_n - a$$

or

$$x_1 = a + d_1,\ x_2 = a + d_2,\ \ldots,\ x_n = a + d_n$$

$$\begin{aligned}\Sigma x &= x_1 + x_2 + \ldots + x_n \\ &= (a + d_1) + (a + d_2) + \ldots + (a + d_n) \\ &= (a + a + \ldots + a) + (d_1 + d_2 + \ldots + d_n)\end{aligned}$$

Thus, the mean $\bar{x} = \frac{\Sigma x}{n}$ can be found by the following formula:

$$\bar{x} = \frac{1}{n}(a + a + \ldots + a) + \frac{1}{n}(d_1 + d_2 + \ldots + d_n) = a + \bar{d}$$

Example 4

(a) Find the mean of the following numbers 0, 1, 2, 3, 4, 5, 6.
(b) Using your result from (a), find the mean of the distribution 18, 19, 20, 21, 22, 23, 24.

Solution

(a) Let d be a member of the set of numbers 0, 1, 2, 3, 4, 5, 6.

$$\sum d = 0 + 1 + 2 + 3 + 4 + 5 + 6 = 21$$

$$\bar{d} = \frac{21}{7} = 3$$

∴ the mean of the seven numbers is 3.

(b) Comparing the set of numbers with that from (a), we find that $(18 - 0) = (19 - 1) = (20 - 2) = \ldots = (24 - 6)$. Hence, if x is a member of the set of numbers 18, 19, 20, 21, 22, 23, 24, then $x = 18 + d$ where $0 \leqslant d \leqslant 6$.

$$\bar{x} = 18 + \bar{d} = 18 + 3 = 21$$

∴ the mean of the distribution is 21.

Example 5

Find the mean of the following numbers: 297, 295, 292, 315, 311.

Solution

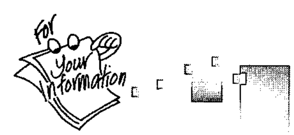
Let the assumed mean be 300, i.e., let $a = 300$.

$d_1 = 297 - 300 = -3$ $\quad d_2 = 295 - 300 = -5$ $\quad d_3 = 292 - 300 = -8$

$d_4 = 315 - 300 = 15$ $\quad d_5 = 311 - 300 = 11$

$$\therefore \quad \bar{d} = \frac{1}{5}(-3 - 5 - 8 + 15 + 11) = 2$$

$$\bar{x} = a + \bar{d} = 300 + 2 = 302.$$



The numbers $d_1, d_2, d_3, \ldots, d_n$ are smaller compared to the corresponding numbers $x_1, x_2, x_3, \ldots, x_n$. It is easier to find d. Thus using the formula $\bar{x} = a + d$, the calculation of $\bar{x}$ is simplified.

Exercise 14c

1. Find the mean of the following numbers: 2, 4, 6, 7, 8, 9.
   Use your answer to find the mean of
   (a) 5, 7, 9, 10, 11, 12
   (b) 8, 10, 12, 13, 14, 15
   (c) 19, 21, 23, 24, 25, 26

2. **(a)** Given the numbers 13, 10, 22, 16, 17, 13, 12, 17, 13
Find **(i)** the mode; **(ii)** the median; **(iii)** the mean.
(b) Use your anwser to **(a)(iii)** to find the mean of
(i) 17, 14, 26, 20, 21, 17, 16, 21, 17
(ii) 28, 25, 37, 31, 32, 28, 27, 32, 28
(iii) 113, 110, 122, 116, 117, 113, 112, 117, 113
(iv) 19, 23, 18, 19, 23, 22, 28, 16, 19

3. Find the mean of the five numbers 4, 6, 8, 9, 11.
Use your answer to find the mean of
(a) 115, 117, 119, 120, 122.
(b) 4, 7, 6, 7, 8, 7, 9, 7, 7, 11.
(c) 122, 12, 120, 12, 119, 12, 117, 12, 115, 12.
(d) 11, 200, 200, 9, 200, 200, 200, 8, 6, 4.

4. Using an assumed mean of 500, find the mean of the six numbers:

501, 503, 505, 506, 508, 513.

5. Using an assumed mean of 1 240, find the mean of the five numbers:

1 242, 1 248, 1 252, 1 244, 1 249.

6. By using an assumed mean of 66.5, find the mean of the following numbers:

54.5, 58.5, 62.5, 66.5, 70.5, 74.5.

7. **(a)** Using an assumed mean of 50, find the mean of the following numbers:

46, 47, 48, 49, 50, 51, 52, 53.

(b) Using the answer in **(a)**, find the mean of
(i) 73, 74, 75, 76, 77, 78, 79, 80
(ii) 103, 102, 101, 100, 99, 98, 97, 96

8. Find the mean of each of the following sets of numbers using an assumed mean:
(a) 425, 412, 432, 408, 418
(b) 1 419, 1 424, 1 404, 1 406, 1 400, 1 402, 1 408
(c) 1 392, 1 396, 1 376, 1 400, 1 394, 1 398, 1 381

For Your Information

When there are one or two extreme values in a set of data, the mean is not a good measure of central tendency.

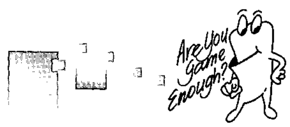
CHECK THIS OUT

Products A and B are manufactured in a factory. A random check on the quality of both types of products shows that $\frac{6}{11}$ of product A and that $\frac{4}{7}$ of product B are defective when they are produced from the first production line. From the other production line, the corresponding proportions are $\frac{1}{3}$ and $\frac{5}{14}$.
Do you agree that most of the defective ones are product B?

If not, what additional information do you need before you can conclude which type of products has a higher probability of being defective?

The Mean of a Frequency Distribution

In this section, we will learn how to use the assumed mean to find the mean of a frequency distribution.



In a town, a mother of 12 children was voted 'The Mother of the Year'. A local newspaper published a photograph of the mother with her husband and their 12 children. The chief editor of the newspaper assigned the photographer, Mr Reynolds, to take a picture of a family with the mean number of children in the town. However, Mr Reynolds was not able to produce the photograph as he could not find a family with $2\frac{1}{2}$ children, which was the mean number of children in that town. Find out whether Mr Reynolds would be able to carry out his assignment if the chief editor were to ask him to produce a photograph of a family with
(a) the modal number of children;
(b) the median number of children in the town.

Example 6

Find the mean of each of the following two distributions:

(a)

d	*0*	*1*	*2*	*3*	*4*	*5*
f	*12*	*18*	*15*	*9*	*3*	*3*

(b)

x	*15*	*16*	*17*	*18*	*19*	*20*
f	*12*	*18*	*15*	*9*	*3*	*3*

Solution

(a)

d	f	fd
0	12	0
1	18	18
2	15	30
3	9	27
4	3	12
5	3	15
	$\Sigma f = 60$	$\Sigma fd = 102$

$$\bar{d} = \frac{\Sigma fd}{\Sigma f} = \frac{102}{60} = 1.7$$

(b)

x	f	fx
15	12	180
16	18	288
17	15	255
18	9	162
19	3	57
20	3	60
	$\Sigma f = 60$	$\Sigma fx = 1\ 002$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{1\ 002}{60} = 16.7$$

In this example, we notice that for each frequency f, $x = a + d$, where $a = 15$.

In addition, we note that $\bar{x} = a + \bar{d}$, where $a = 15$.

NB: The simplified method for finding the mean of a set of numbers by using an assumed mean can be applied to find the mean of a frequency distribution.

The computation of the mean in Example 3 can be made less tedious by using the **assumed mean method**. The assumed mean chosen should be one of the mid-values. It should be as near to the middle with as high a frequency as possible.

The data in the following table is used to illustrate the assumed mean method. The assumed mean of 107 is chosen and it is denoted by a in the table.

Mid-value (x)	*Frequency (f)*	*Deviation (d) = x – a*	*fd*
92	4	–15	–60
97	11	–10	–110
102	15	–5	–75
107(a)	24	0	0
112	18	5	90
117	9	10	90
122	3	15	45
	$\sum f = 84$		$\sum fd = -20$

From the table, $\bar{d} = \frac{\Sigma fd}{\Sigma f} = \frac{-20}{84}$

$\therefore$ the actual mean, $\bar{x} = a + \bar{d} = 107 + \left(\frac{-20}{84}\right) = 106.8$ cents.

Example 7

The marks scored in a test by 500 children are given in the following table:

Mark (x)	*Number of children*
60 < x ≤ 80	*81*
80 < x ≤ 100	*103*
100 < x ≤ 120	*127*
120 < x ≤ 140	*99*
140 < x ≤ 160	*90*

Using an assumed mean of 110, calculate the mean mark.

Solution

Mark	*Mid-value (x)*	*Frequency (f)*	*d = x – 110*	*fd*
60–80	70	81	–40	–3 240
80–100	90	103	–20	–2 060
100–120	110(a)	127	0	0
120–140	130	99	20	1 980
140–160	150	90	40	3 600
		$\sum f = 500$		$\sum fd = 280$

$\bar{d} = \frac{\Sigma fd}{\Sigma f} = \frac{280}{500} = 0.56$

$\bar{x} = 110 + \bar{d} = 110 + 0.56 = 110.56$

$\therefore$ the mean mark scored by 500 children is 110.56.

Exercise 14d

1. Copy and complete the table which uses an assumed mean of 33 kg.

Calculate the mean of the distribution.

Weight (kg) (x)	*Number of pupils* (f)	$d = x - 33$	fd
30	6	–3	–18
31	15		
32	20		
33(a)	25		
34	18		
35	13		
36	3		
	$\Sigma f =$		$\Sigma fd =$

2. Copy and complete the following table which uses an assumed mean of 20.

Mid-value (x)	*Frequency* (f)	$d = x - 20$	fd
8	5		
12	7		
16	12		
20(a)	15	0	0
24	9		
28	2		
	$\Sigma f =$		$\Sigma fd =$

Calculate the mean of the distribution.

3. Copy and complete the following tables. Table (i) uses an assumed mean of *60* and Table (ii) uses an assumed mean of *61*.

Table (i)

Age (x years)	*Frequency* (f)	$d = x - 60$	fd
58	4		
59	9		
60(a)	14		
61	13		
62	6		
63	4		
	$\Sigma f =$		$\Sigma fd =$

Table (ii)

Age (x years)	*Frequency* (f)	$d = x - 61$	fd
58	4		
59	9		
60	14		
61(a)	13		
62	6		
63	4		
	$\Sigma f =$		$\Sigma fd =$

Calculate the mean age in each case. Do you get the same mean?

4. Calculate the mean of each of the following distributions using an assumed mean. For each case state the median and the mode.

(a)

x	35	45	55	65	75	85	95
f	4	6	10	15	8	5	2

(b)

x	150	155	160	165	170	175
f	2	4	3	5	2	1

(c)

x	24.5	34.5	44.5	54.5	64.5	74.5	84.5
f	8	11	16	20	23	10	12

(d)

x	42	47	52	57	62	67	72	77
f	5	23	32	23	11	3	2	1

(e)

x	549	649	749	849	949	1 049	1 149
f	8	14	21	29	15	10	3

5. The mass of 100 pebbles (in grams) picked up by a boy from a beach are as follows:

Mass of pebbles (x g)	*Frequency*
$55 < x \leqslant 65$	2
$65 < x \leqslant 75$	3
$75 < x \leqslant 85$	9
$85 < x \leqslant 95$	23
$95 < x \leqslant 105$	26
$105 < x \leqslant 115$	21
$115 < x \leqslant 125$	10
$125 < x \leqslant 135$	5
$135 < x \leqslant 145$	1

Copy and complete the table below and, hence, find the mean mass of the 100 pebbles.

Mass of pebbles (g)	*Mid-point (x)*	$d = x - 100$	*Frequency (f)*	*fd*
55–65				
65–75				
75–85				
85–95				
95–105	100(a)			
105–115				
115–125				
125–135				
135–145				
			$\Sigma f = 100$	$\Sigma fd =$

6. The heights of 50 plants were measured correct to the nearest centimetre. Copy and complete the table below and, hence, calculate the mean height of these 50 plants.

Heights (cm)	*Mid-point* (x)	$d = x - 25.5$	*Frequency* (f)	fd
1–10	5.5		4	
11–20	15.5		6	
21–30	25.5(a)		14	
31–40	35.5		16	
41–50	45.5		10	
			$\Sigma f = 50$	$\Sigma fd =$

7. Thirty bulbs were life-tested and their lifespan to the nearest hour are as follows:

167 171 179 167 171 165 175 179 169 168 171 177 169 171 177
173 165 175 167 174 177 172 164 175 179 179 174 174 168 171

(a) Find the mean lifespan by dividing their sum by 30.
(b) Find the mean lifespan by grouping the lifespans using class intervals 164–166, 167–169 and so on.

Lifespan (h)	*Tally*	*Mid-value* (x)	*Frequency* (f)	fx
164–166		165		
167–169		168		
170–172		171		
173–175		174		
176–178		177		
179–181		180		
			$\Sigma f = 30$	$\Sigma fx =$

(c) Repeat **(b)** using class intervals 164–165, 166–167 and so on.

Lifespan (h)	*Tally*	*Mid-value* (x)	*Frequency* (f)	fx
164–165		164.5		
166–167		166.5		
168–169		168.5		
170–171		170.5		
172–173		172.5		
174–175		174.5		
176–177		176.5		
178–179		178.5		
			$\Sigma f = 30$	$\Sigma fx =$

Find out from 20 of your schoolmates the amount of pocket money each of them receive for a week. Organise the data into a group frequency distribution table using the following as a guide:

Amount of pocket money ($P)	*Number of students*
$0 < P \leqslant 5$	
$5 < P \leqslant 10$	
$10 < P \leqslant 15$	
$15 < P \leqslant 20$	
⋮	
$P > 50$	

Using data from the table, find the mode and median pocket money of the students in your sample. Is it possible to calculate the mean pocket money for the above table if there are some students in the row with $P > 50$?

Now, use the raw data that you have collected to find the mode, median and mean of the pocket money of your schoolmate.

Are the two sets of answers about the same?

Suppose you are to conduct the same survey on pupils of a primary two class, what results will you expect to get for the mode, median and mean values?

(d) Are the values of the mean of the dsitribution found in **(a)**, **(b)** and **(c)** the same? Repeat **(b)** by taking 171 as the assumed mean.

Lifespan (h)	*Mid-value* (x)	$d = x - 171$	*Frequency* (f)	fd
164–166	165			
167–169	168			
170–172	171(a)			
173–175	174			
176–178	177			
179–181	180			
			$\Sigma f = 30$	$\Sigma fd =$

8. In an examination taken by 400 students, the scores were as shown in the following distribution table:

Marks	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency	8	14	32	56	102	80	54	30	16	8

Find **(a)** the modal class; **(b)** the mean.

9. The following table shows the time taken for 100 lorries to travel between two towns using a certain route.

Time taken (t min)	*Number of lorries*
$116 < t \leqslant 18$	1
$118 < t \leqslant 120$	6
$120 < t \leqslant 122$	23
$122 < t \leqslant 124$	28
$124 < t \leqslant 126$	27
$126 < t \leqslant 128$	9
$128 < t \leqslant 130$	5
$130 < t \leqslant 132$	1

Calculate the mean travelling time. State the modal class and construct a histogram to represent the distribution.

10. The table below shows the distribution of ages of the members of a club.

Age (years)	20–24	25–29	30–34	35–39	40–44	45–49	50–54
Frequency	22	48	60	36	22	10	2

Calculate the mean age of the members of the club.

11. A machine in a factory broke down 100 times in a certain year. The length of time taken to repair the machine each time was recorded. The table below shows the distribution of the lengths of time (t minutes) taken to repair the machine

Repair time	*Frequency*
$0 < t \leqslant 10$	3
$10 < t \leqslant 20$	13
$20 < t \leqslant 30$	30
$30 < t \leqslant 40$	25
$40 < t \leqslant 50$	14
$50 < t \leqslant 60$	8
$60 < t \leqslant 70$	4
$70 < t \leqslant 80$	2
$80 < t \leqslant 90$	1

Calculate the mean length of time taken to repair the machine.

Summary

1. A set of data can be described by numerical quantities called **averages** or **measures of central tendency**. The three common measures of central tendency are the **arithmetic mean, median** and **mode.**
2. The *arithmetic mean*, or the *mean*, of a set of observations is the sum of the observations divided by the number of observations in the set.
3. The *mode* of a set of observations is the observation which occurs most frequently in the set.
4. The *median* of a set of observations (arranged in order of magnitude) is the
   (a) middle observation if the number of observations is odd.
   (b) mean of two middle observations if the number of observations is even.
5. The calculation of the mean of a set of data can be simplified by using the **assumed mean method.**

Review Questions 14

☆1. **(a)** The distribution of 300 values of a variable x is shown in the following table:

x	0	1	2	3	4	5	6	7
Frequency	18	37	69	66	45	26	24	15

For this distribution, find
(i) the mode; **(ii)** the median; **(iii)** the mean.

(b) In a test of general knowledge, 300 pupils were shown 30 pictures of different places in the city where they live. They had to write down the names of as many places as they could on a piece of paper. The results are given in the table above. As an example, 37 pupils correctly identified 19 places and so on.
Using your result from **(a)** above, state the mean number of places correctly identified.

Number of places named correctly	18	19	20	21	22	23	24	25
Number of pupils	18	37	69	66	45	26	24	15

☆**2.** A bag contained six cards, each bearing one of the numbers 1, 2, 3, 4, 5, 6. A card was drawn from the bag, its number noted down and then replaced. This was repeated 60 times and the frequency distribution table below shows the results:

Number	1	2	3	4	5	6
Frequency	m	14	n	5	10	11

Given that the mean of this distribution is 3.6,

(a) find the values of m and n;

(b) state the mode and median of the distribution.

☆**3.** The numbers 24, 22, 34, 28, 29, 24, 25, 29, x and y have a mode of 29 and a median of 27. Find the values of x and y, given that $x < y$.

☆**4.** The heights of a group of 30 children were measured to the nearest centimetre and the readings are recorded as shown below:

122 144 136 136 140 139 126 120 125 129 127 116 132 138 124
135 122 137 135 129 133 130 128 118 131 127 128 147 133 119

(a) Copy and complete the table given. **(b)** Write down the modal class.

(c) Estimate the mean height of the 30 children using the table above.

(d) Estimate the percentage of children whose heights are below 135 cm.

(e) Draw a histogram and frequency polygon representing the distribution of heights on the same diagram.

Height (cm)	*Mid-value (x)*	*Tally*	*Frequency (f)*	$d = x - 132$	*fd*
115–119					
120–124					
125–129					
130–134	132(*a*)				
135–139					
140–144					
145–149					
			$\Sigma f = 30$		$\Sigma fd =$

☆**5.** The scores of 28 pupils in an English test are given below:

69	72	56	68	62	61	58
57	43	66	53	45	72	64
46	50	61	59	52	63	54
62	61	55	52	58	64	63

(a) Calculate the mean score of these 28 pupils in the test.

(b) Copy and complete the table below:

Mark	*Mid-value (x)*	*Tally*	*Frequency (f)*	$d = x - 57$	*fd*
40–44					
45–49					
50–54					
55–59	57(*a*)				
			$\Sigma f = 28$		$\Sigma fd =$

(c) Use the table in **(b)** to estimate the mean score.

(d) Find the difference between the estimated mean score in **(c)** and the actual mean score in **(a)** and express this difference as a percentage of the actual mean score.

(e) Draw a histogram and a frequency polygon representing the distribution using the same diagram.

☆**6.** **(a)** The distribution of 100 values of a variable x is shown in the table below:

x	0	1	2	3	4	5	6
Frequency	20	30	25	16	5	2	2

For this distribution, find

(i) the mode; **(ii)** the median; **(iii)** the mean.

(b) A newsagent recorded the number of requests for a certain weekly magazine over a period of 100 successive weeks. The results are as shown below:

Number of requests per week	17	18	19	20	21	22	23
Frequency	20	30	25	16	5	2	2

Using your result from **(a)(iii)**, state the mean number of requests per week.

The newsagent bought these magazines for 10 cents each and sold them for 20 cents each. He thus made a profit of 10 cents on each magazine sold and incurred a loss of 10 cents on each magazine that remained unsold. He decided to buy 19 copies of this magazine each week. Calculate the profit he made for the

(i) 20 weeks when he had 17 requests per week;

(ii) whole period of 100 weeks.

CHECK THIS OUT

Suppose there are 25 employees in a company. The distribution of their monthly wages is shown in the table below:

Salary	*Frequency*
\$50 000	*1*
\$25 000	*1*
\$15 000	*4*
\$10 000	*2*
\$5 000	*5*
\$1 500	*12*

When asked about the monthly salary of the employees, the following statements are made:

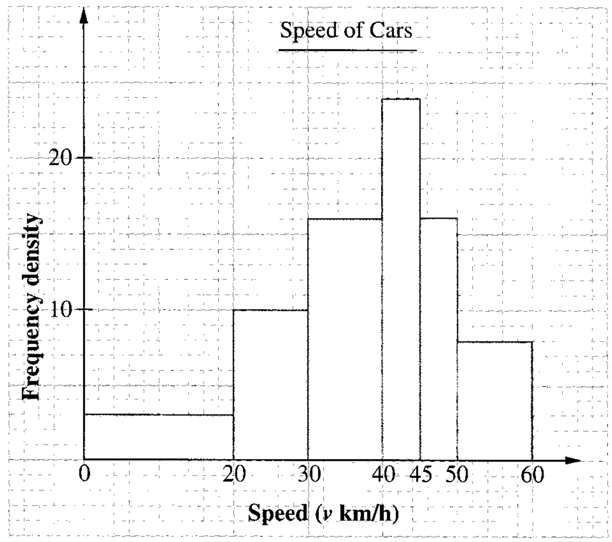
Employer A *says that each employee earns \$7 920 on the average.*

Employer B *says that over 50% of the employee earn at least \$5 000.*

Employer C *says that almost half the number of the employees earn \$1 500. Are their statements correct?*

Give your comments.

☆**7.** A survey is carried out to find the speeds of cars passing a certain road. The histogram illustrates the results of the survey.

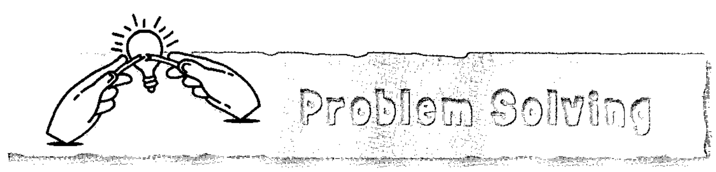


(a) Copy and complete the following table.

Speed (v km/h)	*Number of cars*
$0 < v \leqslant 20$	
$20 < v \leqslant 30$	10
$30 < v \leqslant 40$	16
$40 < v \leqslant 45$	12
$45 < v \leqslant 50$	
$50 < v \leqslant 60$	

(b) Find the number of cars included in the survey.

(c) Work out an estimate of the mean speed of the cars along the road. (*C*)



1. In a survey the speeds of cars passing a point on a road were recorded. The following histogram illustrates the results.

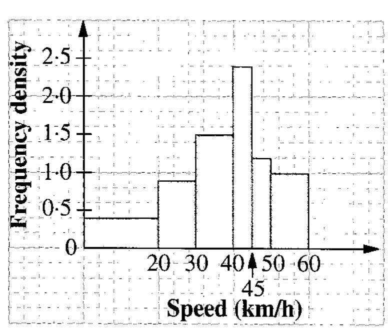
(a) How many cars were involved in the survey?

(b) Which speed interval includes the median speed of these cars?

(c) Will the vertical line drawn through the median speed in the histogram divide the total area of the rectangles into half? If so, why?

(d) Indicate, roughly, the position of the median speed in the histogram.

(e) Challenge yourself to estimate this median speed from the histogram.



2. The table below shows the number of pupils in a class of 36 who scored marks between 3 and 9 inclusive in a test.

Marks	3	4	5	6	7	8	9
Number of pupils	3	4	x	7	y	5	4

(a) If the mean is 6, calculate the values of x and y.
(b) With the values of x and y, state the modal mark and the median mark.

3. Choose any novel. Take any 100 consecutive words from it and count the number of letters in each word. Tabulate your results in the form of a frequency table and draw a histogram. Find the mean word length. Repeat the experiment for another novel by the same author and again for a novel by a different author. Compare and comment on the results.

Repeat the above by considering 20 consecutive sentences and count the number of words in each sentence instead.

4. In an examination, each pupil in a group scores either 5, 10 or 15 marks. The number of pupils scoring each mark is shown in the table below:

Mark	5	10	15
Number of pupils	8	12	x

(a) If the mode is 10, write down the range of values of x.
(b) If the median mark is 10, write down the largest possible value of x.
(c) Using the value of x found in **(b)**, calculate the mean mark.

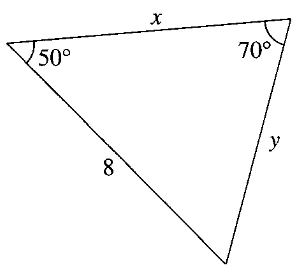
Revision Exercise IV No. 1

1. **(a)** In a sale, the cost of a dress was reduced by 25%. If the original price is \$80, what is the cost price?

(b) If $u = a\sqrt{\frac{2t}{b}}$, calculate the value of u when $a = 5$, $t = 8$ and $b = 25$.

(c) If $h = \frac{7\,gu^2}{16\,R}$, express u in terms of g, h and R.

(d) Find the values of x and y in the diagram below:



2. The table below shows the estimated population distribution of a certain country:

Age (years)	*Number* (million)
0–4	3.6
5–14	7.5
15–29	11.8
30–59	17.8
60–109	5.3

Draw a histogram to represent the above data. Use an assumed mean method to find the mean age of the population.

3. A car, starting from rest, attains a velocity of 18 m/s after it has been travelling for 9 seconds with constant acceleration. It continues at this speed for a further 31 seconds. The brakes are then applied, which brings the car eventually to rest after a distance of 36 m.

(a) Draw a velocity-time graph for the journey.

(b) Find the acceleration and retardation of the car.

(c) Find the average speed of the car for the whole journey.

4. The marks obtained by 48 students in a Science test are as follows:

48 37 78 90 51 76 88 94 33 35 74 78
23 36 54 60 65 46 42 43 45 28 32 36
37 44 68 72 59 48 43 89 78 32 76 84
53 39 39 67 68 83 27 43 75 67 83 57

(a) Construct a frequency table for the above marks using class intervals 21–30, 31–40, 41–50, and so on.

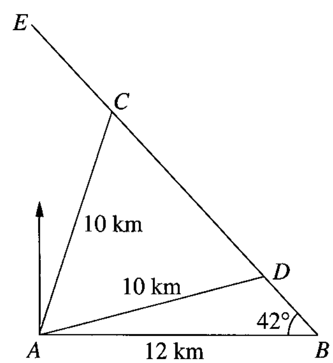
(b) Use your table to calculate the mean.

5. The following diagram shows a point B which lies 12 km due east of A. A straight road BE makes an angle of 42° with AB, C and D are two points on the road such that $AD = AC = 10$ km. Calculate

(a) the bearing of C from A;

(b) the bearing of D from A;

(c) the distance between C and D.



6. $ABCD$ is a quadrilateral in which $AB = AD = 6$ cm, $BD = 9$ cm, $BC = 12$ cm and $C\hat{B}D = 60°$. Find

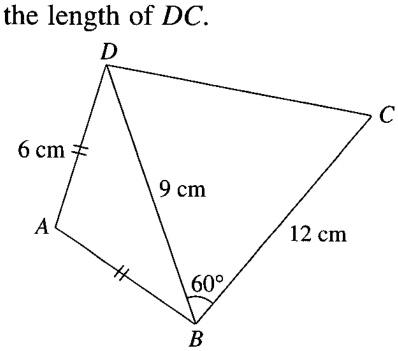
(a) $B\hat{A}D$;

(b) the area of $\triangle ABD$;

(c) the area of the quadrilateral $ABCD$;

(d) $A\hat{B}C$;

(e) the length of DC.



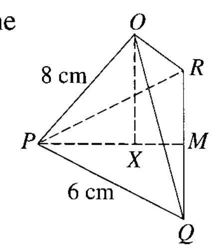
7. The following data represents the marks obtained by 36 pupils in a test:

21	24	25	26	30	27	24	25	25	25	30	36
27	27	31	23	24	23	22	19	24	23	24	24
28	28	32	21	29	26	19	30	26	27	32	25

Taking class intervals 19–20, 21–22, 23–24, etc., construct a frequency distribution table and construct a histogram for the distribution.

8. Two cylindrical jars A and B have diameters $2x$ cm and $5x$ cm respectively. Initially B is empty and A contains water to a depth of 20 cm. If all the water in A is poured into B, find the height of water in jar B.

9. $OPQR$ is a tetrahedron. M is the mid-point of QR, $PQ = QR = PR = 6$ cm and $OP = OQ = OR = 8$ cm. Calculate
   - **(a)** the height OX of the tetrahedron.
   - **(b)** $O\hat{P}X$,
   - **(c)** $O\hat{M}X$.



10. **(a)** The surface area of two cups are in the ratio 9 : 64. If the smaller cup has a height of 25 cm and a volume of 2 400 cm^3, calculate
    - **(i)** the height of the larger cup;
    - **(ii)** the volume of the larger cup.

    (b) An open cylindrical tank has a height of 1.4 m and a diameter of 60 cm. Calculate its total external surface area, giving your answer in m^2.

Revision Exercise IV No. 2

1. **(a)** Find 30% of \$50.00.

   (b) Evaluate $\left(\frac{3}{4}\right)^2 - \left(\frac{1}{8} \div \frac{2}{3}\right)$.

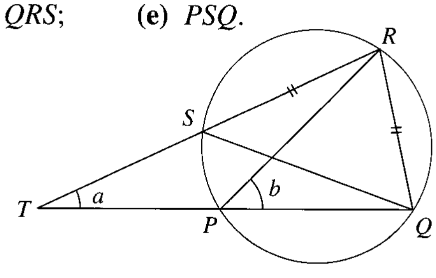
   (c) Evaluate the following by using factors. Show the method clearly.

$$\frac{5.74 \times 63 + 5.74 \times 37}{(7.87)^2 - (2.13)^2}.$$

2. **(a)** Factorise:
   - **(i)** $4x(3a + 4b) - 3y(3a + 4b)$;
   - **(ii)** $\frac{x^4}{64} - \frac{y^4}{9}$.

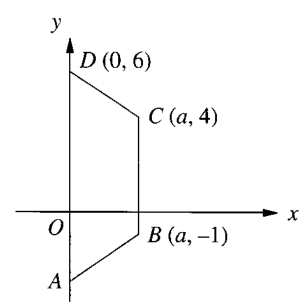
   (b) Solve the following equations:
   - **(i)** $\frac{x+1}{4} - 3 = \frac{35}{12} - \frac{3x+2}{3}$;
   - **(ii)** $3x + \frac{1}{x} = 7 - \frac{3}{x}$.

3. In the diagram, TPQ and TSR are straight lines and $SR = QR$. Given that $Q\hat{T}R = a$ and $Q\hat{P}R = b$, express in terms of a and b,
   (a) $R\hat{Q}S$; **(b)** $P\hat{R}S$; **(c)** $P\hat{S}T$;
   (d) $Q\hat{R}S$; **(e)** $P\hat{S}Q$.



4. In the diagram, as shown in question 3, given that $a = 30°$, $b = 45°$ and $SR = 6$ cm,
   - **(a)** show that SQ is the diameter of the circle $PQRS$;
   - **(b)** calculate
     (i) SQ; **(ii)** PR; **(iii)** TR.

5.



$ABCD$ is a trapezium, in which $DC = \sqrt{13}$ units. If B is the point $(a, -1)$, C is the point $(a, 4)$, D is the point $(0, 6)$ and the gradient of AB is $\frac{2}{3}$, calculate
- **(a)** the coordinates of the point C;
- **(b)** the coordinates of the point A,
- **(c)** the area of the trapezium $ABCD$;
- **(d)** the length of AB;
- **(e)** the equation of AB.

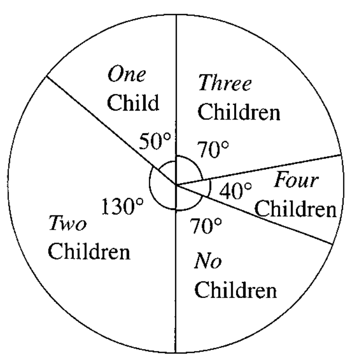
6. The following is an incomplete table of values of the graph of $y = x^2(x - 2)$:

x	−2	−1	0	1	2	3	4
y	−16		0		0		

(a) Calculate the missing values of y.
(b) Using scales of 2 cm to 1 unit on the x-axis and 2 cm to 4 units on the y-axis, draw the graph of $y = x^2(x - 2)$, for $-2 \leqslant x \leqslant 4$.
(c) By drawing a tangent, estimate the gradient of the curve when $x = 1\frac{1}{2}$.
(d) Use your graph to solve the equation $x^2(x - 2) = -1$.
(e) By drawing a suitable straight line on the same axes, use your graph to find three values of x which satisfy the equation $x^2(x - 2) = x - 2$.

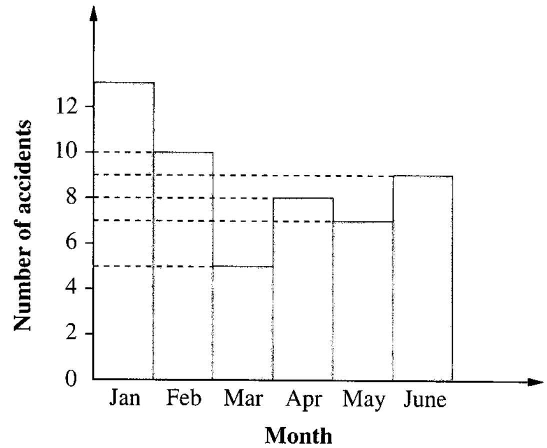
7. Thirty six couples are each asked to indicate how many children they have. The following pie chart represents the results obtained.

(a) How many families have no children?
(b) How many families have two children?
(c) How many families have more than two children?



8. The following chart illustrates the number of accidents on a main road during the first six months of a certain year.

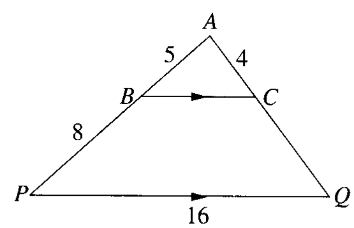
(a) State the modal number of accidents per month. Hence, write down the month with the most accidents.
(b) Find the total number of accidents in the six months.
(c) Calculate the mean number of accidents per month.



9. The volumes of two spheres are 108 cm^3 and 32 cm^3. Find the ratio of their

(a) radii;
(b) surface areas.

10.



In the figure, BC is parallel to PQ. If AB = 5 cm, BP = 8 cm, AC = 4 cm and PQ = 16 cm, calculate the lengths of

(a) BC;
(b) CQ.

Revision Exercise IV No. 3

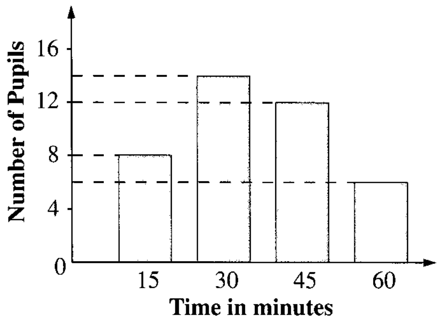
1. Given that $a = 4.8 \times 10^4$ and that $b = 8 \times 10^{-3}$, calculate the following, expressing each answer in standard form.

(a) ab^2; **(b)** $\frac{a}{b}$; **(c)** $\frac{a^2}{b}$.

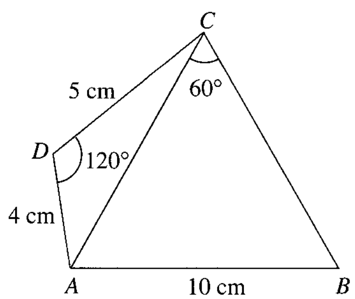
2. The following bar chart shows the time, in minutes, taken by a group of pupils to get to school.

(a) How many pupils were there in the group?

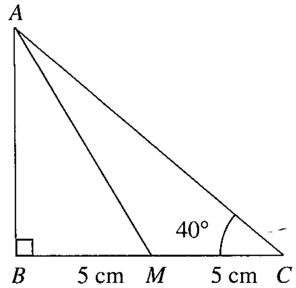
(b) How many pupils took more than 15 minutes to get to school?
(c) What percentage of the pupils spent 30 minutes to travel to school?
(d) Calculate the mean time taken by each pupil.



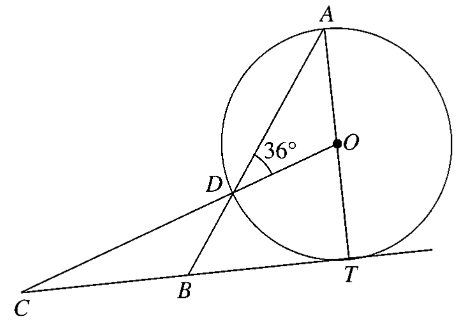
3. In the following diagram, $ABCD$ is a quadrilateral. $AD = 4$ cm, $DC = 5$ cm, $AB = 10$ cm, $A\hat{D}C = 120°$ and $A\hat{C}B = 60°$. Calculate
(a) the length of AC;
(b) DAC;
(c) ABC;
(d) the length BD;
(e) the area of the quadrilateral $ABCD$.



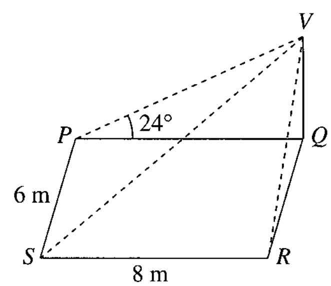
4. In the diagram, M is the mid-point of the side BC of $\triangle ABC$ and $A\hat{B}C = 90°$. Given that $BM = MC = 5$ cm and that $A\hat{C}B = 40°$, calculate
(a) AM;
(b) AC;
(c) $A\hat{M}B$;
(d) $C\hat{A}M$.



5. In the diagram, AOT is a diameter of the circle, centre O and CBT is a tangent to the circle at T. Given that $A\hat{D}O = 36°$, find
(a) $B\hat{A}T$; (b) $A\hat{B}T$; (c) $O\hat{C}T$.



6. At one corner of a horizontal rectangular field stands a vertical pole VQ.



Given that $SR = 8$ m, $PS = 6$ m and the angle of elevation of V from P is 24°, calculate
(a) VQ,
(b) VS,
(c) the angle between VS and VQ,
(d) the angle of elevation of V from R.

7. (a) The table below shows the age in months of television sets in a hospital.

Age in months (x)	*Number of sets (f)*
$0 < x \leqslant 10$	10
$10 < x \leqslant 20$	15
$20 < x \leqslant 30$	25
$30 < x \leqslant 40$	20
$40 < x \leqslant 60$	30
$60 < x \leqslant 90$	15

(i) State the modal class.
(ii) Illustrate the data by a histogram.

(b) Copy and complete the following table, which uses an assumed mean of 25.

Age in months (x)	*Number of sets (f)*	$x - 25$	$f(x - 25)$
5	10	–20	–200
15	15		
25	40		
35	55		
45	20		
55	10		
Total =	150	Total =	

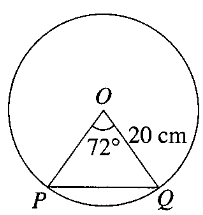
Hence, or otherwise, calculate the mean of the distribution.

8. O is the centre of a circle of radius 20 cm. PQ is a chord and $P\hat{O}Q = 72°$.

(a) Calculate the length of arc PQ.

(b) Calculate the area of the sector OPQ.

(c) Find the area of $\triangle OPQ$ and, hence, write down the area of the shaded segment PQ.



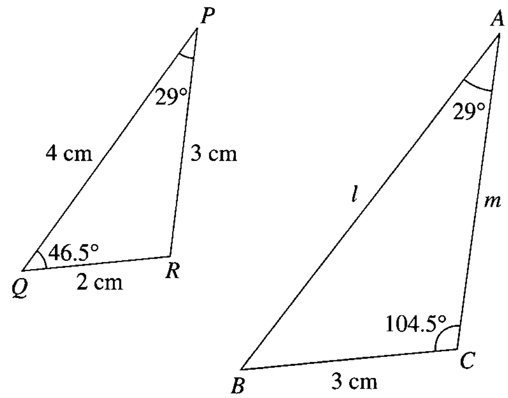
9. A particle, starting from rest, attains a speed of 16 m/s after it has travelled for 4 seconds with uniform acceleration. It continues at this speed for a further 8 seconds and then it slows down with uniform retardation to a certain point in a further 8 seconds.

(a) Draw the speed-time graph.

(b) Find the acceleration of the particle.

(c) Find the total distance travelled.

10.



For the figures given above,

(a) Explain, clearly, why $\triangle PQR$ and $\triangle ABC$ are similar.

(b) Calculate the lengths of the unknown sides l and m.

End-of-Year Examination Specimen Paper 1

Time: $2\frac{1}{2}$ h

*Answer **all** the questions in **Section A** and any **5** questions in **Section B**.*

Section A (50 marks)

*Answer **all** the questions in this section. Calculators are not allowed.*

1. Evaluate

(a) $1\frac{3}{5} + 2\frac{11}{20} - \frac{19}{15}$ **(b)** $8^{-\frac{2}{3}} + 32^{\frac{1}{5}}$

(c) $5.7^0 + 16^{-\frac{3}{4}}$ [4]

2. Express 0.023 984 9

(a) as a decimal correct to 3 decimal places;

(b) correct to 4 significant figures;

(c) in the standard form $A \times 10^n$, where $1 \leqslant A < 10$ and n is an integer. [3]

3. Factorise the following expressions:

(a) $(x - 2)^2 - 25$ **(b)** $21 - x - 2x^2$

(b) $4(ax - ay) + 8(2y - 2x)$ [4]

4. Copy and complete the following sequence of numbers:

(a) 4, 5, 8, 13, 20, ______, ______. [1]

(b) 3, 4, 8, 17, 33, ______, ______. [1]

(c) $\frac{7}{8}, \frac{2}{3}, \frac{1}{2}, \frac{4}{11}, \frac{1}{4}$, ______, ______. [1]

5. If $x + y = 2$ and $x^2 + y^2 = 8$, find the value of

(a) xy; **(b)** $x - y$. [3]

6. Make a the subject of the formula

$F = \dfrac{2ab - 4c}{3a - 5}$. [3]

7. Given that y varies directly as the cube of x, copy and complete the following table, and then express y in terms of x.

x	1	2	
y	3		192

[3]

8. Solve the following equations:

(a) $\frac{x}{2} + 3 = \frac{1}{4}x - 5$ [1]

(b) $(2x - 3)^2 - 4x(x - 3) = 5x + 4$ [2]

9. The difference between the squares of two positive consecutive odd numbers is 48. Taking x as the smaller number, form an equation in x and, hence, find the numbers. [3]

10. A cyclist has to cover a journey of 117 km in $6\frac{1}{2}$ hours. After $2\frac{3}{4}$ hours, he finds that he has travelled 57 km. Find the decrease in speed he has to make in order to arrive at his destination on time. [3]

11. Find the weight, in kg, of a rectangular sheet of metal plate, 1.2 m long, 0.8 m wide and 65 mm thick, given that the density of the metal is $8\frac{1}{4}$ g/cm^3. [3]

12. Given $a + 4b = 7a - b$, and $a \neq 0$, find the value of $\frac{5b}{7a}$. [2]

13. The price of coffee has increased by 25%. What percentage of coffee consumption is to be decreased, so that there would be no increase in the expenditure for a household? [3]

14. The coordinates of A and B are (7, 4) and (–2, –5) respectively. Find

(a) the gradient of AB;

(b) the equation of AB. [3]

15. Two dice were thrown simultaneously and the total score tabulated. The results of 80 throws are shown in the following table:

Score	*Frequency*
2	2
3	4
4	6
5	8
6	17
7	16
8	8
9	7
10	5
11	4
12	3

Find **(a)** the mode; **(b)** the median; **(c)** the mean. [4]

16. A pie chart was drawn to show how Mr Law's total salary is spent. Income tax takes up 15% of his total salary and he pays 10% of the remainder as rent. Calculate the angles of the sectors used to represent the amount he pays

(a) in income tax; **(b)** as rent. [3]

Section B (50 marks)

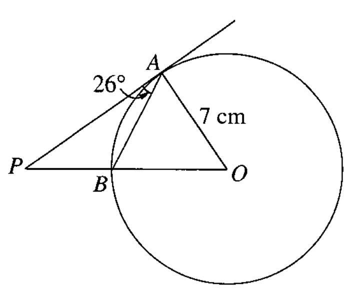
*Answer any **5** questions in this section.*

17. (a) In the diagram O is the centre of the circle and PA is the tangent to the circle at A. If $OA = 7$ cm and $P\hat{A}B = 26°$, calculate

(i) $A\hat{O}B$;

(ii) the length of AP;

(iii) the area of the minor sector AOB.

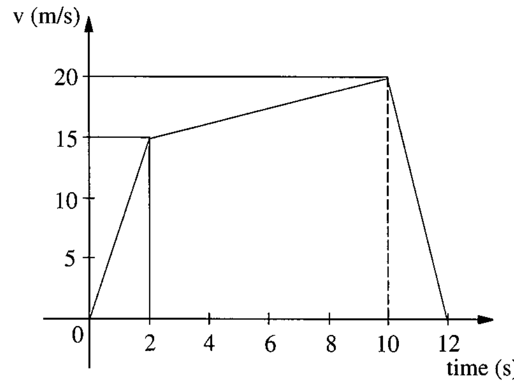


(b) The graph below shows the speed-time graph of a particle over a period of 12 seconds of its motion. Find

(i) its acceleration during the first 2 seconds;

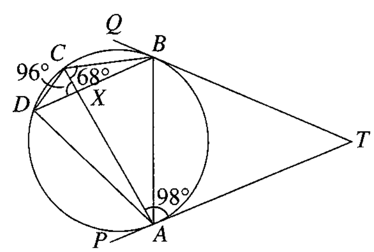
(ii) the total distance moved;

(iii) its average speed during the 12 seconds.

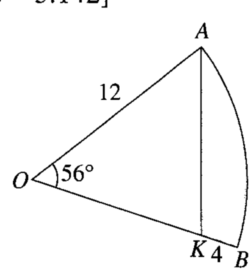


18. (a) In the diagram, TAP and TBQ are tangents to the circle at A and B respectively. AC and BD intersect at the point X. Given that $T\hat{A}C = 98°$, $C\hat{X}D = 96°$ and $A\hat{C}B = 68°$, calculate

(i) $T\hat{B}A$; **(ii)** $A\hat{B}C$; **(iii)** $P\hat{A}D$.



(b) In the diagram, O is the centre of the sector OAB of radius 12 cm. K is a point on OB such that $KB = 4$ cm. If $A\hat{O}B = 56°$, calculate the area of the shaded region, giving your answer correct to 2 decimal places.
[Take $\pi = 3.142$]



19. A table of values for the graph of $y = x^2 - 3x + 9$ is given below.

x	-2	-1	0	1	2	3	4	5
y	a	13	9	7	7	b	13	19

Calculate the values of a and b.
Using a scale of 2 cm to represent 1 unit on the x-axis and a scale of 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 - 3x + 9$.
Use your graph to solve the equations:
(a) $x^2 - 3x + 9 = 11$ **(b)** $x^2 - 3x = 7$
By drawing a suitable straight line graph on the same axes, solve the equation $x^2 - 4x + 3 = 0$.

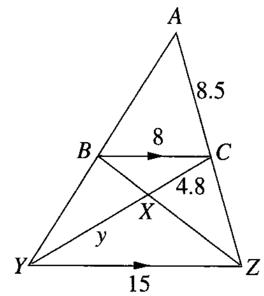
20. The results of an examination taken by 150 pupils are given in the following table:

Marks obtained	*Number of pupils*
0–9	7
10–19	12
20–39	32
40–59	46
60–79	30
80–89	14
90–99	9

(a) Draw a histogram to illustrate these results.
(b) Estimate the median mark.
(c) Calculate the mean mark.

21. **(a)** American Maurice Green ran the 100 m race in a world record time of 9.79 seconds in June 1999. Express his record speed in km/h giving your answer correct to 2 decimal places.
(b) Adam sells a comic book to Bob at a profit of 25%. Bob then sells the same book to Charles at a loss of 15%. If Charles paid $5.10 for the book, how much did Adam pay for it?

(c) In the diagram BC is parallel to YZ.
(i) Name two pairs of similar triangles.
(ii) If BC = 8 cm, CX = 4.8 cm, AC = 8.5 cm and YZ = 15 cm, calculate the lengths of XY and CZ.



22. **(a)** Suppose a bus leaves a town A for another town B at 11 00 and travels at a uniform speed of 50 km/h. An hour later, a car travelling at a uniform speed of 80 km/h leaves town A for town B by the same route. Draw a distance-time graph for the bus and the car and find graphically the time at which the car overtakes the bus.
(b) The ratio of the heights of two similar cones is 2 : 5. If the smaller cone has a curved surface area of 250 cm² and a volume of 480 cm³, calculate the curved surface area and volume of the larger cone.

End-of-Year Examination Specimen Paper 2

Time: $2\frac{1}{2}$ h

*Answer **all** the questions in **Section A** and any 5 questions in **Section B**.*

Section A (50 marks)

*Answer **all** the questions in this section. Calculators are not allowed.*

1. Factorise completely
(a) $4 - 16x^2$;
(b) $2a - 2b + 3c(b - a)$. [4]

2. Solve the simultaneous equations
$0.4x + 0.3y = 1.7$ and $1.4x - 0.5y = 1.3$. [3]

3. If y varies directly as $2x - 3$, and if $y = 21$ when $x = 5$, express y in terms of x. Find y, when $x = 7$, and x, when $y = 63$. [3]

4. Evaluate:

(a) $\left(\frac{7}{9}\right)^0$ (b) $\left(2\frac{3}{4}\right)^{-1}$ (c) $16^{-1.25}$ [3]

5. Two similar containers have capacities of 24 cm^3 and 375 cm^3 respectively. What are the ratios of their

(a) heights; (b) surface areas? [3]

6. The line joining the points $P(1, 2)$ and $Q(3, 8)$ cuts the x-axis at R. Find the

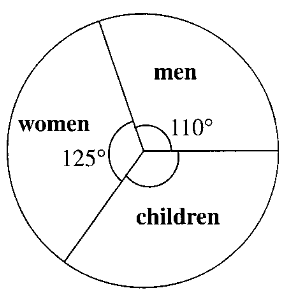
(a) equation of PQ;
(b) coordinates of R. [3]

7. Assume a man walks for 3 hours at a constant speed of 6 km/h and then for another 2 hours at 8 km/h. Find his average speed in

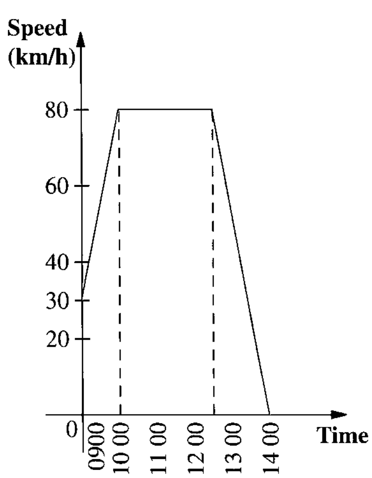
(a) km/h; (b) m/s. [3]

8. Find two consecutive positive even numbers such that the sum of their squares is 244. [3]

9. The pie chart below illustrates the proportion of users of a public swimming pool. If 330 men used the pool, how many children used the pool during the same period? [2]



10.



The speed-time graph shows the movement of a car. Calculate the

(a) acceleration of the car between 09 00 and 10 00; [2]
(b) total distance travelled by the car between 09 00 and 14 00. [2]

11. T_A, T_B, T_C, T_D and T_E are equilateral triangles of sides 1, 1, 3, 3 and 4 cm respectively. If a pie chart is drawn to show their corresponding areas, find the angle representing the area of T_C. [2]

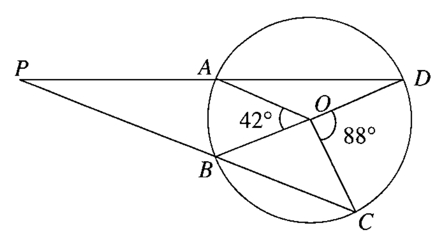
12. The mean of 20 numbers is x. If 10 of these numbers are each increased by 10, express the mean of the new set of numbers in terms of x. [2]

13. (a) Given $\sin x = \sin 32°$, such that $90° < x < 180°$, write down the value of x. [1]
(b) Given $\sin x = \frac{3}{5}$, such that $90° < x < 180°$, write down the values of $\cos x$ and $\tan x$. [2]

14. Given $1 \leqslant x \leqslant 3$ and $3 \leqslant y \leqslant 6$, find

(a) the largest possible value of $x^2 - y$; [1]
(b) the smallest possible value of $\frac{x}{y}$. [2]

15. In the figure below, O is the centre of the circle, $A\hat{O}B = 42°$ and $C\hat{O}D = 88°$. Calculate $A\hat{P}B$.



[3]

16. A map is made to the scale of 1 : 25 000. A representation of a lake on the map has a perimeter of 8 cm.

(a) How many kilometres will I cover if I walk round the perimeter of the lake? [1]

(b) Given that the lake is a square, calculate its surface area in square kilometres. [2]

17. The distribution of marks scored by pupils of a class is as follows:

Marks	4	5	6	7	8	9	10
Number of pupils	7	5	6	3	x	8	7

If the median mark is 7, write down the possible values of x. [3]

Section B (50 marks)

*Answer any **5** questions in this section.*

18. Given that $y = 28 + 4x - 3x^2$, copy and complete the following table of values:

x	–3	–2	–1	0	1	2	3	4	5
y	–11	8		28		24		–4	–27

(a) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis, draw the graph of $y = 28 + 4x - 3x^2$ for $-3 \leqslant x \leqslant 5$.

(b) Find from your graph the solution of the following equations:

(i) $28 + 4x - 3x^2 = 0$;

(ii) $28 + 4x - 3x^2 = 10$.

(c) By drawing a tangent, find the gradient of the curve $y = 28 + 4x - 3x^2$ at the point where $x = 3$.

(d) By drawing a suitable straight line using the same scale and axes, solve graphically the equation $3x^2 - 3x = 18$.

19. (a) Simplify $\dfrac{4}{x-1} + \dfrac{2}{x+1} - \dfrac{5x}{x^2-1}$.

(b) Given that $y = \sqrt{\dfrac{2x+y}{3x-5}}$, express x in terms of y.

(c) Solve $4^{x+2} = 8^{2x-7}$.

20. (a) The diagram shows the speed-time graphs of a goods train and a lorry during a period of 100 seconds.

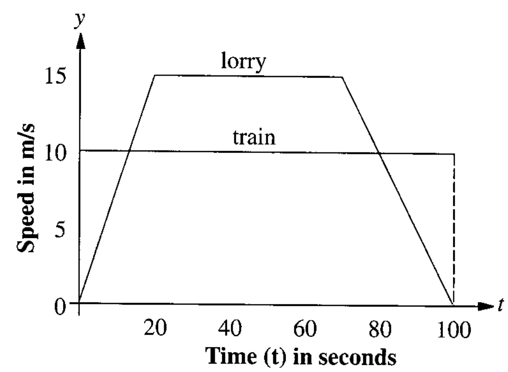
Calculate

(i) the acceleration of the lorry in the first 20 seconds;

(ii) the distance travelled by the lorry during the 100 seconds;

(iii) the distance travelled by the train during the 100 seconds;

(iv) the time when the lorry overtakes the train.



(b) Given that x and y are positive integers, such that $(x + y)$ is an odd number, show that $(x + 3y)$ is an odd number and state your reasoning clearly.

21. (a) A man is three times as old as his son. Eight years ago, the product of their ages was 112. Let x be the present age of the son. Form an equation in x and, hence, solve this equation to find their present ages.

(b) Solve the equation $3x^2 - 7x - 17 = 0$, giving your answer correct to 3 significant figures.

22. (a) The fares paid by bus passengers in one day are shown in the following frequency distribution:

Fare in cents	35	50	60	70	80	90
Number of passengers	47	165	72	34	46	26

Calculate the mean fare per passenger. What is the modal fare?

(b) Pupils in a class of 44 were asked how many children there were in their families and the following data shows their replies:

1	4	7	4	2	3	4	2	3	4	4
2	3	2	3	1	2	3	2	4	5	2
3	4	1	2	1	6	2	5	3	2	1
5	5	1	3	2	1	1	1	6	3	3

(i) Rewrite the above data in a frequency distribution table.

(ii) Draw a histogram to illustrate the data and write down the mode.

(iii) Calculate the percentage of families that have fewer than three children.

23. The water charges for a domestic household in June 1999 is 87 cents for each unit for household which uses less than 22 units per month, plus a water conservation tax of 20%. For household that uses more than 22 units of water per month, each additional unit is 98 cents, plus a water conservation tax of 25%. There is also a water borne fee of 20 cents per unit. Electricity is priced at 13.7 cents per unit.

(a) Mr Ong's family uses 874 units of electricity and 28.5 units of water. Mr Singh's family uses 21.5 units of water and 987 units of electricity while Mr Muhamad's family uses 24.4 units of water and 786 units of electricity. Calculate the amount that Mr Ong, Mr Singh and Mr Mohamad each has to pay for the PUB bill in June 1999.

(b) To encourage the conservation of water and discourage wastage, the Public Utilities Board revised the rates in August 1999. The new water tariff is now \$1.16 per unit for households using 22 units or less per month, plus a water conservation tax of 22%. For household that uses more than 22 units of water per month, the rate is \$1.38 per unit for those units above 22, plus a conservation tax of 27% for these units. The water borne fee is now fixed at 22 cents per unit. The electricity tariff is also revised to 14.2 cents per unit.

(i) If Mr Ong uses the same number of units of water and electricity as he did in June, how much will Mr Ong have to pay?

(ii) Mr Singh now uses 26.8 units of water and 678 units of electricity in August. Find the percentage difference in his PUB bill as compared with June.

(iii) Mr Mohamad hopes to maintain his PUB bill as that of June. How many units of water can he use in August if he uses 876 units of electricity?

End-of-Year Examination Specimen Paper 3

Time: $2\frac{1}{2}$ h

*Answer **all** the questions in **Section A** and any **5** questions in **Section B**.*

Section A (50 marks)

*Answer **all** the questions in this section. Calculators are not allowed.*

1. Express
(a) 1.747 correct to one decimal place;
(b) 740 correct to one significant figure;
(c) 0.002 45 in the standard form. [3]

2. A train travels at a constant speed of 55 km/h for 2 hours and then at 70 km/h for another 3 hours. Find its average speed over the 5-hour journey. [3]

3. Evaluate:
(a) $\left(\frac{1}{2}\right)^{-2}$ **(b)** $\left(3\frac{5}{8}\right)^0 \times 8^{\frac{2}{3}}$
(c) $16^{-\frac{1}{2}}$ [3]

4. If A is an obtuse angle and $\sin A = \frac{7}{25}$, find the values of:
(a) $\tan A$ **(b)** $\cos A$ [3]

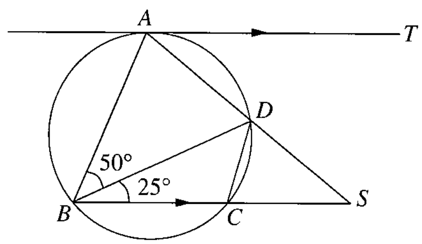
5. Solve the simultaneous equations:
$2x + 3y = 17$ and $3x - y = 9$. [3]

6. The duration of each telephone call made by a businessman on a certain day was recorded (in minutes) as follows:

1, 7, 2, 7, 3, 9, 4, 11, 6, 14, 6,
16, 3, 10, 1, 7, 3, 8, 5, 13.

Find the mean, median and the mode of the duration of his calls. [4]

7. In the figure, AT is the tangent to the circle at A. The chord BC is parallel to AT. The chords AD and BC are produced to meet at S. If $ABD = 50°$ and $DBC = 25°$, calculate ADB and CSD.

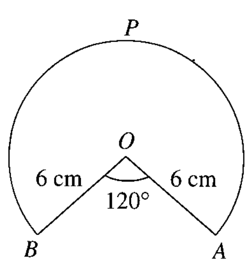


[4]

8. Given that $y = \sqrt{\dfrac{x}{2 + x}}$, express x in terms of y. [3]

9. A, B, C and D are squares of sides 1 cm, 2 cm, 3 cm and 4 cm respectively. In a pie chart showing their areas, what is the angle representing the area of B? [3]

10. In the figure below, $OAPB$ is a sector of a circle of radius 6 cm with its centre at O and $AOB = 120°$.



Find the perimeter of $OAPB$, giving your answer in terms of π. [3]

11. **(a)** Given $\dfrac{2x + 3y}{2x - 3y} = \dfrac{5}{3}$, find the value of $\dfrac{x}{y}$. [2]

(b) Given $\dfrac{a}{3} = \dfrac{b}{5}$, evaluate

(i) $\dfrac{a + b}{b}$;

(ii) $\dfrac{a - b}{a}$;

(iii) $\dfrac{a + 3b}{3a + b}$. [4]

12. Given x and y are integers such that $2 \leqslant x \leqslant 5$ and $-4 \leqslant y \leqslant -1$, find the smallest and greatest possible values of

(a) $x - y$; **(b)** $\dfrac{x}{y}$. [4]

13. If y is inversely proportional to x^2, and if $y = 2$ when $x = 3$, find the equation connecting x and y. Find, also, the value of y when $x = 2$. Draw a sketch graph to illustrate the relation between x and y, showing both positive and negative values of x. [4]

14. P is the point (4, 3). find the image of P under

(a) an enlargement with centre at (2, 1) and a scale factor of 3, followed by a reflection in the line $x = 4$, [2]

(b) a reflection in the line $y = 2$, followed by a 90° clockwise rotation about the point (2, 0). [2]

·**Section B** (50 marks)

Answer any 5 questions in this section.

15. **(a)** Solve the equation $3x^2 + 2x = 13$, giving your answer correct to 2 decimal places.

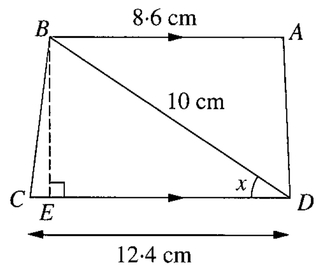
(b) A man cycles from A to B, a total distance of 50 km. For the first 40 km, his average speed is x km/h but for the last 10 km, his average speed is 5 km/h less. If the total journey takes 2 hours and 40 minutes, form

an equation which x must satisfy, and show that it simplifies to $4x^2 - 95x + 300 = 0$.
Solve this equation and, hence, find his average speed for the first 40 km.

16. In the following diagram, $ABCD$ is a trapezium in which BA is parallel to CD. $BA = 8.6$ cm, $BD = 10$ cm, $CD = 12.4$ cm and $BDC = x$. The line BE is perpendicular to CD. Write down expressions in x for

(a) BE;
(b) the area of the trapezium $ABCD$.

Given that the area of the trapezium $ABCD$ is 73.5 cm², calculate x.



17. (*Answer this question on a sheet of graph paper.*)
The variables x and y are connected by the equation $y = x^2 + x - 5$. Some corresponding values of x and y are given in the following table:

x	–4	–3	–2	–1	0	1	2	3
y	7	1	k	–5	–5	–3	h	7

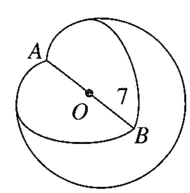
(a) Calculate the values of h and k.
(b) Using a scale of 2 cm to 1 unit on the x-axis and 1 cm to 1 unit on the y-axis, draw the graph of $y = x^2 + x - 5$ for $-4 \leqslant x \leqslant 3$.
(c) By adding a suitable straight line to your graph, solve the equation $x^2 - x - 1 = 0$ graphically.
(d) By drawing a suitable straight line, find the gradient of the graph $y = x^2 + x - 5$ at the point $x = 1$.
(e) Find the range of values of x for which $x^2 + x - 5 \geqslant 2 - x$.
(f) Draw the axis of symmetry on the graph and then write down its equation.

18. There are 100 houses in a certain housing estate. The following table shows the number of children living in each house:

Number of children	1	2	3	4	more than 4
Number of houses	13	37	17	5	7

(a) How many houses in the estate do not have any children?
(b) Explain clearly why it is not possible to calculate the mean number of children living in each house in the housing estate.
(c) There are altogether 200 children in the estate. A house is considered "over-crowded" if it has more than 4 children living in it. Find the mean number of children in "overcrowded" houses.

19. **(a)** A quarter of a sphere of radius 7 cm is removed, with the remaining figure as shown in the diagram below. Taking π to be $\frac{22}{7}$, find the total surface area of the figure.



(b) The bearing of a lighthouse L from a ship A is 319°. From a ship B, 317 m due east of A, the bearing of the lighthouse becomes 288°. Calculate LA. If the height of the lighthouse is 54 m above sea-level, calculate the angle of elevation of the top of the lighthouse from A.

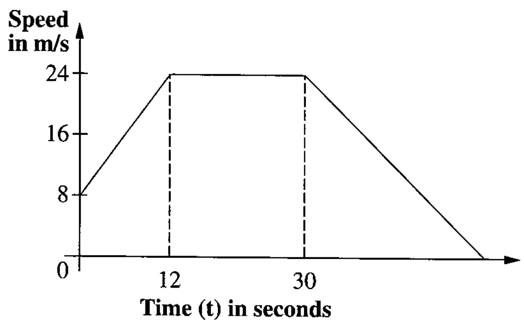
20. The charges for household electricity are as follows:

first 30 units at 25 cents per unit,
next 30 units at 12 cents per unit,
any additional unit at 7 cents per unit.

Using a horizontal axis marked from 0 to 100 units, with a scale of 1 cm to 5 units, and a vertical scale of 1 cm to 1 dollar, draw a graph of cost against number of units used. Use your graph to find the

(a) the cost of 84 units;
(b) the number of units used if the charge is \$8.70;
(c) the number of units used if the charge is \$12.15.

21. (a)



The diagram above shows the speed-time graph of a car.

(i) Calculate the speed of the car when $t = 9$.
(ii) Calculate the distance travelled during the first 30 seconds.
(iii) Given that the magnitude of the deceleration is equal to $\frac{2}{5}$ the rate at which it accelerates during the first 12 seconds, calculate the total time of travel of the car, before it comes to a stop.

(b) The weights of 40 students in a class, correct to the nearest kg, are as follows:

52	67	65	57	52	60	58	59
53	42	51	72	69	57	54	54
58	52	44	47	73	58	62	56
63	57	68	59	63	47	68	58
48	50	64	54	57	59	44	55

(i) Construct a frequency table, using a class interval of 5 kg, starting with 41 kg.
(ii) Use this frequency table to estimate the mean weight.
(iii) List all the weights in the class which contain the median and, hence, find the median.

Answers

Exercise 1a *(Pg 3)*

1. **(a)** 0 or 5 **(b)** 0 or $1\frac{3}{4}$
   (c) 0 or $-\frac{4}{5}$ **(d)** 0 or $\frac{3}{4}$
   (e) 0 or –9 **(f)** 0 or 1
   (g) 1, 1 **(h)** 2 or –9
   (i) $\frac{1}{2}$ or –3 **(j)** 5 or –7
   (k) 4, 4 **(l)** 1 or –4
   (m) 3 or –4 **(n)** $\frac{1}{2}$ or $-\frac{2}{5}$
   (o) 6 or 16
   (p) $2\frac{1}{2}$ or $-1\frac{1}{6}$
2. **(a)** $x^2 - 5x + 6 = 0$
   (b) $x^2 + x - 12 = 0$
   (c) $x^2 - x - 30 = 0$
   (d) $2x^2 - 11x + 5 = 0$
   (e) $15x^2 + 2x - 8 = 0$
   (f) $48x^2 + 2x - 35 = 0$
   (g) $8x^2 + 34x + 35 = 0$
   (h) $6x^2 + 5x - 6 = 0$

Exercise 1b *(Pg 5)*

1. **(a)** 2 or –4
   (b) 1.5 or –2.5
   (c) 1.67 or –3
   (d) 0.325 or –1.075
   (e) –1 or 2.60
   (f) –2.10 or –2.57
   (g) 8.12 or –0.12
   (h) 0.32 or –6.32
   (i) 3.90 or –0.90
   (j) 1.52 or –2.85
   (k) 3.12 or –0.32
   (l) 0.72 or –1.58
2. **(a)** $\frac{49}{4}$ **(b)** $\frac{9}{4}$ **(c)** $\frac{49}{16}$
   (d) $\frac{81}{100}$ **(e)** 1.44 **(f)** $\frac{49}{9}$
   (g) $\frac{4}{25}$ **(h)** $\frac{49}{16}$ **(i)** $25k^2$
   (j) $\frac{25}{4}k^2$ **(k)** $\frac{9}{4}m^2$ **(l)** $\frac{4}{9}$
   (m) $25x^2$ **(n)** $\frac{25}{4}x^2$ **(o)** $\frac{25}{4}n^4$

Exercise 1c *(Pg 7)*

(a) complex
(b) –0.35 or 1.15
(c) –0.31 or –3.19
(d) complex
(e) $\frac{1}{2}$ or –3
(f) –0.28, –2.39
(g) 0.64, 3.36
(h) –0.28, –2.12
(i) 0.85 or –2.35
(j) 6 repeated
(k) 0.55 or –6.55
(l) complex
(m) –1 or $\frac{2}{3}$ **(n)** –2 or 13
(o) –2 or $\frac{1}{3}$ **(p)** complex
(q) –3 or 10 **(r)** 3.65 or –1.65
(s) –2 or 8 **(t)** –1.10, 1.60
(u) 16.60 or –0.60
(v) 0.82 or –9.81
(w) 3.45 or –1.45
(x) 3.38 or –1.78
(y) $-\frac{1}{2}$ or 1
(z) 0.13 or 3.07

Exercise 1d *(Pg 10)*

1. complex
2. complex
3. 2 or $-\frac{1}{3}$
4. –1.37 or 0.37
5. –2.90 or 0.23
6. 3.22 or 0.78
7. 1.96 or –0.76
8. 3 or –8
9. 3 or –5
10. 4.46 or –2.46
11. 3 or $-\frac{7}{2}$ **12.** 2 or 3
13. $\frac{2}{3}$ or $\frac{4}{9}$ **14.** $\frac{4}{7}$ or $5\frac{1}{3}$
15. $\frac{1}{2}$ or $-\frac{1}{2}$ **16.** 3.73 or 0.27
17. 4.58 or –4.58
18. 0 or $\frac{5}{2}$
19. 3.16 or –3.16
20. 4.90 or –4.90
21. 8 or –8 **22.** 2.58 or 1.42
23. 0 or –5 **24.** 3 or $\frac{1}{3}$
25. –3 or 4

Exercise 1e *(Pg 12)*

1. 12, 13 **2.** 15, 17
3. 22, 24 **4.** 6 cm, 5 cm
5. 7, 9 **6.** 12 cm
7. 5 cm, 20 cm **8.** 150 cm
9. 14.3 km/h
10. 5.46, 6.46, 8.46
11. 10
12. **(a)** $\frac{1\,140}{x}$
    (b) $\frac{1\,140}{x+1}$
    (c) $\frac{1\,140}{x} - \frac{1\,140}{x+1} = 3$
    (d) $x = 19$
13. 15 km/h **14.** 25 km/h
15. $\frac{240}{x}$, $\frac{240}{x-6}$; 68.80, 62.80
16. 70 **17.** $\frac{3}{5}$
18. 8 cm, 15 cm, 17 cm; 60 cm²
19. **(a)** $4x + 8$ **(b)** $x + 2$, 5
20. A : 25, 28;
    B : 20, 30
21. 20 min, 25 min
22. 15 or 24

Review Questions 1 *(Pg 14)*

1. **(a)** –2.93 or –0.07
   (b) 0.5 or –2
   (c) 2.2 or –2.29
   (d) no real roots
   (e) no real roots
   (f) 1.54 or –0.869
   (g) –3.35 or –0.149
   (h) no real roots
   (i) no real roots
   (j) 6 or –0.5
   (k) –3.94 or 0.190
   (l) 1.5 or –2.5
   (m) 9 or –9
2. 11.2 km/h, 16.2 km/h
3. 45
4. 16 × 18
5. **(a)** **(i)** \$34.85 **(ii)** \$60
   (b) **(i)** $\frac{60}{x}$ **(ii)** $\frac{60}{x+2}$
   (iii) $\frac{60}{x} + \frac{60}{x+2} = 11$
   (iv) 10; 5

Exercise 2a (Pg 22)

1. (a) 3^{12} (b) 7^{25} (c) 8^6
(d) 15^4 (e) 9^6 (f) 4^{30}
(g) m^7 (h) c^8 (i) y^7
(j) b^2 (k) x^3 (l) n^5
(m) c^{12} (n) p^{16} (o) x^{14}
(p) a^{16} (q) a^7 (r) a^{18}
(s) x^{12} (t) k^{49} (u) h^{72}

2. (a) $3^{13} \times 5^9$ (b) $6^{21} \times 19^9$
(c) $5^{15} \times 7^{12}$ (d) $7^{11} \div 5^{10}$
(e) $3^{10} \div 2^{12}$ (f) $\left(\frac{11}{7}\right)^{12}$
(g) 15^5 (h) $\left(\frac{13}{7}\right)^7$
(i) $(ab)^8$ (j) $(xy)^{13}$
(k) m^8n^4 (l) h^9k^7

3. (a) $4x^8$ (b) $27a^{15}$
(c) $25b^{14}$ (d) $12a^5b^6$
(e) $6a^2b^3c^2$ (f) $35a^7b^9$
(g) $392a^{13}$ (h) $288x^{30}$
(i) $512x^{26}$ (j) $4a^3b$
(k) $3x^5y^4$ (l) $5ab^3$
(m) $-2ab^2$ (n) $\frac{81y^{12}}{4x^{10}}$
(o) $256ab$ (p) $\frac{a^6b^{14}}{4}$
(q) $\frac{a^3b^3}{4}$ (r) $(a+b)^5$
(s) $2(a-b)^{15}$ (t) $5(2a+b)^6$
(u) $9a^5b$ (v) $70a^8b^6$
(w) $24a^{14}b^9$ (x) $20a^3b^4$

4. (a) $\frac{64}{3}x^{13}y$ (b) $\frac{8x^4}{y^2}$
(c) $\frac{15b}{2a^2}$ (d) $\frac{x^6}{2y}$
(e) $\frac{5x^6y^8}{2}$ (f) $\frac{32}{9}x^8y^2$
(g) $\frac{1}{2a}$ (h) $2y^5$
(i) $3a^7b^{14}$ (j) $9x^4y^{11}$
(k) $\frac{64x^4y^2z^2}{3}$

Exercise 2b (Pg 25)

1. (a) $\frac{1}{3^5}$ (b) $\frac{1}{5^2}$ (c) $\frac{1}{8^6}$
(d) 9^{12} (e) $\frac{1}{6^5}$ (f) $\frac{1}{7^9}$
(g) $\frac{1}{5^{12}}$ (h) 9^8 (i) $\frac{1}{a^5}$
(j) $\frac{1}{a^3}$ (k) $\frac{1}{a^6}$ (l) $\frac{1}{a^{11}}$
(m) $\frac{2}{7}\left(\frac{b}{a}\right)^5$ (n) $\frac{b^4}{a^3}$
(o) $5a^2b^3$ (p) $\frac{b^4}{a^6}$
(q) $\frac{3b^9}{a^{12}}$ (r) $\frac{1}{ab^5}$
(s) a^4b^{15} (t) b^3
(u) $\frac{x^{11}y^6}{16}$ (v) ab^2
(w) $\frac{128a^8}{b^2}$ (x) $\frac{a^{20}}{b}$
(y) ab^3 (z) b

2. (a) $\frac{1}{6^{-2}}$ (b) $\frac{1}{7^{-4}}$ (c) 5^{-4}
(d) $\frac{1}{9^{-8}}$ (e) $\frac{1}{3^{-8}}$ (f) 8^{-24}
(g) $\frac{1}{a^{-2}b^{-3}}$ (h) b^{-1}
(i) $\frac{1}{a^{-6}b^{-3}}$ (j) b^{-4}
(k) a^{-3} (l) $\left(\frac{b}{a}\right)^{-5}$
(m) $a^{-4}b^{-3}c^{-1}$ (n) $\left(\frac{a}{b}\right)^{-2}$
(o) $\frac{a^{-5}}{b^{-2}}$ (p) $a^{-3}b^{-6}$
(q) $\frac{1}{a^{-5}b^{-4}}$ (r) $\frac{1}{a^{-2}c^{-6}}$
(s) $\frac{b^{-6}}{a^{-2}}$ (t) $\frac{a^{-3}}{b^{-4}}$
(u) $a^{-2}b^{-8}$ (v) $a^{-7}b^{-2}$
(w) $2^{-1}a^{-4}b^{-6}$ (x) $\frac{a^{-6}}{b^{-4}}$

3. (a) $\frac{1}{16}$ (b) $\frac{1}{81}$ (c) 1
(d) $\frac{1}{49}$ (e) $\frac{1}{8}$ (f) $\frac{1}{25}$
(g) $\frac{1}{16}$ (h) $3\frac{3}{8}$ (i) 1
(j) $-15\frac{5}{8}$ (k) $\frac{81}{256}$
(l) $2\frac{10}{27}$ (m) $\frac{9}{64}$
(n) $\frac{1}{18}$ (o) $1\frac{1}{3}$
(p) $\frac{7}{8}$ (q) $7\frac{1}{9}$
(r) 31 850 496
(s) 1 (t) 3
(u) 16 (v) 12
(w) $\frac{8}{27}$ (x) $\frac{125}{2\ 187}$

4. (a) a^3 (b) a^{-3} (c) a^{-5}
(d) 1 (e) a^4 (f) x^{12}
(g) x^3 (h) x^2 (i) x^{-5}
(j) x^5 (k) $a^{1\ 100}$ (l) a^9
(m) 1 (n) $xy^{-3}z^{-3}$
(o) x^3y^2z (p) $p^8q^3r^6$
(q) a^9 (r) $p^{30}q^4r^{19}s^7$

Exercise 2c (Pg 30)

1. (a) 7 (b) 3 (c) 2
(d) 2 (e) 10 (f) 27
(g) 8 (h) 25 (i) 16
(j) 6 (k) 9 (l) 27
(m) 100 (n) 8 (o) 8
(p) 32 (q) 3 (r) $\frac{1}{2}$
(s) $\frac{1}{3}$ (t) $\frac{1}{2}$ (u) $\frac{1}{16}$
(v) $\frac{1}{64}$ (w) $\frac{1}{2}$ (x) 32
(y) $\frac{1}{5^5}$ (z) 2

2. (a) a^8 (b) a^5 (c) $3a^3$
(d) $2a^3$ (e) x^8 (f) $3x^5$
(g) $5p^3q^5$ (h) $(a+b)^{\frac{5}{2}}$
(i) $2a^{\frac{1}{2}}b^{\frac{3}{2}}$ (j) $a^{\frac{5}{3}}b^2c^{-\frac{4}{3}}$
(k) $2a^{\frac{1}{3}}(a+b)^{-\frac{2}{3}}$
(l) $b^{\frac{n}{p}}a^{-\frac{m}{p}}$

3. (a) $\sqrt{21}$ (b) $\sqrt[3]{15}$
(c) $9\sqrt{2}$ (d) $\sqrt[28]{2^{37}}$
(e) $\sqrt[5]{2^9}$ (f) $\sqrt[15]{3^{19}}$
(g) $\sqrt[12]{3^{25}}$ (h) $\sqrt[12]{5^{11}}$
(i) $\sqrt[15]{5^7}$ (j) $\frac{1}{\sqrt[14]{2^3}}$
(k) $\sqrt[15]{4^{16}}$ (l) $\sqrt[6]{3^7}$

4. (a) $a^{\frac{11}{15}}$ (b) $a^{\frac{1}{15}}$ (c) $a^{\frac{49}{30}}$
(d) $a^{-\frac{26}{99}}$ (e) $a^{\frac{9}{14}}$ (f) $4a^{\frac{5}{8}}$
(g) $a^{\frac{1}{5}}$ (h) $a^{\frac{8}{3}}$ (i) $a^{\frac{11}{12}}$
(j) $a^{\frac{4}{3}}b^{\frac{1}{6}}$ (k) $a^{\frac{7}{6}}b^{\frac{11}{12}}$
(l) $a^3b^2c^{\frac{3}{2}}$ (m) $a^{\frac{5}{6}}b^{\frac{3}{2}}$
(n) $a^{\frac{1}{2}}b^{\frac{23}{72}}$

5. (a) $\sqrt[12]{a^{11}}$ (b) $\sqrt[14]{a^{13}}$
(c) $\sqrt[12]{x^{11}}$ (d) $\sqrt[4]{x^5}$
(e) $\sqrt{x^5}\sqrt[3]{y^5}$ (f) $y^2\sqrt{x^9}$
(g) $\sqrt[6]{a}$ (h) $ab\sqrt[4]{b}$
(i) $\sqrt[10]{x^3}\sqrt{y^3}$ (j) $\frac{\sqrt[24]{x^{37}}}{\sqrt[4]{y^{11}}}$
(k) $\sqrt[20]{x^{13}}\sqrt[10]{y^{21}}$ (l) $\sqrt[10]{x^9}\sqrt[12]{y}$

Exercise 2d (Pg 32)

1. (a) 4 (b) 3 (c) 3
(d) 2 (e) 5 (f) 6

(g) 3 (h) 2 (i) 2

(j) ±3 (k) ±4 (l) ±3

(m) 4 (n) 0 (o) 0

(p) 3 (q) 2 (r) 4

(s) 20 (t) 2 (u) 9

(v) 15 (w) 7 (x) –1

(y) 15 (z) 6.5

2. (a) 9 (b) $\frac{1}{4}$ (c) $\frac{1}{25}$

(d) 0

Exercise 2e *(Pg 33)*

1. (a) 4 (b) 8 (c) 8

(d) 3 (e) 3 (f) 7

(g) 4 (h) 4 (i) 6

(j) 5 (k) 5 (l) 5

(m) 6 (n) 6 (o) 6

(p) 1 (q) 7 (r) 3

(s) 8 (t) 4 (u) 2

(v) 9 (w) 1 (x) 1

2. (a) 6 (b) 7 (c) 9

(d) 7 (e) 5 (f) 8

Exercise 2f *(Pg 34)*

1. (a) $a = b^2$ (b) $a = \frac{1}{2}b^2$

(c) $a = b^2 - m$

(d) $a = \frac{1}{5}(e^2 + 8)$

(e) $a = 2b^2$ (f) $a = \frac{k}{ml^2}$

(g) $a = \frac{5cx^2}{2}$ (h) $a = \frac{2b}{3b - 1}$

(i) $a = \frac{2k + z^2}{3}$

(j) $a = \pm\sqrt{\frac{b - 3}{2}}$

(k) $a = \pm\sqrt{\frac{2 + 3c}{3}}$

(l) $a = \pm\sqrt{\frac{k - z}{b}}$

(m) $a = \pm\sqrt{5b^2c}$

(n) $a = \pm\sqrt{\frac{b(m - n)}{n}}$

(o) $a = \pm\sqrt{\frac{A}{4\pi}}$

(p) $a = b + c^3$

(q) $a = \pm\sqrt{\frac{3V}{\pi h}}$

(r) $a = \sqrt[3]{\frac{3V}{4\pi}}$

(s) $a = \pm\sqrt{\frac{2Knb}{m}}$

(t) $a = \sqrt[3]{\frac{2c}{x^3 - b}}$

(u) $a = \pm\sqrt{\frac{y(R - r)}{x}}$

2. (a) $b = \frac{1}{2}(a^2 - a)$

(b) $b = \pm\sqrt{c^2 - a^2}$

(c) $y = \pm\sqrt{x} - x$

(d) $c = \frac{1}{3}(e^2 + 7a)$

(e) $w = \pm\sqrt{\frac{x - b}{2}}$

(f) $y = z^3 + 1$

(g) $b = \pm y\sqrt{1 - \left(\frac{a}{x}\right)^2}$

(h) $x = \pm\sqrt{\frac{y^3}{2z^3} + \frac{7}{2}}$

(i) $x = \pm\frac{t}{2}\sqrt{m - 3}$

(j) $m = \frac{5t^4 + 2}{t^4 - 1}$

(k) $c = \frac{ab + 2b - 2a}{b - a}$

(l) $x = \frac{3ay}{4ay - n}$

(m) $g = \frac{3x}{12kt + 8m}$

(n) $s = \frac{q(r - p - q)}{r + p - q}$

(o) $b = \pm\sqrt{D^2 + 4ac}$

(p) $s = \pm\sqrt{\frac{5p}{2p - 3q}}$

(q) $r = \pm\sqrt{3q - p^2x^2}$

(r) $h = \frac{2kq^2 - 9p^2n^2k}{27p^2n^2 - q^2}$

(s) $x = \frac{-ay}{b}$

(t) $v = \pm\frac{u}{y}\sqrt{y^2 - x^2}$

Exercise 2g *(Pg 36)*

1. $\frac{b^2(a + b)}{a^2(a - b)}$ **2.** $\frac{a}{b}$

3. –1 **4.** –1

5. $\frac{1}{d(c - d)}$ **6.** $\frac{e + f}{(e - f)(e - 2f)}$

7. $b(a - 2b)$ **8.** $\frac{2}{y}$

9. $\frac{d + 2}{d}$ **10.** $\frac{3b}{b + c}$

11. $\frac{m}{m + 3}$ **12.** $\frac{5a}{(a + b)(a - b)}$

13. $\frac{(b - 2c)^2}{c(b + 2c)}$ **14.** $\frac{(1 + 3x)(3z + x)}{xz}$

15. $\frac{-a}{a + 2}$ **16.** $\frac{x - y}{x - 3y}$

17. $\frac{p(b - 2)}{2}$ **18.** $\frac{-m^2}{p^2}$

19. $\frac{y - 2}{3(1 - 3y)}$ **20.** $\frac{6 - a}{x + y}$

21. $\frac{(x + 1)(x + 2)}{x(x - 2)}$

22. $\frac{d}{2}$ **23.** $\frac{y - 3}{y + 1}$

24. x **25.** $\frac{2x + 1}{2x + 3}$

26. 1 **27.** 1

28. $\frac{7(x - 3)}{x(x + 3)}$ **29.** 1

30. $x + y$

Exercise 2h *(Pg 37)*

1. $\frac{x - 1}{2}$ **2.** $\frac{5y - 6x}{15xy}$

3. $\frac{x + 9}{6}$ **4.** $\frac{7x + 19y}{5x}$

5. $\frac{12}{x - y}$ **6.** $\frac{b - a}{a(a + b)}$

7. $\frac{x + 1}{a - b}$ **8.** $\frac{8x + 2}{3x}$

9. $\frac{-9 - 4x}{24x^2}$ **10.** $\frac{8a + 10}{a(a + 2)}$

11. $\frac{5 - 2x}{x(x - 6)}$ **12.** $\frac{8}{a - b}$

13. $\frac{3}{x - y}$ **14.** $\frac{2}{p - 3r}$

15. $\frac{-n - m}{n}$ **16.** $\frac{2}{(r + 5)(r - 2)}$

17. $\frac{a^2 + 6ab - 6b^2}{(a + b)(a - b)}$

18. $\frac{37a + 1}{(a - 3)(a - 9)}$

19. $\frac{2(2x - 3y)}{2x + 3y}$

20. $\frac{x^2 + 3xy - y^2}{y(x + y)(x - y)}$

21. $\frac{12 - m}{m(m - 3)(m - 4)}$

22. $\frac{20m + 1}{(2m - 1)(m + 2)}$

23. $\frac{2a^2 - 3a + 3}{(a - 1)^3}$

24. $\frac{13 - 5a}{(a - 1)(a - 2)(a - 3)}$

25. $\frac{-2}{a + 1}$ **26.** $\frac{3}{3 + 2a}$

27. $\frac{x + 1}{x}$ **28.** $\frac{1}{(1 + x)^2}$

29. $\frac{2x(2-x)}{(x+1)(x-1)}$

30. $\frac{x-8}{(x-2)(x-3)(x+4)}$

31. $\frac{3x^2-8x+1}{(2-3x)(2+3x)}$

32. $\frac{2(4x-21)}{(x-6)(x+7)(x-6)}$

33. $\frac{2x^2-5x-11}{(x+1)(x+2)(x-3)}$

34. $\frac{x^2-5x-9}{(x+3)(x-3)}$

Exercise 2i *(Pg 39)*

1. $c = 5$ or -2 **2.** $m = 1$ or 14
3. $a = 2$ or 4 **4.** $e = -3\frac{3}{4}$
5. $x = -6$ or -1 **6.** $b = 1$ or 5
7. $m = -2$ or -1 **8.** $d = 0$
9. $c = 1$ **10.** $a = -6$ or 4
11. $n = 2$ **12.** $m = -9$
13. $c = 3$ **14.** $m = -2$
15. $a = -3$ **16.** $d = -10$

Review Questions 2 *(Pg 40)*

1. (a) a^8 **(b)** a^6
(c) $4x^{10}$ **(d)** $-x^{12}$
(e) $-3x^4$ **(f)** $64x^{18}$
(g) p^6q^2 **(h)** $-a^6b^{12}$
(i) a^5 **(j)** 1
(k) 1 **(l)** 1
(m) a^{-5} **(n)** a^{-6}
(o) a^{-6} **(p)** a^4
(q) a^3 **(r)** a^{-15}
(s) $16a^{-4}$ **(t)** $a^2b^6c^4$
(u) $x^{16}y^8z^4$

2. (a) 1 **(b)** 1 **(c)** $\frac{1}{64}$
(d) -4 **(e)** $1\frac{9}{16}$ **(f)** $3\frac{1}{16}$
(g) $2\frac{14}{25}$ **(h)** $3\frac{3}{8}$ **(i)** $28\frac{1}{6}$
(j) $\frac{1}{49}$ **(k)** 1 **(l)** $\frac{32}{243}$
(m) 4 **(n)** 7 **(o)** 243
(p) 0.2 **(q)** $\frac{4}{9}$ **(r)** $\frac{1}{64}$
(s) 2 **(t)** $\frac{1}{2}$ **(u)** 12
(v) $\frac{1}{2}$ **(w)** 4 **(x)** $\frac{1}{128}$

3. (a) $4x^7y^6$ **(b)** $12a^3b^3$
(c) $x^8y^{12}z^{16}$ **(d)** 1
(e) 1 **(f)** a
(g) a^3 **(h)** $\frac{p^8}{r}$
(i) r^2 **(j)** $2x^2y^2$
(k) 1 **(l)** z
(m) $6a^{\frac{5}{6}}$ **(n)** $\frac{1}{2}x^5$
(o) $15x^{\frac{1}{6}}$ **(p)** $128x^8y^5$
(q) $12a$
(r) $x + 1 + 2x^{\frac{1}{2}}$
(s) $2a - 3a^{\frac{1}{2}} - 2$
(t) $x - y$

Exercise 3a *(Pg 44)*

1. (a) < **(b)** < **(c)** >
(d) > **(e)** = **(f)** >
(g) > **(h)** > **(i)** >
(j) < **(k)** > **(l)** >
2. (a) < **(b)** < **(c)** <
(d) > **(e)** > **(f)** >
(g) > **(h)** > **(i)** <
(j) = **(k)** = **(l)** =

Exercise 3b *(Pg 46)*

1. (a) > **(b)** > **(c)** <
(d) < **(e)** > **(f)** >
(g) > **(h)** < **(i)** <
(j) < **(k)** < **(l)** >
2. (a) $x < 2$ **(b)** $x > 4$
(c) $x > 1$ **(d)** $x < 6$
(e) $x < 2\frac{1}{3}$ **(f)** $x < -2$
(g) $x > -2$ **(h)** $x < \frac{1}{2}$
(i) $x > 4$ **(j)** $x > -1\frac{1}{3}$
(k) $x < 1\frac{1}{2}$ **(l)** $x < 7$
3. (a) $x > -5$ **(b)** $x > -3$
(c) $x > 5$ **(d)** $x < 8$
(e) $x < -1$ **(f)** $x < -10$
(g) $x < 2$ **(h)** $x > -\frac{17}{9}$
(i) $x < -21$ **(j)** $x > -12\frac{5}{7}$

Exercise 3c *(Pg 48)*

1. $x \leqslant 5$ **2.** $x \geqslant 1$
3. $x \geqslant 0$ **4.** $x \geqslant -2$
5. $x \geqslant 4$ **6.** $x \leqslant \frac{2}{3}$
7. $x \geqslant 1\frac{1}{2}$ **8.** $x \geqslant 6$
9. $x \geqslant 3\frac{1}{2}$ **10.** $x \leqslant 4$
11. $x \leqslant 6\frac{2}{3}$ **12.** $x \leqslant 1$
13. $x \geqslant -1$ **14.** $x \leqslant 1\frac{7}{18}$
15. $x \leqslant -10$ **16.** $x \geqslant -4$
17. $x \geqslant -3$ **18.** $x \geqslant 1\frac{1}{2}$
19. $x \leqslant 108$ **20.** $x \geqslant 0$
21. $x > 12\frac{23}{26}$

Exercise 3d *(Pg 50)*

1. \$3 000; \$1 500
2. \$32 500; \$27 500
3. between 5 and 78 years old
4. 400 cm^2
5. \$9
6. 96
7. 15
8. \$27 500

Exercise 3e *(Pg52)*

1. (a) $x \geqslant -2$ **(b)** $x < 1\frac{1}{3}$
(c) $x < 2\frac{3}{4}$ **(d)** $x \geqslant 30$
(e) $x > -2\frac{38}{41}$ **(f)** $x \geqslant 36$
2. (a) $-2 \leqslant x \leqslant 7$
(b) $-\frac{4}{3} < x < 5$
(c) $\frac{21}{4} < x \leqslant 8$
(d) $-2 \leqslant x < 1$
(e) $x > 9$
(f) $x > 48$
3. (a) $x \geqslant 2$
(b) $3 < x \leqslant 5$
(c) $x < -9$
(d) $-1 < x \leqslant 1\frac{1}{2}$
(e) $\frac{1}{2} < x \leqslant 6$
(f) $4 \leqslant x \leqslant 6$
4. (a) $x \geqslant 5$
(b) $1 < x < 4$
(c) $-2 < x < 6$
(d) $-1 < x < 4$
(e) $0 \leqslant x < 3$
(f) $-\frac{1}{4} \leqslant x < 3$

5. **(a)** $-4 \leqslant x \leqslant 4$
(b) $x \leqslant -6$
(c) $-5 \leqslant x < 1$
(d) $3\frac{1}{3} \leqslant x \leqslant 3\frac{1}{7}$
(e) $-13 \leqslant x \leqslant 7$
(f) $-2 \leqslant x < 8\frac{1}{3}$

6. **(a)** 8, 9, 10
(b) 5
(c) 2, 3
(d) 4, 5, 6, 7, 8
(e) 7, 8
(f) 3, 4, 5, 6, 7, 8, 9, 10

7. **(a)** 24, 25
(b) 4, 5, 6
(c) 1, 2, 3, 4, 5, 6
(d) –2, –1, 0, 1, 2, 3, 4, 5
(e) 7, 8, … 15
(f) 4, 5, 6, 7

8. **(a)** 3 **(b)** –2 **(c)** $3\frac{1}{2}$

9. **(a)** 2 **(b)** 13 **(c)** 1

10. **(a)** 9 **(b)** –20 **(c)** $-\frac{3}{4}$
(d) 10

11. 5 **12.** 8
13. 18 **14.** 14
15. (24, 1), (12, 2), (8, 3), (6, 4)
16. $12 \leqslant z \leqslant 40$
17. **(a)** \$18, \$14.40
(b) 9

Review Exercise 3 *(Pg 54)*

1. **(a)** $x > 1\frac{1}{5}$ **(b)** $x < 11\frac{1}{2}$
(c) $x > 2$ **(d)** $x \leqslant 1\frac{1}{10}$
(e) $x \geqslant 25$ **(f)** $x > 4\frac{3}{5}$
(g) $x \leqslant 10\frac{1}{2}$ **(h)** $x \leqslant 7$
(i) $x \geqslant 2\frac{1}{2}$ **(j)** $x < 11$
(k) $x \geqslant 8$ **(l)** $x \leqslant 30\frac{1}{3}$

2. **(a)** $x < -24$ **(b)** $x < 7\frac{1}{5}$
(c) $x \leqslant -1\frac{1}{4}$ **(d)** $x > 1\frac{7}{9}$
(e) $x > -28\frac{3}{8}$ **(f)** $x \geqslant 2\frac{4}{9}$

3. **(a)** 14 **(b)** 13 **(c)** $14\frac{1}{2}$

4. **(a)** $9\frac{1}{2}$ **(b)** 10

5. **(a)** 8 **(b)** 11
(c) 17 **(d)** 5

6. **(a)** 10, 11
(b) –5, –6

7. **(a)** –13
(b) $1\frac{3}{4}$
(c) 33
(d) 37

8. 5 sheets
9. 54 fruit, 10 cents
10. \$2 250, \$1 350

Exercise 4a *(Pg 61)*

1. $\triangle MNL \equiv \triangle GIH$
$\triangle ABC \equiv \triangle ZXY$
$\triangle DEF \equiv \triangle TSU$
$\triangle PQR \equiv \triangle JLK$

2. **(a)** Yes, SSS
(b) Yes, SAS
(c) No
(d) Yes, SAS

3. **(a)** $\triangle AOC \equiv \triangle BOD$, *SAS*
(b) $A\hat{O}C = B\hat{O}D$,
$A\hat{C}O = B\hat{D}O$

4. **(a)** $\triangle PQR \equiv \triangle SRQ$
(b) 5cm, 50°

5. $\triangle FED$ and $\triangle LMN$

6. AD, DC, AC, $\triangle ADC$, $D\hat{A}C$, $\hat{D}$, $A\hat{C}D$

7. SP, $S\hat{P}T$, PT, $\triangle PQR$, *SAS*, $P\hat{S}T$, $S\hat{T}P$, PR

Exercise 4b *(Pg 64)*

1. **(a)** Yes, they are opposite sides of equal angles.
(b) True, AAS property

2. $\triangle DEF \equiv \triangle JLK$

3. (a) and (e), (b) and (h), (c) and (d), (f) and (g)

4. **(b)** $\triangle PQR \equiv \triangle ABC$, *SSS*
(c) $\triangle ABG \equiv \triangle CDG$, *SAS*
(d) $\triangle PQS \equiv \triangle RSQ$, *SSS*
(e) $\triangle ABD \equiv \triangle ACD$, *SSS*
(f) $\triangle ABC \equiv \triangle RST$, *SAS*
(g) $\triangle ABC \equiv \triangle BAD$, *SSS*
(h) $\triangle PQR \equiv \triangle RSP$, *RHS*
(i) $\triangle EFG \equiv \triangle XYZ$, *RHS*
(j) $\triangle ABD \equiv \triangle ACD$, *AAS*
(k) $\triangle ABO \equiv \triangle CDO$, *AAS*
(l) $\triangle LMN \equiv \triangle PHN$, *AAS*
(m) $\triangle XYW \equiv \triangle XZW$, *SAS*
(n) $\triangle CDF \equiv \triangle EDF$, *AAS*

Exercise 4c *(Pg 67)*

1. $P\hat{R}Q$, PR, $\triangle PQR$, $Q\hat{P}R$, PQ, QR

2. $P\hat{Q}R = 90°$, PR, QR, $\triangle PQR$, *RHS*, $Q\hat{P}R$, $P\hat{R}Q$, PQ

3. **(b)** $P\hat{R}Q = X\hat{Y}Z$, $PR = XY$, $QR = YZ$

4. **(b)** $A\hat{B}C = Y\hat{X}Z$, $A\hat{C}B = Y\hat{Z}X$, $AC = YZ$

5. **(a)** *AAS*, $A\hat{C}B = D\hat{C}E$, $BC = EC$, $AC = DC$
(b) *AAS*, $O\hat{P}Q = O\hat{R}S$, $OP = OR$, $OQ = OS$
(c) *RHS*, $W\hat{X}Z = Y\hat{Z}X$, $W\hat{Z}X = Y\hat{X}Z$, $WX = YZ$
(d) *AAS*, $S\hat{Q}P = R\hat{P}Q$, $SQ = RP$, $SP = RQ$
(e) *AAS*, $E\hat{B}F = E\hat{C}D$, $BF = CD$, $EF = ED$
(f) *AAS*, $O\hat{H}G = O\hat{I}J$, $OH = OI$, $GH = JI$

6. **(a)** 4.6 cm **(b)** 7.8 cm
(c) 94° **(d)** 36°
(e) 50°

7. **(a)** 12° **(b)** 72°
(c) No, $\hat{A} \neq \hat{L}$

Exercise 4d *(Pg 70)*

1. $\triangle ABC \equiv \triangle A'B'C'$ (*SAS*)
5. length of AB'

Exercise 4e *(Pg 75)*

1. (a), (d) **2.** (a), (b)
3. (a), (c), (d)
4. **(a)** $a = 12$, $b = 10$
(b) **(i)** $\frac{m}{n} = \frac{1}{3}$ **(ii)** $l = 6$
5. **(a)** $l = 12$, $m = 9$
(b) $a = 2$, $b = 4$
(c) $c = 2.5$, $d = 10$
(d) $e = 6$, $f = 15$

Exercise 4f *(Pg 82)*

1. $A\hat{B}C = A\hat{D}E$, $A\hat{C}B = A\hat{E}D$
(corr. ∠s, DE // BC)

2. **(a)** $A\hat{O}B = D\hat{O}C$
(vert. opp. ∠s),
$O\hat{A}B = O\hat{D}C$
(alt. ∠s, AB // CD),
$O\hat{B}A = O\hat{C}D$
(alt. ∠s, AB // CD),
(b) $\frac{AB}{DC} = \frac{OB}{OC} = \frac{OA}{OD}$

3. **(a)** $\frac{7}{4} = \frac{PR}{QR} = \frac{TR}{SR}$

4. **(a)** $x = 4$
(b) $y = 3, z = 12$
(c) $u = 9, v = 18$
(d) $k = 15, h = 10$
(e) $s = 15, t = 12$
(f) $p = 20, q = 16$

5. **(a)** $l = 6, m = 12$
(b) $p = 21\frac{1}{3}$
(c) $q = 2\frac{2}{3}, r = 6\frac{3}{4}$

6. $a = 6, b = 12, c : d = 1 : 2$

7. $AZ = 7\frac{1}{2}$ cm, $CX = 1.92$ cm

8. **(a)** $a = 5.92$
(b) $b = 15\frac{5}{7}, c = 12\frac{4}{7}$

9. **(a)** $a = 6, b = 11\frac{2}{3}$
(b) $c = 4, d = 10$
(c) $e = 12\frac{6}{7}$
(d) $f = 3\frac{1}{3}, g = 24$
(e) $h = 12$
(f) $i = 13\frac{1}{2}$

10. 12 m

11. 8 m

12. **(a)** $\triangle RQP$
(b) 6.4 cm

Review Questions 4 *(Pg 86)*

1. **(a)** Yes, *RHS* **(b)** No
(c) Yes, *AAS* **(d)** Yes, *SAS*
(e) Yes, *AAS* **(f)** Yes, *SAS*

2. **(a)** $a = 6\frac{3}{7}, b = 12\frac{6}{7}$
(b) $p = 4, q = 8$
(c) $x = 25.2, y = 28.8, z = 9.6$
(d) $c = 5, d = 7, e = 10, f = 8\frac{4}{7}$

3. **(a)** $\triangle ADF \equiv \triangle CBE$, *AAS* or $\triangle ABE \equiv \triangle CDF$, *AAS*
(b) $\triangle PMR \equiv \triangle PNQ$, *SAS*
(c) $\triangle SRP \equiv \triangle XPQ$, *SAS*
(d) $\triangle PLM \equiv \triangle QLN$, *SAS*

4. **(a)** **(i)** $\triangle RLN$ **(ii)** 18 cm
(b) **(i)** $\triangle NMS$ **(ii)** 9 cm
(c) $\triangle PLM$ and $\triangle RLQ$, $\triangle PQM$ and $\triangle SNM$, $\triangle PQM$ and $\triangle RNQ$

5. **(a)** $\triangle CAB$ and $\triangle CPQ$, $\triangle PBQ$ and $\triangle DBC$
(b) $BQ = 7$ cm, $DC = 8\frac{4}{7}$ cm
(c) $AP : PC = 7 : 5$, $BP : BD = 7 : 12$

6. $BD = 12$ cm, $EF = 9\frac{3}{5}$ cm

Exercise 5a *(Pg 93)*

1. **(a)** 8.06 **(b)** 8.54 **(c)** 11.4
(d) 10.8 **(e)** 14 **(f)** 27

2. **(a)** $\left(\frac{1}{2}, 2\right)$ **(b)** $(1, 2)$
(c) $\left(-1, 4\frac{1}{2}\right)$ **(d)** $(3, 1)$
(e) $\left(2\frac{1}{2}, -3\right)$ **(f)** $\left(\frac{15}{4}, \frac{17}{8}\right)$
(g) $\left(\frac{5}{2}, 4\right)$
(h) $\left(\frac{ap + aq}{2}, \frac{ap + aq}{2pq}\right)$

3. 32, 48; 9.6

4. 2.57

5. $p = 4, q = 13$

6. 9

7. **(a)** $(4, -1)$
(b) $(4, -11)$
(c) 14.56
(d) 49.04

8. $(5, 5)$, $(4, 8)$

9. $(-3, 4)$

10. $\pm\sqrt{50} = \pm 7.07$

11. $\left(2\frac{3}{11}, 0\right)$

12. $\left(0, -3\frac{21}{36}\right)$

Exercise 5b *(Pg 98)*

1. **(a)** $-\frac{1}{2}$ **(b)** -10 **(c)** $-1\frac{1}{3}$
(d) $\frac{1}{5}$ **(e)** $3\frac{1}{4}$ **(f)** -1
(g) 2 **(h)** $\frac{1}{6}$
(i) $\frac{2}{h + k}$

2. $-2\frac{4}{5}$

3. $1\frac{5}{6}$

4. $1\frac{1}{2}$

5. 3

7. 0, $\frac{1}{2}$, not defined, $-\frac{1}{6}$, $-\frac{5}{6}$

8. 5

9. 1 or -2

10. 9

11. -1 or $1\frac{1}{2}$

Exercise 5c *(Pg 101)*

1. **(a)** $y = -x$
(b) $y = 2x + 1$
(c) $4y = x + 14$
(d) $y = -x - 6$
(e) $y = 7 - x$
(f) $3y = 2x - 1$

2. **(a)** $3y = x$
(b) $y = 3x - 2$
(c) $y = ax + a$
(d) $y = 1 - 3x$
(e) $2y = 19 - x$
(f) $y = 4$

3. **(a)** 0, 1, $y = 1$
(b) 1, -1, $y = x - 1$
(c) not defined, $x = 1\frac{1}{2}$
(d) $-\frac{1}{2}$, 1, $2y + x = 2$

4. $y = 2x$ **5.** $y = 3x - 8$

6. $2y = x - 6$ **7.** -3

8. **(a)** $-\frac{2}{3}$ **(b)** $(3, 0)$

9. -4

10. **(a)** **(i)** 3 **(ii)** $y = 3x + 3$
(b) $(6, 3)$

11. $k = -\frac{4}{7}$ **12.** 3 or -4

13. $2x + 3y = 9, a = 0$

14. $2y = 5x - 19$

15. **(a)** $(-6, 0)$ **(b)** $\left(2, -6\frac{2}{3}\right)$
(c) $6y + 5x = 9$
(d) $y = -1$

16. **(a)** $2x + 17 = 7y$
(b) $(-8.5, 0)$ **(c)** $(2, 7)$
(d) 14 **(e)** 7.28, 3.85

17. $m = n = 0$ **18.** $-1\frac{2}{7}$

Review Questions 5 *(Pg 104)*

1. **(a)** $y = 2x - 3$ **(b)** 5

2. **(a)** -3 **(b)** 14

3. **(a)** 5 **(b)** 7
(c) $y + 8x = 0$

4. **(a)** $\frac{3}{4}$
(b) 10
(c) $(4, -3)$
(d) $4y + 3x = -24$

5. $4y + 3x = 24$ **(a)** $\left(2\frac{6}{7}, 3\frac{6}{7}\right)$
(b) $y = 3\frac{6}{7}$ **(c)** $x = 2\frac{6}{7}$

6. **(a)** $\frac{2}{3}$ **(b)** $3y = 2x + 5$
(c) $(2, 3)$ **(d)** 10
(e) 4.12

7. **(a)** 10 units **(b)** $\frac{1}{3}$
(c) $y = \frac{1}{3}x + 7$
(d) (12, 7) **(e)** 60 units^2
8. **(a)** **(i)** (6, 9) **(ii)** $y = -3x + 7$
(b) 40 units^2
9. **(a)** $y = 6$ **(b)** $y = 2x + 16$
(c) (–5, 6) **(d)** 25 units^2

Exercise 6a *(Pg 111)*

1. **(a)** 6 cm^2 **(b)** 4 cm^2
(c) 20 cm^2 **(d)** 48 cm^2
(e) 0.6 m^2 **(f)** 27 cm^2
2. **(a)** 6 cm **(b)** 15 cm
(c) 20 cm **(d)** 4 cm
3. $\frac{16}{49}$ **4.** 128 cm^2
5. 3 : 5 **6.** $\frac{5}{6}$
7. 812.5 m^2 **8.** 7 cm^2
9. 3.62 cm
10. **(a)** $66\frac{2}{3}$ cm^2
(b) $42\frac{2}{3}$ cm^2
11. $\frac{p^2}{(p+q)^2}$ **12.** 784 cm^2
13. **(a)** 18 cm **(b)** $\frac{4}{9}$
14. **(a)** $\frac{4}{25}$ **(b)** 100 cm^2
15. **(a)** 4 cm **(b)** $2\frac{2}{7}$ cm
(c) 7 : 4 **(d)** 16 : 49
16. **(a)** 50 cm^2 **(b)** 12 cm^2
(c) 30 cm^2

Exercise 6b *(Pg 117)*

1. **(a)** 576 cm^3 **(b)** 162 cm^3
(c) 324 cm^3 **(d)** 38.5 cm^3
(e) 0.4 m^3 **(f)** 16 cm^3
2. **(a)** 4 cm **(b)** 9 cm
(c) 21 cm **(d)** 5 cm
(e) 6 cm
3. **(a)** 3 : 4 **(b)** 189 cm^3
4. **(a)** 5 : 3 **(b)** 500 cm^3
5. **(a)** 2 : 5 **(b)** 40 cm^3
6. **(i)** **(a)** 25 : 16 **(b)** 125 : 64
(ii) **(a)** 9 : 16 **(b)** 27 : 64
(iii) **(a)** 4 : 9 **(b)** 8 : 27
7. 1 111.11 g
8. **(a)** 4 : 3 **(b)** 16 : 9
9. 16 cm^3 **10.** 20.8 cm
11. 4.76 cm **12.** 148.1 g
13. 10.27 cm
14. **(a)** 4.608 kg
(b) 13 281.25 litres

Review Questions 6 *(Pg 119)*

1. **(a)** 9 : 25 **(b)** 1 : 4
(c) 4 : 9
2. its area will be quadrupled, 100 cm^2
3. **(a)** $\frac{4}{9}$ **(b)** $\frac{49}{64}$
4. **(a)** 2 : 3 **(b)** 216 cm^2
5. **(a)** 3 : 4 **(b)** 45 cm^2
6. 35.08 tonnes
7. 6.144 tonnes
8. 675 kg **9.** 175 cm^2
10. $k = 64$
11. **(a)** $\triangle SLP$
(b) **(i)** $\frac{1}{4}$ **(ii)** $\frac{2}{3}$ **(iii)** $\frac{2}{3}$
12. **(a)** $\frac{21}{25}$
(b) **(i)** $\triangle NOM$ **(ii)** $\frac{2}{5}$
13. **(a)** 9 **(b)** 400 litres
14. **(a)** $\frac{25}{4}$ **(b)** $31\frac{1}{2}$ cm^2

Revision Exercise 1 No. 1 *(Pg 123)*

1. **(a)** $5\frac{3}{8}$ **(b)** $1\frac{3}{5}$
(c) 1.895 **(d)** 0.167 05
2. **(a)** $2x^2 + x - 1$
(b) $2x^3 + 3x^2 - 19x + 15$
3. **(a)** 10 **(b)** $-3\frac{1}{2}$ **(c)** $12\frac{5}{7}$
4. **(a)** $(3x + 4)(x - 2)$
(b) $(3x - 1)(4x + 1)$
(c) $(4x - 5y)^2$
5. **(a)** $4\frac{1}{2}$%
(b) 90 cm, 160 cm
6. **(a)** **(i)** $\frac{47}{60}$ **(ii)** $\frac{47x}{60}$
(b) **(i)** 16 km **(ii)** 16 km^2
7. $A(9, 0)$, $B(0, 9)$
8. **(a)** x^3y^6 **(b)** y
(c) a^2 **(d)** p^2
9. **(a)** **(i)** 32, 40 **(ii)** $8n$
(b) **(i)** 63, 99 **(ii)** $4n^2 - 1$
(iii) 899 **(iv)** 10
(c) 140
10. **(a)** $x \leqslant 3$ **(b)** $x < 4$
(c) $8 > x > 7\frac{1}{3}$

Revision Exercise 1 No. 2 *(Pg 124)*

1. **(a)** 10.0 **(b)** 10.8
(c) 0.804
2. **(a)** $(2x - 3)^2$
(b) $(x - y + 4)(x - y - 4)$
(c) $(x - 2)(x - 3y)$

3. **(a)** 0 or $-\frac{1}{2}$ **(b)** 20%
4. **(a)** 0.03
(b) $9.999\ 76 \times 10^{11}$
(c) 1.8×10^5 **(d)** $4\frac{7}{12}$
5. **(a)** $s = \frac{v^2 - u^2}{2a}$
(b) 55 cm, 30 cm
6. **(a)** 8.66 cm **(b)** 880 cm^3
7. **(a)** $3y = 5x - 4$
(b) 3, 4, 5, 6
8. **(a)** $\frac{(x-y)^2 + x - 5y}{x^2 - y^2}$
(b) $\frac{x - 9}{x - 5}$
(c) $\frac{-2x^2 + 17x - 7}{(x-5)(x+2)}$
9. \$250 000, \$375 000, 4% loss
10. **(a)** $5x + 7y + 25 = 0$
(b) **(i)** $-1\frac{1}{3}$ **(ii)** $5\frac{1}{2}$
(iii) $\pm\frac{1}{3}$

Revision Exercise 1 No. 3 *(Pg 124)*

1. **(a)** 4 or $1\frac{1}{3}$ **(b)** 3 or –1
2. **(a)** $2\frac{1}{10}$ **(b)** $1\frac{7}{8}$
(c) $\frac{104}{105}$
3. 0 or 5
4. **(a)** $(x - 4)^2$
(b) $(x + 2)(x^5 + 1)$
5. 13 yrs 4 mths **6.** 6
7. **(a)** 1 unit^2
(b) $y + 3x = 5$, $\left(1\frac{2}{3}, 0\right)$
8. **(a)** $\frac{1}{2}x$ **(b)** $a^{\frac{7}{12}}$
9. **(a)** $x = 121$, $y = 15$
(b) $2t - 1$
10. $x = 1.63$
rectangle, 13.81 cm

Exercise 7a *(Pg 128)*

1. 10 **2.** 40
3. 60 **4.** $P = 2Q$
5. $M = \frac{7}{3}L$, $5\frac{1}{7}$
6. $y = \frac{3}{2}$, $x = 8$
7. $A = \frac{1}{3}$, $B = 90$
8. 15, 21, 39

9. **(a)** $x = 3, y = 12, x = 8, y = 32, y = 40, x = 10, y = 6, x = \frac{3}{2}$
(b) $y = 27.5, x = 11; y = 35, x = 14; x = 17, y = 42.5; x = 22, y = 55; y = 74, x = 29.6$
(c) $x = 3, y = 6.9; x = 6, y = 13.8; y = 20.17, x = 9, y = 29.9, x = 13$

10. $F = \frac{3}{13}W$
(a) $F = 55\frac{5}{13}$ **(b)** $W = 338$

11. $I = \frac{3}{50}P$
(a) $I = 48$ **(b)** $P = 1\,200$

12. **(a)** $C = 65$ **(b)** $x = 204$

Exercise 7b *(Pg 131)*

1. $y = 13\frac{1}{2}, x = \frac{2}{3}\sqrt{6}$
2. $9\frac{3}{8}$ **3.** $W = 4d^2$, 64
4. 4.19, 14.1, 33.5
5. 160 **6.** $y = 2x^2$, 18
7. 42.6 **8.** $y = \frac{1}{5}\sqrt{x}$, 2
9. $V = 3, r = 1; V = 375, r = 5; r = 1\frac{1}{3}, V = 7\frac{1}{9}$
10. **(a)** 612.5 cm **(b)** 2 s
(c) $l = 24\frac{1}{2}T^2$
11. **(a)** \$1 360 **(b)** 135
(c) $D = 600 + 8n$
12. **(a)** $y = 32$ **(b)** $x = 3$
13. 5 cm, 36 hours **14.** 95 tonnes

Exercise 7c *(Pg 135)*

1. $y = \frac{240}{x}$, 48 **2.** $y = \frac{18}{x}$, 3
3. $8\frac{4}{7}$
4. $y = \frac{72}{x}$, 9, 144
5. 10
6. $y = \frac{1\,089}{x^2}$
(a) $30\frac{1}{4}$ **(b)** $16\frac{1}{2}$
7. $y = \frac{18}{x^2}$, 8
8. $y = \frac{12}{\sqrt{x}}$, $1\frac{7}{9}$, 4.24
9. $Q = 3, P = 4; P = 12, Q = 1; Q = \frac{1}{3}, P = 36$

10. $x = 2, y = 12; y = \frac{1}{3}, x = 12; y = 27, x = 1\frac{1}{3}; x = 8, y = \frac{3}{4}$
11. $t = 4, s = 4; s = 1\frac{1}{3}, t = 36; s = 16, t = \frac{1}{4}; t = 16, s = 2$
12. $x = 4, y = 1\frac{1}{4}; y = \frac{1}{100}, x = 20; k = 80, n = 3$
13. $f = 600$ kHz, 375 m
14. **(a)** 200 N/m² **(b)** $1\frac{1}{3}$ m²
15. 32, 4 hours **16.** 20 cm

Review Questions 7 *(Pg 137)*

1. $-28, s = -6\frac{1}{14}$
2. $x = 5, v = 8$
3. **(a)** $y = 3x$ **(b)** $x = 4$
4. $x = 13\frac{1}{2}$
5. $A = \frac{2}{3}, B = 3\frac{3}{4}$
6. $V = 4x^3, V = 864, x = 10$
7. **(a)** $a + 300b = 29$
$a + 700b = 57$
(b) $a = 8, b = 0.07$
(c) \$30.40
8. **(a)** $y = 28$ **(b)** $x = \pm 10$
9. $d = 3t^2$ **10.** $m = 81$
11. $\frac{3}{4}$ **12.** \$225
13. $s = 5t^2$ **14.** $\frac{1}{25}$
15. 16 **16.** 7
17. \$45, \$80

Exercise 8a *(Pg 149)*

1. −27, −1, 0, 8
(a) 3.4 **(b)** −2.3
2. −13, −5, −4, 3
(a) −15.6 **(b)** 2.7
3. −2, 2, 0, 2
(a) 0.2 **(b)** 2.4
4. 8, 4, 1.3, 0.8
(a) 1.1 **(b)** 2.7
5. 2.5, 1.1, 0.4
(a) 1.3 **(b)** 1.5
6. −5, −2.3, −1.7, −1.5
(a) − 1.8 **(b)** 3.3

7. 1, −1.81, −1.88
(a) 0.7
(b) 2.5
8. **(a)** −5, −1, −2, 20
(b) **(i)** −1 or 1.6, −2.4, 2.8
(ii) −11.6, −1.9, 5.2
9. **(a)** 0, 10, 20
(b) **(i)** 9.4, 15, 27
(ii) 0.8, 3.5, 4.15
10. **(a)** 0.9, 1.75, 5.2
(b) −0.3, 2.5, 4.7
11. **(a)** 24, 20, 19.2
(b) **(i)** 5.4, 3.2, 1.15
(ii) 27, 20.7, 19
12. 2.5, 7.7
(b) **(i)** 2.6, 8.5
(ii) 1.72, 2.46
13. 2.4, 5.8, 10.7
(b) **(i)** 3, 9.5
(ii) 0.18, 1.62
14. **(a)** −0.6, −1.3,
(b) 1, −2.26

Exercise 8b *(Pg 153)*

1. $x = 5, y = -4$
2. $x = \frac{1}{2}, y = 2$
3. $x = -2, y = 1$
4. $x = -1, y = -3$
5. $x = 0, y = 2$
6. $x = 1, y = 1$
7. No, the two graphs are parallel.
8. 1, Yes, $\left(0, 1\frac{1}{3}\right)$, infinite
9. $A(1, 1), B(2, 2), C(5, -1)$
10. $A(4, 4), B(0, 5), C(0, 2), D(1, 0)$
11. $A(0, 1), B(4, 1), C(4, 2), D(2, 5), E(0,2)$

Exercise 8c *(Pg 159)*

1. **(a)** 1, 2
(b) 0, 6
(c) no real x
(d) $\frac{1}{2}$, 4
(e) $\frac{1}{2}$, 1
(f) 1.2, 0.3
2. **(a)** 3.77, −0.27 **(b)** 3, $\frac{1}{2}$
(c) 0 **(d)** $\frac{1}{2}$, 2
3. **(a)** −4, 2
(b) −3.45, 1.45
(c) 2.83, −2.83
(d) 1.41

4. **(a)** −2, 3
(b) 2.79, −1.79
(c) 3.37, −2.37
(d) 0.56, −3.56
5. **(a)** 5.7, −0.7 **(b)** −2.3, 1.3
(c) 0, −7
6. **(a)** 1, 3 **(b)** 1, −3
(c) 3.3, −0.3 **(d)** no real x
(e) −3.3, 0.3
7. **(a)** 4.65, −0.65
(b) 3.42, 0.59
(c) 5.32, −1.32
(d) 2.82, 1.18
(e) 0.5, 4
8. **(a)** 6.2 **(b)** 1.7, −1.2
9. 24.4, 5.65
10. **(a)** −5.13 **(b)** 2.85, −0.35
(c) 4.21, −0.71

Exercise 8d *(Pg 166)*

1. 2.4, −0.5, −2
2. y = 1.3 or −1.8
3. 3.6
4. **(a)** −2.2, −1.2, 2.5
(b) 0.1 **(c)** 1.3
(d) −3.3 **(e)** −2.8, 1, 1.8
(f) 2.5, 2, 1
5. **(a)** 1.46, −0.45
(b) 0.33 or −2
6. 2, −1.3
7. 1.3, 2, −1.5, −6; 4.5; 3.2
8. 2, 0, 2, 6
(a) −0.25 **(b)** 2, 3
(c) 1.35, 3.65 **(d)** 4.3, 0.7
9. **(a)** 3.1 **(b)** −0.68
(c) −1 **(d)** 2.85
(e) 3, 0
10. −4, 0, 5, −1.6, 0.4
11. **(a)** 1.8, 4.7
(b) **(i)** 2.3 **(ii)** 1.6
(iii) 1.4 **(iv)** 1.3
12. **(a)** −1.6, −6.5
(c) **(i)** 0, 3.5 **(ii)** 0.9, 2.9
(iii) −0.4 or 3.7
13. 0.93 m
14. $0.63 \leqslant x \leqslant 2.37$

Review Questions 8 *(Pg 169)*

1. **(a)** −12, −8, −12, −8
(b) **(i)** −9.6 **(ii)** 3.1
2. **(a)** 3, 0, 15
(b) **(i)** −2.9 **(ii)** 2.4
(iii) −2, 0 or 2

3. **(a)** 7.3, 5.5, 4
(b) **(i)** −2.85, −3.9
(ii) 3.8, 6.4, 8.8
4. −1.7, −1.4, −0.3, 1
(b) **(i)** −1.2, 1.7
(ii) 0.6, 1.8
5. 2.3, 0.2
(b) **(i)** −1.9, 1.8
(ii) 2.6, 0.3
6. **(a)** $x = 2, y = -2$
(b) $x = 3, y = 0$
(c) $x = 3, y = 4$
7. **(a)** 2.2 or −2.2 **(b)** 1 or 2
(c) 3.5 or −1.5 **(d)** no real x
(e) 3.1 or −1.1 **(f)** no real x
8. **(a)** 1.3, −2.3 **(b)** 1.8, −2.8
(c) 0.4, −2.4 **(d)** no solution
(e) 1.7, −2.7 **(f)** 1.2, −3.2
9. 7, 1, 0.33, 1, 2.2
(a) 4, 1.6
(b) $1 < x < 1.6$ or $4 < x < 6$
10. −14, −5, 1, −5, −9; −1.60, 2.35; $4x^2 - 3x - 15 = 0$
11. **(b)** $a = 35, b = 0$
(d) 96 **(e)** 108 **(f)** 84
12. **(a)** **(i)** 14, −4
(ii) the horizontal distance from the cliff after projection
(b) **(i)** $x = 5$ **(ii)** $y = 81$
(d) 4.4 m

Revision Exercise II No. 1 *(Pg 175)*

1. **(a)** 100 **(b)** 60
2. **(a)** 60 cm², $7\frac{1}{17}$ cm
(b) 55.9°
3. **(a)** $1\frac{1}{3}$
(b) $\frac{6x + 11}{(3x - 1)(4x + 3)}$
4. 40 × 12, 30 × 16 cm
5. **(a)** $\frac{62}{15x}$ **(b)** $\frac{x + 8}{x - 3}$
(c) $\frac{6xy}{9x^2 - y^2}$
6. **(a)** $h = 6, k = 7$
(b) $5y = x + 3$
(c) 26 units²
7. **(a)** −3.3 **(b)** 1.8
8. 4 cm, 2.4 cm **9.** yes
10. 9 : 4; 24 cm

Revision Exercise II No. 2 *(Pg 175)*

1. 36 **2.** 56.3 m

3. **(a)** $\frac{13}{20x}$
(b) $\frac{3x - 10}{(x - 1)^2}$
(c) $\frac{2x + 1}{(x - 1)(x - 2)}$
(d) $\frac{x - 7}{x^2 - 9}$
4. $(AS)^2 = y^2 + 16$
$(AB)^2 = y^2 - 16y + 73$
y = 2, 6, 15 cm², 13 cm²
5. 352.8 cm²
6. **(a)** $-\frac{3}{4}$ **(b)** 24
7. **(a)** 15
(b) **(i)** 0 **(ii)** $\frac{1}{4}$
8. **(a)** **(i)** $y = \frac{48}{x^2}$
(ii) $5\frac{1}{3}$ **(iii)** ±4
(b) **(i)** $\frac{4}{5}$ **(ii)** 1 000
(iii) 27
9. 4 cm, 4.5 cm
10. $a = 9, b = 1$
(a) **(i)** 2.7 or −0.7
(ii) 0.25
(b) (1, 0), (0, 1)

Revision Exercise II No. 3 *(Pg 177)*

1. 12 cm
2. **(a)** $(x + 2)(x - 1)$
(b) $(5x - 2)(x - 1)$
(c) $(4x - 1)(2x - 1)$
(d) $(x - 5)(x^2 + 1)$
3. 10 000
4. **(a)** **(i)** p^7 **(ii)** $\frac{1}{p^{3.5}}$
(iii) $\frac{x^{25}}{y^{15}}$
(b) $y = 3(2x + 5)$
5. a = 7.56 cm, b = 6.48 cm
6. **(a)** 3 240 cm³
(b) 1 440 cm³
7. $(r + 5)^2 + (r + 4)^2 = 81$; 1.84 or −10.84;
OX = 6.84 cm
OY = 5.84 cm
8. 6
9. $7.5 - x$; $y = 14.1$, $x = 3.75$; 3.75 × 3.75
10. **(a)** 5, 8, 8, 5
(c) 3.4, −1.4
(d) 9
(e) $0 < x < 2$

Mid-Year Examination
Specimen Paper 1 *(Pg 179)*

1. 109
2. **(a)** $3\frac{2}{3}$
(b) $m = \frac{l(5l - 12)}{3l + 1}$
3. **(a)** 4 **(b)** $y = 4x + 3$
4. **(a)** 12 **(b)** $\frac{1}{16}$
5. **(a)** 2.4 **(b)** 6
6. **(a)** 9π cm^2 **(b)** 8
7. **(a)** 3 : 5 **(b)** 9 : 25
8. **(a)** $10\frac{2}{3}$ **(b)** $27\frac{9}{16}$
9. 6%
10. **(a)** 108.6° **(b)** –34.9°
11. 154 cm^2
12. **(a)** $6\frac{2}{3}$ cm^3 **(b)** $\frac{\pi r^3}{48}$
13. **(a)** $(2x - y)(2x + y)$
(b) **(i)** $x = 2$ or 8
(ii) $x = 0$ or -4
14. **(a)** 5.22 cm **(b)** 10.44 cm
15. **(a)** $\frac{2x + 9}{x^2 - 4}$
(b) $\frac{-3x^2 - 41x}{(x + 7)(x - 3)}$
16. **(a)** –14, –5, 1, –9
(c) $x = -0.3$ or 1.8
(d) $x = -1.2$ or 2.2
17. **(a)** 40 073
(b) **(i)** $3x(4x - 9)$
(ii) $(x - 3y)(4 + z)$
(c) 3 or –4
18. **(a)** **(i)** 748 cm^2
(ii) 12.32 kg
(b) **(i)** 14.36 cm
(ii) 10.90 cm^2
(iii) 9.58 cm^2
19. **(a)** $k = 11$, $l = -9$
(b) 4.94
20. **(a)** 0.608 or –4.11
(b) 2 h, 4 h
21. **(a)** $x = \frac{5a^2 + 3b^2}{2b^2 - 3a^2}$
(b) $x - 4y$
(c) **(i)** 28 **(ii)** 20

Mid-Year Examination
Specimen Paper 2 *(Pg 181)*

1. **(a)** 6 **(b)** 1.5×10^{-6}
2. **(a)** $\frac{y^2}{3xz}$ **(b)** $\frac{m - 1}{m + 5}$
3. **(a)** $14.70 **(b)** 20%
4. 90 **5.** 7.1 cm^2
6. **(a)** 5 **(b)** 108° **(c)** No
7. **(a)** $\frac{1}{2}$ **(b)** $2y = x + 6$
8. **(a)** $3(a + 2)(2a - 5)$
(b) $(2x - 3y^2)(2x + 3y^2)(4x^2 + 9y^4)$
9. 8.4 cm
10. **(a)** 12.12 cm **(b)** 14 cm
(c) $51\frac{1}{3}$ cm^2
11. $408, $640
12. **(a)** $y = 2\frac{1}{4}$
(b) **(i)** 95° **(ii)** 40°
13. **(a)** $6\frac{1}{4}$ cm **(b)** 62.5 cm^3
14. **(a)** $1\frac{7}{9}$ **(b)** $\frac{17}{72}$
15. **(a)** $6^3 - 6 = 210 = 5 \times 6 \times 7$
(b) $18 \times 19 \times 20$
(c) $(x - 1)x(x + 1)$
16. $\frac{20}{v}$, $\frac{20}{v - 1}$, $v = 5.7$ km/h, 7 h 46 min
17. **(a)** **(i)** 11 **(ii)** 10
(iii) $9\frac{3}{4}$
(b) 35×18; 30×21
18. **(a)** **(i)** 880 cm^2
(ii) 21.56 kg
(b) 218 cm^3 **(c)** 7.54 *l*
19. **(a)** **(i)** 6 or $-1\frac{1}{3}$
(ii) –2 **(iii)** $\frac{1}{8}$
(b) $3\frac{22}{23}$
20. **(a)** $h = \frac{2\pi g}{\pm\sqrt{t^2 - 4\pi^2}}$
(b) **(i)** $\frac{1}{3}(x + 6)$
(ii) $\frac{x^2 + 2x + 2}{1 - x^2}$
(c) 7.07
21. **(a)** $\frac{2}{3}kx + \frac{k}{3} - n$
(b) 13.5 km/h, 16.5 km/h
(c) –0.4 or –4.6

Mid-Year Examination
Specimen Paper 3 *(Pg 183)*

1. **(a)** $x = -\frac{1}{2}$ or $\frac{2}{3}$
(b) $x = -\frac{6}{7}$
2. 3 **3.** $3.92 **4.** 4
5. **(a)** 27 **(b)** $\frac{1}{49}$
(c) $9\frac{1}{8}$
6. **(a)** 1.7 m
(b) 3 000 000 m^2
7. **(a)** 49 cm^2 **(b)** 28 cm^2
8. **(a)** $m = \frac{l}{n - 2}$ **(b)** $\frac{1}{2x - 3}$
9. 25.2 cm; 2 449 cm^3
10. $\frac{1}{3}$
11. **(a)** 251.36 cm^2
(b) 628.4 cm^3
12. **(a)** $3(x + 2y)(x - 2y)$
(b) $(2a + b)(3x - 2y)$
13. **(a)** $a = 1\frac{1}{3}$, $b = 1\frac{2}{3}$
(b) $y = 3x - 2$, $k = 7$
14. **(a)** $\frac{14tw}{5}$ **(b)** $550 000
15. **(a)** $\frac{1}{2}$ **(b)** $y = \frac{1}{2}x + 3$
16. **(a)** 0.362 or –0.790
(b) $\frac{330}{x - 5} = \frac{330}{x} + \frac{1}{2}$, $5\frac{1}{2}$ h, 6 h
17. **(a)** 3 or $-\frac{1}{5}$
(b) $y = \pm\sqrt{\frac{2 - x^2}{1 + x^2}}$
(c) 0
18. **(a)** 5.14 cm **(b)** 69.1°
(c) 9.89 cm
19. **(a)** 3, $\frac{11}{12}$, 1
(c) **(i)** 0.8 **(ii)** 1.15, 2.43
(d) 0.71 or 2.64
20. **(a)** 1 125 **(b)** 14 mm
(c) 126
21. **(a)** $67 283.48, $8 116.52
(b) **(i)** 3 **(ii)** (1, 10), (4, 5)

Exercise 9a *(Pg 193)*

1. **(a)** 86.4 km/h **(b)** 64.8 km/h
(c) 144 km/h **(d)** 3.6*a* km/h
2. **(a)** 4.44 m/s **(b)** 8.89 m/s
(c) 12.5 m/s **(d)** 0.28*b* m/s
3. **(a)** 29.63 **(b)** 8.23
4. 11 52 **5.** 19.44 m/s
6. **(a)** 60 km **(b)** 56 km/h
(c) $1\frac{1}{2}$ h **(d)** 52.6 km/h
7. 11 52
8. **(a)** 120° **(b)** 83.8 cm
9. 65 min
10. **(a)** from $1\frac{1}{2}$ h to $2\frac{1}{2}$ h
(b) 30 km/h **(c)** 30 km/h
11. **(b)** $v = 45$ km/h, $u = 72$ km/h

12. 10 45
13. 1 h 11 min, 1 h and 1 h 22 min
14. **(a)** 2 **(b)** 40
15. 10 11, 8.3 km
16. **(a)** 5, 8, 5
(c) 4
(d) 2.8 or –1.8
(e) $0 < x < 3$
17. –1, 15, 4, –13
(b) 10, –14
(c) $-1.1 \leqslant x \leqslant 4.1$
18. **(a)** $a = -3.2$
(c) **(i)** $x = 1.2$ or 8.3
(ii) $\frac{1}{10}\left(120 - x^2 - \frac{96}{x}\right) = 4$
(d) $x = 6.4$ **(e)** –1.4
19. **(b)** 3.6 m/s **(c)** 19 m
20. 0.65 km/min, 2.88 km
21. **(a)** 2.3 min **(b)** 0.38 km/min
(c) 2.75 min

Exercise 9b *(Pg 0199)*

1. **(a)** 5 m/s² **(b)** 30 m
(c) 7.5 m/s
2. **(a)** 2 m/s² **(b)** 3 m/s²
(c) 26 m **(d)** $4\frac{1}{3}$ m/s
3. **(a)** 9 **(b)** 30 m/s
4. **(a)** acceleration
(b) 7 m/s **(c)** 420 m
5. **(a)** 4 m/s² **(b)** 60 m
(c) 10 m/s
6. **(a)** 32.5 km **(b)** 78 km/h
7. **(a)** 1.5 m/s² **(b)** 1 500 m
(c) 100 s
8. **(a)** 20 m/s **(b)** 15 m/s
9. **(a)** 2 m/s² **(b)** 153 m
10. 70 m
11. **(a)** $6\frac{2}{3}$ m/s **(b)** 2 925 m
12. **(a)** 42 km/h **(b)** 126 km
(c) 36 km/h
13. **(a)** $\frac{1}{3}$ m/s² **(b)** 720 m
(c) 25.7 m/s **(d)** 18 m/s
14. **(a)** 12 m/s **(b)** 2 m/s²
(c) 10 m/s **(d)** 13 m/s
15. **(a)** 2.4 m/s², 4.8 m/s²
(b) 9.5 m/s, 34 m/s
16. **(a)** 5.2 m/s², –3.5 m/s²
(b) $t = 3.5$ sec

Review Questions 9 *(Pg 202)*

1. **(a)** 60 km/h **(b)** 30 km/h
(c) 24 km/h

2. **(a)** $1 \leqslant t \leqslant 2$ **(b)** 40 km/h
(c) 17.14 km/h
3. 70.4 m/min **4.** 2 376 km/h
5. **(a)** 15 km **(b)** 15 min
6. 17 00, 360 km
7. 62.5 km/h, 12 42
8. $a = 54$, $b = 45$
(a) 6.7 s **(b)** 2.3 s
(c) 5 m/s²
9. 1.5, 2 **(b)** 1.5, 1.7
(c) $0.9 < x < 3.3$
(d) 0.67 **(e)** 3.6
10. **(b)** 4 500
(c) **(i)** 1 500
(ii) rate of production of bacteria at the particular moment
(d) $t = 4.95$ h **(e)** $k = 50$
11. **(a)** $a = 3$, $b = 4.4$
(c) 1.3 or 4.7
(d) $1.6 \leqslant x \leqslant 5.1$
(e) 1.25
12. 0, 50
(a) 4.2 **(b)** 2
(c) –4 m/s², 12 m/s²
13. **(a)** 2.5 m/s² **(b)** 20 m
14. **(a)** 60 km/h **(c)** 7.13 a.m.
(d) 4

Exercise 10a *(Pg 211)*

1. **(a)** sin 70° = 0.939 7
(b) sin 4° = 0.069 8
(c) sin 82° = 0.990 3
(d) –cos 81° = –0.156 4
(e) –cos 73° = –0.292 4
(f) –cos 5° = –0.996 2
(g) –cos 48° = –0.669 1
(h) –cos 24° = –0.913 5
(i) –tan 87° = –19.08
(j) –tan 62° = –1.881
(k) –tan 5° = –0.087 5
(l) –tan 37° = –0.753 6
2. **(a)** 31.3° **(b)** 48.6°
(c) 61.0° **(d)** 20.2°
3. **(a)** 47.9° **(b)** 40.9°
(c) 60° **(d)** 9.9°
4. **(a)** 7.0° **(b)** 50.9°
(c) 70.0° **(d)** 7.2°
5. **(a)** 48.8°, 131.2°
(b) 72.1°, 107.9°
(c) 28.1°, 151.9°
(d) 10.8°, 169.2°
(e) 103.7° **(f)** 141.5°

(g) 121.6° **(h)** 173.7°
(i) 142° **(j)** 104.8°
(k) 131.8°
(l) 117.1°
6. **(a)** 3.5 **(b)** 0.7 **(c)** –2.1
7. 27, 153
8. **(a)** $\frac{24}{7}$ **(b)** $\frac{7}{25}$
9. **(a)** $\frac{4}{5}$ **(b)** $-\frac{4}{3}$
10. **(a)** $\frac{8}{17}$ **(b)** $-\frac{15}{17}$
11. **(a)** $-\frac{24}{25}$ **(b)** $-\frac{7}{24}$
12. **(a)** sin 60°, $\frac{\sqrt{3}}{2}$
(b) –cos 60°, $-\frac{1}{2}$
(c) –tan 60°, $-\sqrt{3}$
(d) sin 45°, $\frac{1}{\sqrt{2}}$
(e) –cos 45°, $-\frac{1}{\sqrt{2}}$
(f) –tan 45°, –1
(g) sin 30°, $\frac{1}{2}$
(h) –cos 30°, $-\frac{\sqrt{3}}{2}$
(i) –tan 30°, $-\frac{1}{\sqrt{3}}$

Exercise 10b *(Pg 214)*

1. **(a)** 34.24 cm²
(b) 29.21 cm²
(c) 52.48 cm²
(d) 17.35 cm²
(e) 27.39 cm²
(f) 70.67 cm²
2. 116.7 cm²
3. 9 035 cm²
4. 633 cm², 29.5 cm
5. 10.2°, 169.8°
6. **(a)** 800 cm²
(b) 774 cm²
7. 22 973 m²
8. **(a)** 30°
(b) 10 cm
(c) 346.4 cm²
9. **(a)** 27.5°
(b) 10.5 cm
(c) 6.22 cm²
10. 30 cm²
11. 116.2 cm²
12. **(a)** 10.8 cm²
(b) 104.4°

Exercise 10c *(Pg 218)*

1. **(a)** $C = 62°$, $b = 10.7$ cm, $c = 9.8$ cm
(b) $R = 79.3°$, $p = 4.4$ cm, $r = 7.0$ cm
(c) $M = 38°$, $l = 11.5$ cm, $n = 5.3$ cm
(d) $Y = 37.1°$, $x = 9.7$ cm, $z = 15.0$ cm
2. **(a)** $C = 57.2°$, $a = 6.35$ cm, $b = 5.14$ cm
(b) $A = 48.7°$, $b = 9.87$ cm, $c = 7.23$ cm
(c) $B = 98°$, $a = 4.81$ cm, $c = 2.87$ cm
(d) $B = 26.9°$, $C = 61.1°$, $c = 13.4$ cm
(e) $A = 55.6°$, $C = 26.4°$, $c = 7.81$ cm
(f) $B = 31.7°$, $A = 113.3°$, $a = 15.2$ cm
(g) $B = 11.8°$, $C = 43.2°$, $c = 10.0$ cm
3. 11.8 cm
4. 28.3°, 61.7°, 10.3 cm
5. 15.6 cm
6. **(a)** No **(b)** Yes **(c)** No **(d)** Yes **(e)** Yes **(f)** No **(g)** No
7. $Q = 48.6°$, $R = 101.4°$ $r = 15.68$ cm or $Q = 131.4°$, $R = 18.6°$ $r = 5.10$ cm
8. $B = 68.9°$, $C = 53.1°$, $C = 13.2$ cm or $B = 111.1°$, $C = 10.9°$, $c = 3.12$ cm
9. 60.36 cm^2
10. **(a)** 6.92 cm **(b)** 40.1 cm^2
11. 49.9°, 130.1°, $6\frac{2}{3}$
12. **(a)** 5 cm **(b)** 9.40 cm **(c)** 4.92 cm
13. 4.32 cm, 5.49 cm, 40.5°
14. **(a)** 2.64 cm **(b)** 55.8° **(c)** 49.3°
15. **(a)** 1.65 cm **(b)** 4.79 cm
16. **(a)** 37.0° **(b)** 9.47 cm **(c)** 5.62 cm
17. **(a)** 9.18 cm **(b)** 0.734 2 km **(c)** 0.321 2
18. **(a)** 8.60 cm **(b)** 35.5° **(c)** 6.33
19. **(a)** 12.54 m **(b)** 41.7° **(c)** 13.44 m
20. 1 110 m

Exercise 10d *(Pg 222)*

1. 6.24 cm **2.** 12.17 cm
3. 4.57 cm **4.** 96.9 cm
5. 9.45 cm **6.** 93.4°
7. 120°
8. 88.5°, 32.6°, 58.9°
9. **(a)** 3.464 **(b)** 5.292 **(c)** 90°
10. $c = 10.6$ cm, $A = 44.4°$, $B = 63.7°$
11. 48.2° **12.** 99.6°
13. $\frac{37}{208}$; 7.09
14. **(a)** 9 **(b)** 15.1
15. 93.8°, 9.29 cm
16. 6.78
17. 5.57, 52.1°
18. **(a)** 6.12 cm **(b)** 7 cm
19. $-\frac{1}{20}$, 6.58 cm
20. **(a)** 73.4° **(b)** 1.92 cm **(c)** 2.18 cm
21. **(a)** 22.6° **(b)** 4.84 m **(c)** 6.86 m

Exercise 10e *(Pg 226)*

1. **(a)** 033° **(b)** 118° **(c)** 226° **(d)** 321°
2. **(a)** 055° **(b)** 165° **(c)** 317° **(d)** 235° **(e)** 345° **(f)** 137°
3. **(a)** 036° **(b)** 216° **(c)** 073° **(d)** 253° **(e)** 296° **(f)** 116°
4. **(a)** 310° **(b)** 270° **(c)** 220°
5. 028° or 216°
6. **(a)** 315° **(b)** 003° or 267° **(c)** 238° or 032°
7. 34.62 km, 35.51 km
8. 71.12 m
9. **(a)** 218 m **(b)** 180 m **(c)** 435.6 m
10. 40.2 km
11. 31.24 km, 080.2°
12. 7.97 km
13. **(a)** 53.4° **(b)** 126.6° **(c)** 385.5 m^2 **(d)** 28.8 m
14. **(a)** 1 925 **(b)** 79.6 km **(c)** 191.4°

Exercise 10f *(Pg 231)*

1. **(a)** 6 cm **(b)** 33.1° **(c)** 45°
2. **(a)** 33.7° **(b)** 10 cm **(c)** 21.8°
3. 13 cm **(a)** 53.1° **(b)** 67.4°
4. 22.4°, 44.8°
5. 53.1°, 57.0°
6. **(a)** 14.1 cm, 28.7 cm **(b)** 63.8°
7. **(a)** 26.6° **(b)** 57.5°
8. **(b)** 347 m **(c)** **(i)** 42.5 m **(ii)** 29.6°

Exercise 10g *(Pg 234)*

1. **(a)** 50 cm **(b)** 17.2 cm **(c)** 58.4°
2. **(a)** 56.3° **(b)** 5.55 m **(c)** 10.6 m **(d)** 52.0°
3. 38.6 m
4. **(a)** 053.1° **(b)** 59.9 m **(c)** 4.6°
5. 207 m
6. **(a)** 053.1° **(b)** 326.3°, 12.1 m, 7.7°
7. **(a)** 38.7° **(b)** 027.1°, 14.0°
8. 170 m, 8.4°

Review Questions 10 *(Pg 236)*

1. **(a)** 8 cm **(b)** 60 cm^2 **(c)** 7.06 cm
2. **(a)** 10.4 cm **(b)** 38.2 cm^2 **(c)** 7.34 cm
3. **(a)** 6.67 cm **(b)** 4.39 cm **(c)** 44.4° **(d)** 14.34 cm^2 **(e)** 17.24 cm^2
4. **(a)** 4.46 cm **(b)** 2.46 cm^2
5. **(a)** 8.04 cm **(b)** 25.32 cm^2
6. **(a)** 40 cm^2 **(b)** 53.1° **(c)** 8.94 cm
7. **(a)** 8.63 cm **(b)** 7.66 cm **(c)** 33.6 cm^2
8. **(a)** 14.0 cm **(b)** 5.99 cm **(c)** 70.6°
9. **(a)** 10.41 cm, 8.43 cm
10. 106.3°
11. **(a)** 40.1° **(b)** 43.3° **(c)** 26.99 cm^2
12. **(a)** 103.5° **(b)** 4.43 cm

13. **(a)** 18.49 cm **(b)** 11.18 cm
14. $-\frac{17}{48}$
15. 1 188 m, 93 543 m²
16. 40.5°
(a) 24 m **(b)** 139.4 m²
17. 6.36 cm, 4.10 cm²
18. **(a)** 17.4 cm, 10.6 cm
(b) 83.1 cm²
19. **(a)** **(i)** 73.3 m **(ii)** 18.5°
(iii) 021.5°
(b) 42.8 m
20. **(a)** **(i)** 035° **(ii)** 4.59 km
(iii) 12.3 km
(b) **(i)** 7.51 km **(ii)** 4.53 km
21. **(a)** **(i)** 311.2 km²
(ii) 31.8 km
(b) **(i)** 17.2 km **(ii)** 26 mins
22. **(a)** 14.8 cm **(b)** 46.6°
(c) 22.5 cm
23. **(a)** 17.1 m **(b)** 16.8 m
(c) 35.5°
24. **(a)** 77.2 m **(b)** 52.1 m
(c) 29.5°
25. **(a)** 10 cm **(b)** 24 cm
(c) 45.2° **(d)** 24.7 cm
26. 122.6 m
27. **(a)** 20.7 m **(b)** 77.6 m
(c) 1 760 m² **(d)** 45.3 m
(e) 24.6°
28. **(a)** **(i)** 111° **(ii)** 61.7°
(iii) 082.7°
(b) **(i)** 57.4 m **(ii)** 20.7°
(c) 44.9 m
29. **(a)** **(i)** 108° **(ii)** 40.9 m
(b) **(i)** 60° **(ii)** 30°
(iii) 14.9 m
(c) 27.7 m

Exercise 11a *(Pg 245)*

1. **(a)** $x = 12, y = 90$
(b) $x = 11, y = 90$
(c) $x = 12, y = 67.4$
(d) $x = 10.95, y = 61.9$
(e) $x = 16, y = 53.1$
(f) $x = 6, y = 50.2$
2. 15 cm **3.** 13 cm
4. 13.75 cm **5.** 5.7 cm
6. 28.8 cm
7. 7 cm or 1 cm
8. 8.97 cm or 1.76 cm
9. 8.4 cm
10. 17 cm

Exercise 11b *(Pg 249)*

1. **(a)** 80° **(b)** 30° **(c)** 40°
(d) 125° **(e)** 50° **(f)** 45°
2. **(a)** 30° **(b)** 50° **(c)** 115°
(d) 155° **(e)** 35° **(f)** 28°
3. **(a)** 60° **(b)** 70° **(c)** 50°
(d) 12° **(e)** 40° **(f)** 70°
4. **(a)** 38° **(b)** 52° **(c)** 52°
5. 65°, 55° 60° **6.** 125°
7. 31° **8.** 123°
9. 32° **10.** 186°
11. 37° **12.** 62°, 47°
13. 66°, 114° **14.** 150°, 15°

Exercise 11c *(Pg 253)*

1. 36° **2.** 47°
3. 130°, 50° **4.** 50°
5. 70°, 70° **6.** 60°, 35°
7. 270° **8.** 5 cm
9. 60° **10.** 45°
11. 90°, 55° **12.** 40°
13. 60°, 40°
14. 26°, 64°, 64°
15. 30°

Exercise 11d *(Pg 257)*

6. 35°

Review Questions 11 *(Pg 258)*

1. **(a)** $x = 41, y = 49$
(b) $x = 118, y = 62$
(c) $x = 74, y = 103$
(d) $x = 57.5, y = 115$
(e) $x = 50$
(f) $x = 26, y = 82$
(g) $x = 100°$
(h) $x = 95$
(i) $x = 28, y = 72$
2. **(a)** $x = 46, y = 116$
(b) $x = 121, y = 59$
(c) $x = 108, y = 144$
(d) $x = 41$
(e) $x = 22, y = 48$
(f) $x = 44, y = 46, z = 68$
(g) $x = 24, y = 42$
(h) $x = 63$
3. **(a)** $x = 72, y = 32$
(b) $x = 57, y = 40$
(c) $x = 22.5, y = 135$
(d) $x = 78, y = 30$
(e) $x = 80$
(f) $x = 80$
(g) $x = 103, y = 45$

(h) $x = 42, y = 90$
(i) $x = 59, y = 29$
4. $x + y$
5. **(a)** 1 cm **(b)** $\frac{1}{2}y$
6. 24°, 49°
7. **(a)** 69° **(b)** 124.5°

Exercise 12a *(Pg 267)*

1. **(a)** $x = 49, y = 14$
(b) $x = 58, y = 15$
(c) $x = 34, y = 14.8$
(d) $x = 35, y = 55$
(e) $x = 8, y = 67.4$
(f) $x = 12.6, y = 50.0$
(g) $x = 35, y = 33$
(h) $x = 40, y = 40$
(i) $x = 8, y = 28.1$
2. 24°
3. **(a)** 54° **(b)** 66°
(c) 36° **(d)** 112°
4. 138° **5.** 51°
6. 94° **7.** $\frac{90° + x}{2}$
8. 61° or 119° **9.** 46°, 134°
10. 9 cm **11.** 64°
12. 3.5 cm **13.** 45 cm

Exercise 12b *(Pg 271)*

1. **(a)** $x = 20$
(b) $x = 50, y = 40$
(c) $x = 50, y = 62, z = 68$
(d) $x = 36, y = 70, z = 20$
(e) $x = 35, y = 20$
(f) $x = 40, y = 46$
(g) $x = 65, y = 32.5$
(h) $x = 62, y = 34$
(i) $x = 107$
2. **(a)** 56° **(b)** 50° **(c)** 100°
3. 90°, 42°, 48° **4.** 98°
5. 50° **6.** 28°, 28°
7. 48°, 66° **8.** $y - x$
9. 50° **10.** 105°
11. 90° **12.** 130°
13. 25° **14.** 40°
15. 32°

Review Questions 12 *(Pg 274)*

1. **(a)** $x = 37, y = 53$
(b) $x = 65, y = 40$
(c) $x = 30, y = 30$
(d) $x = 25, y = 113$
(e) $x = 40, y = 55$
(f) $x = 42, y = 28$

(g) $x = 90, y = 86$
(h) $x = 50, y = 160$
(i) $x = 65, y = 124$

2. **(a)** $x = 50, y = 25$
(b) $x = 34, y = 114$
(c) $x = 28.5, y = 16.5$
(d) $x = 8, y = 38$
(e) $x = 32, y = 70$
(f) $x = 55, y = 130$
(g) $x = 26, y = 38$
(h) $x = 26, y = 148$
(i) $x = 118, y = 62$

3. 105°, 60°

4. 20°

5. **(a)** $x = 70, y = 35$
(b) $x = 34, y = 56$
(c) $x = 132, y = 114$
(d) $x = 105, y = 30$
(e) $x = 6.43, y = 25$
(f) $x = 54, y = 72$

6. 20°

7. 60°

8. 55°, 60°, 65°

9. 32°, 70°, 26°, 46°

10. 50°

Revision Exercise III
No. 1 *(Pg 277)*

1. **(a)** $3\frac{13}{14}$
(b) 0.209 9

2. $n = \frac{4m + 3}{2 - m}$
(a) $\frac{12}{17}$ **(b)** $\frac{8}{9}$

3. 15, 2.6, 2.6

4. **(a)** $P(-4, 0)$, $Q(0, 2)$
(b) $M(-2, 1)$, $x = -2$
(c) $x + 2y = 4$

5. **(a)** -6 **(b)** 35

6. $u = 20, v = 70, w = 30, x = 70,$
$y = 130, z = 80$

7. **(a)** **(i)** 2 m/s
(ii) $\frac{1}{3}$ m/s^2
(iii) 210 m
(b) **(i)** 20, 26
(ii) 37, 69

8. **(a)** $x = 54, y = 36$
(b) $x = 56, y = 90$

9. **(a)** $x = 25, y = 16$
(b) $x = 128, y = 64$

10. $y = \frac{1}{3}x^2$; $21\frac{1}{3}$

Revision Exercise III
No. 2 *(Pg 278)*

1. **(a)** **(i)** 0.039
(ii) 0.038 60
(iii) $3.859\,6 \times 10^{-2}$
(b) 0.000 35 **(c)** 10%

2. 128, $W = 32h$

3. **(a)** 72° **(b)** 42°

4. **(a)** **(i)** 128° **(ii)** 88°
(b) **(i)** 9.81 cm **(ii)** 6.01 cm

5. **(a)** 60° **(b)** 10.4 cm
(c) 46.1°

6. **(a)** 1.75 or -1.00
(b) $k = \frac{5 + 5a}{2 + 6a}$

7. **(a)** $3\frac{1}{2}$ **(b)** 81

8. 4, 3.5, 4.7, 6.3, 8, 9.8
(a) 1.4
(b) **(i)** 3.2, 0.8 **(ii)** 1, 5

9. **(a)** 2.5 m/s^2 **(b)** 600 m
(c) 41 sec **(d)** 20.12 m/s

10. **(a)** 6.78 km **(b)** 074.4°
(c) 331°

Revision Exercise III
No. 3 *(Pg 279)*

1. **(a)** $\frac{3}{5}, \frac{5}{8}, \frac{7}{10}$
(b) **(i)** 14 **(ii)** 10

2. **(a)** 5 **(b)** 9
(c) $-4, \frac{3}{2}$

3. **(a)** $\frac{3}{4}$
(b) $4y = 3x + 4$, $\left(-\frac{4}{3}, 0\right)$
(c) 10 units^2
(d) 6.67 units

4. **(a)** $\frac{4}{5}$
(b) $y = \frac{1}{3}z^2$, 12, 12

5. **(a)** 50°
(b) 100°

6. $\frac{1}{2}(a + 1)(5a + 3)$; 1.11, 5.32 cm

7. **(a)** $a = 35.3$
(b) $x = y = z = 40$

8. 156.5 cm^2

9. **(a)** **(i)** 70°
(ii) 69°
(iii) 55°
(b) **(i)** $180° - 6x$
(ii) $3x$
(iii) $90° - x$

10. 55

Exercise 13a *(Pg 282)*

1. **(c)** 30%

3. **(b)** $26\frac{2}{3}$% **4.** **(b)** $73\frac{1}{3}$%

Exercise 13b *(Pg 291)*

1. **(a)** 55.5, 60.5, 7;
60.5, 65.5, 6;
65.5, 70.5, 5;
70.5, 75.5, 10;
75.5, 80.5, 5;
80.5, 85.5, 5;
85.5, 90.5, 2;
90.5, 95.5, 3;
95.5, 100.5, 3
(b) 90.9 belongs to 91 – 95;
66.2 belongs to 66 – 70;
81.5 belongs to 81 – 85

2. 4, 4, 6, 8, 10, 5, 4, 4, 3
(a) 5 mm

5. **(a)** 84

7. **(a)** 10, 2 standard, 4;
5, 1 standard, 6;
5, 1 standard, 11;
5, 1 standard, 13;
15, 3 standard, 1

8. **(a)** 10, 1.2; 10, 1.1;
20, 0.4; 30, 0.2

11. **(a)** 619

Exercise 13c *(Pg 296)*

1. 32, 37, 42, 47, 52, 57

2. **(a)** 45

Review Questions 13 *(Pg 298)*

2. **(a)** 8, 11, 21, 12 **(b)** 80

3. **(a)** 4th day, 35 workers
(b) 10th day, 13 workers
(c) 4th and 8th days

4. **(a)** $20 < x \leqslant 25$, 12;
$30 < x \leqslant 35$, 7;
$35 < x \leqslant 40$, 4;
$40 < x \leqslant 45$, 3
(b) 39

Exercise 14a *(Pg 306)*

1. **(a)** 4.5 **(b)** 3 **(c)** 4.7

2. **(a)** 8 **(b)** 11.5 **(c)** 10.5

3. **(a)** 60 cents **(b)** 75 cents
(c) 74 cents **(d)** 70

4. 18.5

5. **(a)** 16 **(b)** 16
(c) 15.25

6. **(c)** 16, 4
(d) **(i)** 2 **(ii)** 2
7. **(a)** 0 **(b)** 1.5 **(c)** 2.1
8. **(a)** **(i)** $x \leqslant 38$ **(ii)** 2, 2
(b) 71, 0

Exercise 14b *(Pg 309)*

1. **(a)** 3, 141; 5, 250; 9, 216; 11, 154; 13, 104; 1 230
(b) 6.15 units **(c)** $4 < x \leqslant 6$
2. **(a)** $12 < l \leqslant 16$
(b) 2, 2; 5, 15; 8, 40; 397
(c) 11.3 cm
3. 50–59, 51.03 years
4. 3.34–3.36, 3.33 cm
5. $50 < v \leqslant 60$, 51.5 km/h

Exercise 14c *(Pg 312)*

1. 6
(a) 9 **(b)** 12 **(c)** 23
2. **(a)** **(i)** 13
(ii) 13
(iii) 14.8
(b) **(i)** 18.8 **(ii)** 29.8
(iii) 114.8 **(iv)** 20.8
3. 7.6
(a) 118.6 **(b)** 7.3
(c) 65.3 **(d)** 103.8
4. 506 **5.** 1 247
6. 64.5
7. **(a)** 49.5
(b) **(i)** 76.5 **(ii)** 99.5
8. **(a)** 419
(b) 1 409
(c) 1 391

Exercise 14d *(Pg 316)*

1. –2, –30; –1, –20; 0, 0; 1, 18; 2, 26; 3, 9; 100; 32.85 kg
2. –12, –60; –8, –56; –4, –48; 4, 36; 8, 16; 50; –112; 17.76
3. Table (i): –2, –8, –1, –9; 0, 0; 1, 13; 2, 12; 3, 12; 50; 20; 60.4
Table (ii): –3, –12; –2, –18; –1, –14; 0, 0; 1, 6; 2, 8; 50; –30; 60.4; Yes
4. **(a)** 63, 65, 65
(b) 161.18, 160, 165
(c) 56.2, 54.5, 64.5
(d) 53.7, 52, 52
(e) 820, 849, 849

5. 60, –40, 2, –80; 70, –30, 3, –90; 80, –20, 9, –180; 90, –10, 23, –230; 0, 26, 0; 110, 10, 21, 210; 120, 20, 10, 200; 130, 30, 5, 150; 140, 40, 1, 40; 20; 100.2 g
6. –20, –80; –10, –60; 0, 0; 10, 160; 20, 200; 220; 29.9 cm
7. **(a)** 172.1 hours
(b) 3, 495; 7, 1 176; 6, 1 026; 7, 1 218; 3, 531; 4 720; 5 166; 172.2 hours
(c) 3, 493.5; 3, 499.5; 4, 674; 5, 852.5; 2, 345; 6, 1 047; 3, 529.5; 4, 714; 5 155; 171.8 hours
(d) No; –6, 3, –18; –3, 7, –21; 0, 6, 0; 3, 7, 21; 6, 3, 18; 9, 4, 36; 36; 172.2 hours
8. **(a)** 40–50 **(b)** 49.7
9. 123.52 minutes, $122 < t \leqslant 124$
10. 32.65 years **11.** 34 minutes

Review Questions 14 *(Pg 320)*

1. **(a)** **(i)** 2 **(ii)** 3
(iii) 3.1
(b) 21.1
2. **(a)** 4, 16 **(b)** 3, 3
3. $x = 26$, $y = 29$
4. **(a)** 117, 3, –15, –45; 122, 4, –10, –40; 127, 8, –5, –40; 5, 0, 0; 137, 7, 5, 35; 142, 2, 10, 20; 147, 1, 15, 15; –55
(b) 125–129
(c) 130.2 cm
(d) 66.7%
5. **(a)** 58.8
(b) 42.1, –15, –15; 47, 2, –10, –20; 52, 5, –5, –25; 6, 0, 0; 62, 9, 5, 45; 67, 3, 10, 30; 72, 2, 15, 30; 45
(c) 58.6 **(d)** 0.34%
6. **(a)** **(i)** 1 **(ii)** 1.5
(iii) 1.7
(b) **(i)** 18.7 **(ii)** \$30
(iii) \$176

7. **(a)** 6; 8; 8 **(b)** 77
(c) 39.74 km/h

Revision Exercise IV
No. 1 *(Pg 325)*

1. **(a)** \$60 **(b)** 4
(c) $\pm 4\sqrt{\frac{hR}{7g}}$
(d) $x = 7.4$, $y = 6.5$
2. 34.3
3. **(b)** 2 m/s², $4\frac{1}{2}$ m/s²
(c) $15\frac{15}{44}$ m/s
4. **(b)** 56.5
5. **(a)** 005.4° **(b)** 078.6°
(c) 11.9 km
6. **(a)** 97.2° **(b)** 17.9 cm²
(c) 64.6 cm² **(d)** 101.4°
(e) 10.8 cm
8. $3\frac{1}{5}$ cm
9. **(a)** 7.2 cm **(b)** 64.3°
(c) 76.5°
10. **(a)** **(i)** $66\frac{2}{3}$ cm
(ii) $45\,511\frac{1}{9}$ cm³
(b) 2.92 m²

Revision Exercise IV
No. 2 *(Pg 326)*

1. **(a)** \$15.00 **(b)** $\frac{3}{8}$
(c) 10
2. **(a)** **(i)** $(3a + 4b)(4x - 3y)$
(ii) $\left(\frac{x^2}{8} + \frac{y^2}{3}\right)\left(\frac{x^2}{8} - \frac{y^2}{3}\right)$
(b) **(i)** $x = 4$
(ii) $x = 1$ or $1\frac{1}{3}$
3. **(a)** b **(b)** $(b - a)$
(c) $(2b - a)$ **(d)** $(180 - 2b)$
(e) $(180 + a - 3b)$
4. **(b)** **(i)** 8.49 cm **(ii)** 7.35 cm
(iii) 10.4 cm
5. **(a)** (3, 4) **(b)** (0, –3)
(c) 21 units²
(d) $\sqrt{13}$ **(e)** $3y + 9 = 2x$
6. –3, –1, 9, 32 **(c)** 0.75
(d) $x = 1$ **(e)** $x = -1, 1, 2$
7. **(a)** 7 **(b)** 13 **(c)** 11
8. **(a)** 13, January
(b) 52 **(c)** $8\frac{2}{3}$
9. **(a)** 3 : 2 **(b)** 9 : 4
10. **(a)** $6\frac{2}{13}$ cm **(b)** $6\frac{2}{5}$ cm

Revision Exercise IV
No. 3 *(Pg 327)*

1. (a) 3.072×10^0
(b) 6×10^6 **(c)** 2.88×10^{11}
2. (a) 40 **(b)** 32
(c) 35% **(d)** 36 min
3. (a) 7.8 cm **(b)** 33.7°
(c) 42.6° **(d)** 12 cm
(e) 46.8 cm²
4. (a) 9.77 cm **(b)** 13.1 cm
(c) 59.2° **(d)** 19.2°
5. (a) 36° **(b)** 54°
(c) 18°
6. (a) 3.56 m **(b)** 10.62 m
(c) 70.4° **(d)** 30.7°
7. (a) $20 < x \leqslant 30$
(b) 31
8. (a) 25.1 cm **(b)** 251 cm²
(c) 190 cm², 61.1 cm²
9. (b) 4 m/s² **(c)** 224 m
10. (b) $m = 4\frac{1}{2}$ cm, $l = 6$ cm

End-of-Year Examination
Specimen Paper 1 *(Pg 330)*

1. (a) $2\frac{53}{60}$ **(b)** $2\frac{1}{4}$ **(c)** $1\frac{1}{8}$
2. (a) 0.024 **(b)** 0.023 98
(c) $2.398\ 49 \times 10^{-2}$
3. (a) $(x + 3)(x - 7)$
(b) $(7 + 2x)(3 - x)$
(c) $4(x - y)(a - 4)$
4. (a) 29, 40
(b) 58, 94
(c) $\frac{2}{13}, \frac{1}{14}$
5. (a) –2 **(b)** $2\sqrt{3}$
6. $a = \frac{4c - 5F}{2b - 3F}$
7. $y = 24, x = 4, y = 3x^3$
8. (a) $x = -32$ **(b)** $x = 1$
9. 11, 13 **10.** $4\frac{8}{11}$ km/h
11. 514.8 kg **12.** $\frac{6}{7}$
13. 20%
14. (a) 1 **(b)** $y = x - 3$
15. (a) 6 **(b)** 7 **(c)** 6.89
16. (a) 54° **(b)** 30.6°
17. (a) (i) 52° **(ii)** 8.96 cm
(iii) 22.24 cm²
(b) (i) $7\frac{1}{2}$ m/s²
(ii) 175 m
(iii) $14\frac{7}{12}$ m/s

18. (a) (i) 68° **(ii)** 82°
(iii) 54°
(b) 30.59 cm²
19. $a = 19, b = 9$
(a) 3.55 or –0.55
(b) 4.55 or –1.55, $x = 1$ or 3
20. (b) 49.5 **(c)** 50.3
21. (a) 36.77 km/h
(b) $4.80
(c) (i) $\triangle ABC$ and $\triangle AYZ$, $\triangle BCX$ and $\triangle ZYX$
(ii) 9cm, $7\frac{7}{16}$ cm
22. (a) 13 40
(b) 1 562.5 cm², 7 500 cm³

End-of-Year Examination
Specimen Paper 2 *(Pg 332)*

1. (a) $4(1 + 2x)(1 - 2x)$
(b) $(a - b)(2 - 3c)$
2. $x = 2, y = 3$
3. $y = 3(2x - 3)$, 33, 12
4. (a) 1 **(b)** $\frac{4}{11}$ **(c)** $\frac{1}{32}$
5. (a) 2 : 5 **(b)** 4 : 25
6. (a) $y = 3x - 1$ **(b)** $\left(\frac{1}{3}, 0\right)$
7. (a) 6.8 km/h **(b)** $1\frac{8}{9}$ m/s
8. 10, 12 **9.** 375
10. (a) 50 km/h² **(b)** 315 km
11. 90° **12.** $x + 5$
13. (a) 148 **(b)** $-\frac{4}{5}, -\frac{3}{4}$
14. (a) 6 **(b)** $\frac{1}{6}$
15. 23°
16. (a) 2 km **(b)** $\frac{1}{4}$ km²
17. 1, 2, 3, 4, 5
18. 21, 29, 13
(b) (i) –2.45, 3.8
(ii) –1.9, 3.2
(c) –14
(d) –2 or 3
19. (a) $\frac{x + 2}{(x + 1)(x - 1)}$
(b) $x = \frac{5y^2 + y}{3y^2 - 2}$
(c) $6\frac{1}{4}$
20. (a) (i) $\frac{3}{4}$ m/s²
(ii) 1 125 m
(iii) 1 000 m
(iv) $t = 30$s

21. (a) 36, 12
(b) 3.82 or –1.48
22. (a) 58 cents, 50 cents
(b) (i) 2
(iii) 45.5%
23. (a) $156.37, $161. 97, $138.47
(b) (i) $172.90
(ii) 12.5% decrease
(iii) 8.6 units

End-of-Year Examination
Specimen Paper 3 *(Pg 335)*

1. (a) 1.7 **(b)** 700
(c) 2.45×10^{-3}
2. 64 km/h
3. (a) 4 **(b)** 4 **(c)** $\frac{1}{4}$
4. (a) $-\frac{7}{24}$ **(b)** $-\frac{24}{25}$
5. $x = 4, y = 3$
6. 6.8, 6.5, 3 or 7
7. 75°, 50°
8. $x = \frac{2y^2}{1 - y^2}$
9. 48°
10. $12 + 8\pi$
11. (a) 6
(b) (i) $1\frac{3}{5}$ **(ii)** $-\frac{2}{3}$
(iii) $1\frac{2}{7}$
12. (a) 3, 9 **(b)** $-5, -\frac{1}{2}$
13. $y = \frac{18}{x^2}, 4\frac{1}{2}$
14. (a) (0, 7)
(b) (3, –2)
15. (a) 1.77 or –2.44
(b) 20 km/h
16. (a) $10 \sin x$
(b) $105 \sin x$, 44.4°
17. (a) $h = 1, k = -3$
(c) –0.6 or 1.6
(d) 3
(e) $x \leqslant -3.8$ or $x \geqslant 1.8$
(f) $x = -0.5$
18. (a) 21 **(c)** 6
19. (a) 616 cm²
(b) 190 m, 15.9°
20. (a) $12.78 **(b)** 40
(c) 75
21. (a) (i) 20 m/s
(ii) 624 m
(iii) 75 sec
(b) (ii) 56.9
(iii) 57